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Existence of solutions for some classes of integro-differential equations via measure of noncompactness

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Abstract. In this present paper, we introduce a new measure of noncompactness on the space consisting of all real functions which are n times bounded and continuously differentiable on $\mathbb{R}_+$. As an application, we investigate the problem of the existence of solutions for some classes of the functional integral-differential equations which enables us to study the existence of solutions of nonlinear integro-differential equations. In our considerations we apply the technique of measures of noncompactness in conjunction with Darbo's fixed point theorem. Finally, we give some illustrative examples to verify the effectiveness and applicability of our results.

Keywords: measures of noncompactness, Darbo's fixed point theorem, integro-differential equations.

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1 Introduction

Integro-differential equation (IDE) have a great deal of application in different branches of sciences and engineering. It arises naturally in a variety of models from biological science, applied mathematics, physics, and other disciplines, such as the theory of elasticity, biomechanics, electromagnetic, electrodynamics, fluid dynamics, heat and mass transfer, oscillating magnetic field, etc, see [14, 18, 21, 23].

Many papers have been devoted to the study of (IDE) and its versions by using different techniques, for example, some of the existing numerical methods can be found in [6, 7, 12, 13, 19, 22, 27, 30–32] and the references therein. The tools utilized in these papers are: the tau method, direct methods, collocation methods, Runge–Kutta methods, wavelet methods and spline approximation.

On the other hand, measures of noncompactness are very useful tools in the theory of operator equations in Banach spaces. They are frequently used in the theory of functional

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equations, including ordinary differential equations, equations with partial derivatives, integral and integro-differential equations, optimal control theory, etc. In particular, the fixed point theorems derived from them have many applications. There exists an enormous amount of considerable literature devoted to this subject (see for example [1,8–11,15,16,20,21,28,29]). The first measure of noncompactness was introduced by Kuratowski [24] in the following way.

$$\alpha(S) := \inf\left\{\delta > 0 \,\Big|\, S = \bigcup_{i=1}^{n} S_i \text{ for some } S_i \text{ with } \operatorname{diam}(S_i) \le \delta \text{ for } 1 \le i \le n \le \infty\right\}.$$

Here $\operatorname{diam}(T)$ denotes the diameter of a set $T \subset X$, namely $\operatorname{diam}(T) := \sup\{d(x,y) \,|\, x,y \in T\}$. Another important measure is the so-called Hausdorff (or ball) measure of noncompactness is defined as follows

$$\chi(X) = \inf\{\varepsilon : X \text{ has a finite } \varepsilon\text{-net in } E\}. \tag{1.1}$$

These measures share several useful properties [5,9]. These measures seem to be nice, but they are rather rarely applied in practice. Hence, in order to resolve this problem, Banaś presented some measures of noncompactness being the most frequently utilized in applications (see [10,11]).

The principal application of measures of noncompactness in the fixed point theory is contained in Darbo's fixed point theorem [9]. The technique of measures of noncompactness in conjunction with it turned into a tool to investigate the existence and behavior of solutions of many classes of integral equations such as Volterra, Fredholm and Urysohn type integral equations (see [2–4,9,11,16,17,25]).

Now, in this paper, as a more effective approach, similar to the measures of noncompactness considered in [10,11], in the first place we introduce a new measure of noncompactness on the space consisting of all real functions which are n times bounded and continuously differentiable on $\mathbb{R}_+$. Then we study the problem of existence of solutions of the functional integral-differential equation

$$x(t) = p(t) + q(t)x(t) + \int_0^t g(t,s,x(\xi(s)),x'(\xi(s)),\dots,x^{(n)}(\xi(s)),Tx(s))\,ds \tag{1.2}$$

on this space.

As a special case of (1.2) we can refer to the integro-differential equation

$$\begin{aligned} x^{(n)}(t) = f_1\Big(&t, x(\xi(t)), x'(\xi(t)),\dots,x^{(n-1)}(\xi(t)),\\ &\int_0^\infty k(t,s) f_2(s,x(s),x'(s),\dots,x^{(n-1)}(s)\,ds\Big), \end{aligned} \tag{1.3}$$
$$x(0) = x_0,\ x'(0) = x_1,\dots,\ x^{(n-1)}(0) = x_{n-1},$$

and the integro-differential equation

$$\begin{aligned} x^{(n)}(t) = f_1\Big(&t, x(\xi(t)),\ x'(\xi(t)),\dots,x^{(n-1)}(\xi(t)),\\ &\int_0^t k(t,s) f_2(s,x(s),x'(s),\dots,x^{(n-1)}(s)\,ds\Big), \end{aligned} \tag{1.4}$$
$$x(0) = x_0,\ x'(0) = x_1,\dots,\ x^{(n-1)}(0) = x_{n-1},$$

which will be investigated in this paper. In our considerations, we apply Darbo's fixed point theorem associated with this new measure of noncompactness. Finally, some examples are presented to verify the effectiveness and applicability of our results.

2 Preliminaries

In this section, we recall some basic facts concerning measures of noncompactness, which are defined axiomatically in terms of some natural conditions. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. Let $\overline{B}(x,r)$ denote the closed ball centered at x with radius r. The symbol $\overline{B}_r$ stands for the ball $\overline{B}(0,r)$. For X, a nonempty subset of E, we denote by $\overline{X}$ and Conv X the closure and the closed convex hull of X, respectively. Moreover, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets of E.

Definition 2.1 ([9]). A mapping $\mu\colon \mathfrak{M}_E \longrightarrow \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1° the family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$;

2° $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$;

3° $\mu(\overline{X}) = \mu(X)$;

4° $\mu(\operatorname{Conv} X) = \mu(X)$;

5° $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$;

6° if $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1,2,\ldots$, and if $\lim_{n\to\infty}\mu(X_n) = 0$, then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$.

In what follows, we recall the well known fixed point theorem of Darbo type [9].

Theorem 2.2. *Let Ω be a nonempty, bounded, closed and convex subset of a space E and let $F\colon \Omega \longrightarrow \Omega$ be a continuous mapping such that there exists a constant $k \in [0,1)$ with the property*

$$\mu(FX) \leq k\mu(X) \tag{2.1}$$

for any nonempty subset X of Ω. Then F has a fixed point in the set Ω.

Here $BC(\mathbb{R}_+)$ is the Banach space of all bounded and continuous function on $\mathbb{R}_+$ equipped with the standard norm

$$\|x\|_u = \sup\{|x(t)| : t \geq 0\}.$$

For any nonempty bounded subset X of $BC(\mathbb{R}_+)$, $x \in X$, $T > 0$ and $\varepsilon \geq 0$ let

$$\begin{aligned}
\omega^T(x,\varepsilon) &= \sup\{|x(t) - x(s)| : t,s \in [0,T],\ |t-s| \leq \varepsilon\}.\\
\omega^T(X,\varepsilon) &= \sup\left\{\omega^T(x,\varepsilon) : x \in X\right\},\\
\omega_0^T(X) &= \lim_{\varepsilon\to 0}\omega^T(X,\varepsilon), \quad \omega_0(X) = \lim_{T\to\infty}\omega_0^T(X),\\
X(t) &= \{x(t) : x \in X\},
\end{aligned}$$

and

$$\mu_1(X) = \omega_0(X) + \limsup_{t\to\infty} \operatorname{diam} X(t).$$

It was demonstrated in [9] that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$.

3 Main results

In this section, we introduce a measure of noncompactness on $BC^n(\mathbb{R}_+)$.

Let $BC^n(\mathbb{R}_+) = \{f \in C^n(\mathbb{R}_+) : e^{-t}|f^{(i)}(t)| \text{ is bounded for all } t \geq 0,\ i = 1, 2, \dots, n\}$, where $f^{(0)} = f$. It is easy to see that $BC^n(\mathbb{R}_+)$ is a Banach space with norm

$$\|f\|_{BC^n(\mathbb{R}_+)} = \max_{0 \leq k \leq n} \|hf^{(k)}\|_u,$$

where $h(t) = e^{-t}$ for all $t \in \mathbb{R}_+$.

Theorem 3.1. *Suppose $1 \leq n < \infty$ and X be a bounded subset of $BC^n(\mathbb{R}_+)$. Then $\mu\colon \mathfrak{M}_{BC^n(\mathbb{R}_+)} \longrightarrow \mathbb{R}_+$ given by*

$$\mu(X) = \max_{0 \leq k \leq n} \mu_1(X^{(k)}) \tag{3.1}$$

defines a measure of noncompactness on $BC^n(\mathbb{R}_+)$, where $X^{(k)} = \{hx^{(k)} : x \in X\}$.

The proof relies on the following useful observations.

Lemma 3.2 ([5]). *Suppose $\mu_1, \mu_2, \dots, \mu_n$ are measures of noncompactness in Banach spaces $E_1, E_2, \dots, E_n$, respectively. Moreover assume that the function $F\colon \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$ is convex and $F(x_1, \dots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \dots, n$. Then*

$$\mu(X) = F(\mu_1(X_1), \mu_2(X_2), \dots, \mu_n(X_n)),$$

defines a measure of noncompactness in $E_1 \times E_2 \times \cdots \times E_n$ where X_i denotes the natural projection of X into E_i, for $i = 1, 2, \dots, n$.

Lemma 3.3 ([26]). *Let $(E_i, \|\cdot\|_i)$, for $i = 1, 2$ be Banach spaces and let $L\colon E_1 \longrightarrow E_2$ be a one-to-one, continuous linear operator of E_1 onto E_2. If μ_2 is a measure of noncompactness on E_2, define, for $X \in \mathfrak{M}_{E_1}$,*

$$\widetilde{\mu}_2(X) := \mu_2(LX).$$

Then $\widetilde{\mu}_2$ is a measure of noncompactness on E_1.

Proof of Theorem 3.1. First, consider $E = (BC(\mathbb{R}_+))^{n+1}$ equipped with the norm

$$\|(x_1, \dots, x_n, x_{n+1})\| = \max_{1 \leq i \leq n+1} \|x_i\|_u.$$

Also, $F(x_1, \dots, x_{n+1}) = \max_{1 \leq i \leq n+1} x_i$ for any $(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_+$, therefore all the conditions of Lemma 3.2 are satisfied. Then

$$\mu_2(X) := \max_{1 \leq i \leq n+1} \mu_1(X_i),$$

defines a measure of noncompactness in the space E where X_i denotes the natural projection of X, for $i = 1, 2, \dots, n+1$. Now, we define the operator $L\colon BC^n(\mathbb{R}_+) \longrightarrow E$ by the formula

$$L(x) = (hx, hx', hx'', hx^{(3)}, \dots, hx^{(n)}).$$

Obviously, L is a one-to-one and continuous linear operator. We show that $L(BC^n(\mathbb{R}_+))$ is closed in E. To do this, let us choose $\{x_n\} \subset BC^n(\mathbb{R}_+)$ such that $L(x_n)$ is a Cauchy sequence in E. Thus, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $k, m > N$ we have

$$\|L(x_k - x_m)\| < \varepsilon.$$

So, we deduce

$$\begin{aligned}\|x_k - x_m\|_{BC^n(\mathbb{R}_+)} &= \max_{0\le i\le n} \|h(x_k^{(i)} - x_m^{(i)})\|_u \\ &= \|(h(x_k - x_m), h(x_k' - x_m'), \ldots, h(x_k^{(n)} - x_m^{(n)}))\| \\ &= \|L(x_k - x_m)\| \\ &< \varepsilon.\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence of $BC^n(\mathbb{R}_+)$, and there exists $x \in BC^n(\mathbb{R}_+)$ such that $x_n \longrightarrow x$. Since L is continuous, so we have $L(x_n) \longrightarrow L(x)$. This implies that $Y = L(BC^n(\mathbb{R}_+))$ is closed. Thus, the operator $L\colon BC^n(\mathbb{R}_+) \longrightarrow Y$ be a one-to-one and continuous linear operator of $BC^n(\mathbb{R}_+)$ onto Y. Since Y is a closed subspace of X, so μ_2 is a measure of noncompactness on Y. Hence, for $X \in \mathfrak{M}_{BC^n(\mathbb{R}_+)}$,

$$\widetilde{\mu_2}(X) = \mu_2(LX) = \max_{0\le k\le n} \mu_1(X^{(k)}) = \mu(X).$$

Now using Lemma 3.3, the proof is complete. □

Corollary 3.4. *Let $\mathcal{F}$ be a bounded set in $BC^n(\mathbb{R}_+)$ with $1 \le n < \infty$. Also, assume that for every $\varepsilon > 0$ and $T > 0$, there exist $\delta > 0$ such that for all $0 \le k \le n$, $t, s \in [0, T]$ with $|t - s| < \delta$ and $f \in \mathcal{F}$*

$$|f^{(k)}(t) - f^{(k)}(s)| < \varepsilon,$$

and

$$\lim_{t\longrightarrow\infty} |e^{-t} f^{(k)}(t) - e^{-t} g^{(k)}(t)| = 0 \tag{3.2}$$

uniformly with respect to $f, g \in \mathcal{F}$, for all $0 \le k \le n$. Then $\mathcal{F}$ will be a totally bounded subset of $BC^n(\mathbb{R}_+)$.

Proof. It is enough to show that $\mu(\mathcal{F}) = 0$. Take an arbitrary $\varepsilon > 0$ and $T > 0$, there exists $\delta > 0$ such that for all $0 \le k \le n$, $t, s \in [0, T]$ with $|t - s| < \delta$ and $f \in \mathcal{F}$, we have

$$|f^{(k)}(t) - f^{(k)}(s)| < \varepsilon.$$

Thus, we obtain

$$\max_{0\le k\le n} \omega^T(hf^{(k)}, \delta) = \max_{0\le k\le n} \sup \left\{e^{-t}|f^{(k)}(t) - f^{(k)}(s)| : t, s \in [0, T], |t - s| \le \delta\right\} \le \varepsilon$$

for all $f \in \mathcal{F}$, and we deduce

$$\max_{0\le k\le n} \omega^T(\mathcal{F}^{(k)}, \delta) \le \varepsilon.$$

Therefore, we obtain ${\omega_0}^T(\mathcal{F}^{(k)}) = 0$ for all $0 \le k \le n$, and finally

$$\max_{0\le k\le n} \omega_0(\mathcal{F}^{(k)}) = 0. \tag{3.3}$$

On the other hand, take $\varepsilon > 0$, by 3.2, there exists $T > 0$ such that

$$e^{-t}|f^{(k)}(t) - g^{(k)}(t)| < \varepsilon$$

for all $t > T$ and $f, g \in \mathcal{F}$. Thus, we have

$$\max_{0\le k\le n} \operatorname{diam} \mathcal{F}^{(k)}(t) \le \max_{0\le k\le n} \sup \left\{e^{-t}|f^{(k)}(t) - g^{(k)}(t)| : f, g \in \mathcal{F}\right\} \le \varepsilon$$

for all $t > T$, so we deduce

$$\max_{0\le k\le n} \limsup_{t\to\infty} \operatorname{diam} \mathcal{F}^{(k)}(t) = 0. \tag{3.4}$$

Further, combining (3.3) and (3.4), we get

$$\mu(\mathcal{F}) = \max_{0\le k\le n} \mu_1(\mathcal{F}^{(k)}) = \max_{0\le k\le n} \left(\omega_0(\mathcal{F}^{(k)}) + \limsup_{t\to\infty} \operatorname{diam} \mathcal{F}^{(k)}(t)\right) = 0, \tag{3.5}$$

and consequently $\mathcal{F}$ will be a totally bounded subset of $BC^n(\mathbb{R}_+)$. □

4 Existence of solutions for some classes of integro-differential equations

In this section we study the existence of solutions for (1.2)–(1.4). Further, we present some illustrative examples to verify the effectiveness and applicability of our results.
We will consider the Equation (1.2) under the following assumptions:

(i) $\xi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous function;

(ii) $p, q \in BC^n(\mathbb{R}_+)$ and

$$\lambda := \sup\left\{\sum_{i=0}^{i=k} \binom{k}{i} \|q^{(k-i)}\|_u : 0 \le k \le n\right\} < 1; \tag{4.1}$$

(iii) $g\colon \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{n+2} \longrightarrow \mathbb{R}$ is continuous and has a continuous derivative of order n with respect to the first argument such that

$$g(t, t, x_0, x_1, \dots, x_{n+1}) = 0 \quad \text{for } t \in \mathbb{R}_+,\ x_0, x_1, \dots, x_{n+1} \in \mathbb{R}, \tag{4.2}$$

and there exists a nondecreasing and continuous function $\psi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\begin{cases} \sup\left\{e^{-t}\left|\int_0^t \frac{\partial^k g}{\partial t^k}\big(t, s, x_0(\xi(s)), x_1(\xi(s)), \dots, x_{n+1}(s)\big)\, ds\right| : t \in \mathbb{R}_+, \right.\\ \qquad \left. \|x_i\|_u \le r, 1 \le k \le n\right\} \le \psi(r), \\ \sup\left\{e^{-t}\left|\int_0^t g\big(t, s, x_0(\xi(s)), x_1(\xi(s)), \dots, x_{n+1}(s)\big)\, ds\right| : t \in \mathbb{R}_+,\ \|x_i\|_u \le r\right\} \le \psi(r) \end{cases} \tag{4.3}$$

for any $r \in \mathbb{R}_+$.
Moreover, for any $r \in \mathbb{R}_+$ and $1 \le k \le n$

$$\begin{cases} \lim\limits_{t\to\infty} e^{-t}\left|\int_0^t \left[g\big(t, s, x_0(\xi(s)), \dots, x_{n+1}(s)\big) - g\big(t, s, y_0(\xi(s)), \dots, y_{n+1}(s)\big)\right] ds\right| = 0, \\ \lim\limits_{t\to\infty} e^{-t}\left|\int_0^t \left[\frac{\partial^k g}{\partial t^k}\big(t, s, x_0(\xi(s)), \dots, x_{n+1}(s)\big) - \frac{\partial^k g}{\partial t^k}\big(t, s, y_0(\xi(s)), \dots, y_{n+1}(s)\big)\right] ds\right| = 0 \end{cases} \tag{4.4}$$

uniformly with respect to $x_i, y_i \in \bar{B}_r$;

(iv) $T\colon BC^n(\mathbb{R}_+) \longrightarrow BC^0(\mathbb{R}_+)$ is a continuous operator such that for any $x \in BC^n(\mathbb{R}_+)$ we have

$$\|T(x)\|_{BC^0(\mathbb{R}_+)} \leq \|x\|_{BC^n(\mathbb{R}_+)}; \tag{4.5}$$

(v) there exists a positive solution r_0 to the inequality

$$\|p\|_{BC^n(\mathbb{R}_+)} + \lambda r + \psi(r) \leq r.$$

Theorem 4.1. *Under assumptions (i)–(v), the equation (1.2) has at least a solution in the space $BC^n(\mathbb{R}_+)$.*

Proof. First of all we define the operator $F\colon BC^n(\mathbb{R}_+) \longrightarrow BC^n(\mathbb{R}_+)$ by

$$Fx(t) = p(t) + q(t)x(t) + \int_0^t g\big(t, s, x(\xi(s)), x'(\xi(s)), \dots, x^{(n)}(\xi(s)), Tx(s)\big)\, ds. \tag{4.6}$$

First, notice that the continuity of $Fx(t)$ for any $x \in BC^n(\mathbb{R}_+)$ is obvious. Also, for any $t \in \mathbb{R}_+$, $1 \leq k \leq n$ and by (4.2), we have

$$\begin{aligned}
\frac{d^k(Fx)}{dt^k}(t) &= p^{(k)}(t) + \sum_{i=0}^{i=k} \binom{k}{i} x^{(i)}(t) q^{(k-i)}(t) \\
&\quad + \int_0^t \frac{\partial^k g}{\partial t^k}\big(t, s, x(\xi(s)), x'(\xi(s)), \dots, x^{(n)}(\xi(s)), Tx(s)\big)\, ds,
\end{aligned}$$

and Fx has continuous derivative of order k $(1 \leq k \leq n)$. Using conditions (i)–(iv), for arbitrarily fixed $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
e^{-t}|Fx(t)| &\leq e^{-t}|p(t)| + e^{-t}|q(t)||x(t)| \\
&\quad + e^{-t}\left|\int_0^t g\big(t, s, x(\xi(s)), x'(\xi(s)), \dots, x^{(n)}(\xi(s)), Tx(s)\big)\, ds\right| \\
&\leq \|p\|_{BC^n(\mathbb{R}_+)} + \|q\|_u \|x\|_{BC^n(\mathbb{R}_+)} + \psi(\|x\|_{BC^n(\mathbb{R}_+)}),
\end{aligned}$$

and similarly

$$\begin{aligned}
e^{-t}\left|\frac{d^k(Fx)}{dt^k}(t)\right| &\leq e^{-t}|p^{(k)}(t)| + \sum_{i=0}^{i=k} \binom{k}{i} e^{-t}|x^{(i)}(t)||q^{(k-i)}(t)| \\
&\quad + e^{-t}\left|\int_0^t \frac{\partial^k g}{\partial t^k}\Big(t, s, x(\xi(s)), x'(\xi(s)), \dots, x^{(n)}(\xi(s)), Tx(s)\Big)\, ds\right| \\
&\leq \|p^{(k)}\|_u + \sum_{i=0}^{i=k} \binom{k}{i} \|q^{(k-i)}\|_u \|x\|_{BC^n(\mathbb{R}_+)} + \psi(\|x\|_{BC^n(\mathbb{R}_+)}).
\end{aligned}$$

Hence

$$\|Fx\|_{BC^n(\mathbb{R}_+)} \leq \|p\|_{BC^n(\mathbb{R}_+)} + \lambda\|x\|_{BC^n(\mathbb{R}_+)} + \psi(\|x\|_{BC^n(\mathbb{R}_+)}). \tag{4.7}$$

Due to Inequality (4.7) and using (v), the function F maps $\bar{B}_{r_0}$ into $\bar{B}_{r_0}$. We also show that the map F is continuous. For this, take $x \in BC^n(\mathbb{R}_+)$, $\varepsilon > 0$ arbitrarily and consider $y \in BC^n(\mathbb{R}_+)$

with $\|x-y\|_{BC^n(\mathbb{R}_+)} < \varepsilon$, then we obtain

$$\begin{aligned}
e^{-t}|Fx(t)-Fy(t)| &\le e^{-t}|q(t)||x(t)-y(t)| \\
&\quad + e^{-t}\bigg|\int_0^t g(t,s,x(\xi(s)),\dots,x^{(n)}(\xi(s)),Tx(s))\,ds \\
&\qquad - \int_0^t g(t,s,y(\xi(s)),\dots,y^{(n)}(\xi(s)),Ty(s))\,ds\bigg| \\
&\le \|q\|_u\|x-y\|_{BC^n(\mathbb{R}_+)} \\
&\quad + e^{-t}\int_0^t \Big|g(t,s,x(\xi(s)),\dots,x^{(n)}(\xi(s)),Tx(s)) \\
&\qquad - g(t,s,y(\xi(s)),\dots,y^{(n)}(\xi(s)),Ty(s))\Big|\,ds,
\end{aligned} \tag{4.8}$$

and similarly

$$\begin{aligned}
e^{-t}\left|\frac{d^k(Fx)}{dt^k}(t)-\frac{d^k(Fy)}{dt^k}(t)\right| &\le \sum_{i=0}^{i=k}\binom{k}{i}\|q^{(k-i)}\|\,\|x-y\|_{BC^n(\mathbb{R}_+)} \\
&\quad + e^{-t}\int_0^t \left|\frac{d^kg}{dt^k}(t,s,x(\xi(s)),\dots,x^{(n)}(\xi(s)),Tx(s))\right. \\
&\qquad \left. - \frac{d^kg}{dt^k}(t,s,y(\xi(s)),\dots,y^{(n)}(\xi(s)),Ty(s))\right|ds.
\end{aligned} \tag{4.9}$$

Furthermore, considering condition (iii), there exists $T>0$ such that for $t>T$, we have

$$e^{-t}|Fx(t)-Fy(t)| \le \|q\|_u\|x-y\|_u + \varepsilon,$$

and similarly

$$e^{-t}\left|\frac{d^k(Fx)}{dt^k}(t)-\frac{d^k(Fy)}{dt^k}(t)\right| \le \sum_{i=0}^{i=k}\binom{k}{i}\|q^{(k-i)}\|\,\|x-y\|_{BC^n(\mathbb{R}_+)} + \varepsilon.$$

Also, if $t\in[0,T]$, then from (4.8) and (4.9), it follows that

$$e^{-t}|Fx(t)-Fy(t)| \le \|q\|_u\|x-y\|_u + T\theta_T(\varepsilon),$$

and similarly

$$e^{-t}\left|\frac{d^k(Fx)}{dt^k}(t)-\frac{d^k(Fy)}{dt^k}(t)\right| \le \left(\sum_{i=0}^{i=k}\binom{k}{i}\|q^{(k-i)}\|\right)\|x-y\|_{BC^n(\mathbb{R}_+)} + T\vartheta_T(\varepsilon),$$

where

$$\begin{aligned}
\theta_T(\varepsilon) &= \sup\Big\{e^{-t}\big|g(t,s,x_0,x_1\dots,x_{n+1})-g(t,s,y_0,y_1,\dots,y_{n+1})\big| : t,s\in[0,T], \\
&\qquad x_i,y_i\in[-b,b],|x_i-y_i|\le\varepsilon\Big\}, \\
\vartheta_T(\varepsilon) &= \sup\left\{e^{-t}\left|\frac{d^kg}{dt^k}(t,s,x_0,x_1\dots,x_{n+1})-\frac{d^kg}{dt^k}(t,s,y_0,y_1,\dots,y_{n+1})\right| : t,s\in[0,T],\right. \\
&\qquad \left. x_i,y_i\in[-b,b],|x_i-y_i|\le\varepsilon\right\}, \\
b &= e^T(\|x\|+\varepsilon).
\end{aligned}$$

By using the continuity of g and $\frac{d^k g}{dt^k}$ on $[0,T]\times[0,T]\times[-b,b]^{n+2}$, we have $\theta_T(\varepsilon)\longrightarrow 0$ and $\vartheta_T(\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$.Thus F is a continuous operator on $BC^n(\mathbb{R}_+)$ into $BC^n(\mathbb{R}_+)$. Now, let X be a nonempty and bounded subset of $\bar{B}_{r_0}$, and assume that $T>0$ and $\varepsilon>0$ are arbitrary constants. Let $t_1,t_2\in[0,T]$, with $|t_2-t_1|\le\varepsilon$ and $x\in X$. We obtain

$$\begin{aligned}
&e^{-t}|Fx(t_1)-Fx(t_2)|\\
&\quad= e^{-t}\bigg|p(t_1)+q(t_1)x(t_1)+\int_0^{t_1} g(t_1,s,x(\xi(s)),x'(\xi(s)),\dots,x^{(n)}(\xi(s)))ds\\
&\qquad\quad-\left(p(t_2)+q(t_2)x(t_2)+\int_0^{t_2} g(t_2,s,x(\xi(s)),x'(\xi(s)),\dots,x^{(n)}(\xi(s)))ds\right)\bigg|\\
&\quad\le e^{-t}|p(t_1)-p(t_2)|+e^{-t}|q(t_1)-q(t_2)||x(t_1)|+|q(t_2)||x(t_1)-x(t_2)|\\
&\qquad+e^{-t}\bigg|\int_0^{t_1} g(t_1,s,x(\xi(s)),x'(\xi(s)),\dots,x^{(n)}(\xi(s)))ds\\
&\qquad\quad-\int_0^{t_2} g(t_2,s,x(\xi(s)),x'(\xi(s)),\dots,x^{(n)}(\xi(s)))ds\bigg|\\
&\quad\le e^{-t}|p(t_1)-p(t_2)|+e^{-t}|q(t_1)-q(t_2)||x(t_1)|+e^{-t}|q(t_2)||x(t_1)-x(t_2)|\\
&\qquad+e^{-t}\left|\int_{t_1}^{t_2} g(t_1,s,x(\xi(s)),x'(\xi(s)),\dots,x^{(n)}(\xi(s)))ds\right|\\
&\qquad+e^{-t}\left|\int_0^{t_2}\left[g(t_1,s,x(\xi(s)),\dots,x^{(n)}(\xi(s)))-g(t_2,s,x(\xi(s)),\dots,x^{(n)}(\xi(s)))\right]ds\right|\\
&\quad\le \omega^T(p,\varepsilon)+r_0\omega^T(q,\varepsilon)+\lambda\omega^T(hx,\varepsilon)+U_{r_0}\varepsilon+T\omega^T_{r_0}(g,\varepsilon),
\end{aligned}\tag{4.10}$$

and similarly

$$\begin{aligned}
e^{-t}\left|\frac{d^k(Fx)}{dt^k}(t_1)-\frac{d^k(Fx)}{dt^k}(t_2)\right| &\le \omega^T(p^{(k)},\varepsilon)+r_0\omega^T(q^{(k)},\varepsilon)+\lambda\max_{0\le i\le k}\omega^T(hx^{(i)},\varepsilon)\\
&\quad+W_{r_0}\varepsilon+T\omega^T_{r_0}\left(\frac{d^k g}{dt^k},\varepsilon\right)
\end{aligned}\tag{4.11}$$

where

$$\begin{aligned}
U_{r_0} &= \sup\left\{e^{-t}|g(t,s,x_0,x_1\dots,x_{n+1})| : t,s\in[0,T],|x_i|\le r_0\right\},\\
W_{r_0} &= \sup\left\{e^{-t}\left|\frac{d^k g}{dt^k}(t,s,x_0,x_1\dots,x_{n+1})\right| : t,s\in[0,T],1\le k\le n,|x_i|\le r_0\right\},\\
\omega^T_{r_0}(g,\varepsilon) &= \sup\Big\{e^{-t}|g(t_1,s,x_0,x_1\dots,x_{n+1})-g(t_2,s,x_0,x_1\dots,x_{n+1})| : s,t_1,t_2\in[0,T],\\
&\qquad |x_i|\le r_0,|t_1-t_2|\le\varepsilon\Big\},\\
\omega^T_{r_0}\left(\frac{d^k g}{dt^k},\varepsilon\right) &= \sup\Big\{e^{-t}\left|\frac{d^k g}{dt^k}(t_1,s,x_0,x_1\dots,x_{n+1})-\frac{d^k g}{dt^k}(t_2,s,x_0,x_1\dots,x_{n+1})\right| :\\
&\qquad s,t_1,t_2\in[0,T],\ 1\le k\le n,|x_i|\le r_0,|t_1-t_2|\le\varepsilon\Big\}.
\end{aligned}$$

Since x was arbitrary element of X in (4.10) and (4.11), we obtain

$$\omega^T([F(X)]^{(0)},\varepsilon)\le\omega^T(p,\varepsilon)+r_0\omega^T(q,\varepsilon)+\lambda\omega^T(X^{(0)},\varepsilon)+U_{r_0}\varepsilon+T\omega^T_{r_0}(g,\varepsilon),$$

and similarly

$$\omega^T([F(X)]^{(k)},\varepsilon) \le \omega^T(p^{(k)},\varepsilon) + r_0\omega^T(q^{(k)},\varepsilon) + \lambda \max_{0\le i\le k} \omega^T(X^{(i)},\varepsilon) + W_{r_0}\varepsilon + T\omega^T_{r_0}\left(\frac{d^k g}{dt^k},\varepsilon\right)$$

for all $1 \le k \le n$. Thus, by the uniform continuity of $p^{(i)}$ and $q^{(i)}$ on the compact set $[0,T]$ for all $0 \le i \le n$, and g, $\frac{\partial^k g}{\partial t^k}$ on the compact sets $[0,T]\times[0,T]\times[-b,b]^{n+2}$, we have $\omega(p^{(i)},\varepsilon) \longrightarrow 0$, $\omega(q^{(i)},\varepsilon) \longrightarrow 0$, $\omega(g,\varepsilon) \longrightarrow 0$ and $\omega(\frac{\partial^k g}{\partial t^k},\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Therefore, we obtain

$$\omega_0^T([F(X)]^{(k)}) \le \lambda \max_{0\le i\le k} \omega_0^T(X^{(i)}),$$

for any $0 \le k \le n$, and finally

$$\max_{0\le k\le n} \omega_0([F(X)]^{(k)}) \le \lambda \max_{0\le k\le n} \omega_0(X^{(k)}). \tag{4.12}$$

On the other hand, for all $x, y \in X$ and $t \in \mathbb{R}_+$ we get

$$\begin{aligned}
&e^{-t}|Fx(t) - Fy(t)| \\
&\quad\le e^{-t}|q(t)||x(t) - y(t)| \\
&\qquad + e^{-t}\left|\int_0^t [g(t,s,x(s),x'(s),\ldots,x^{(n)}(s)) - g(t,s,y(s),y'(s),\ldots,y^{(n)}(s))]ds\right| \\
&\quad\le \|q\|_u \operatorname{diam}(X^{(0)}) + \zeta_0(t),
\end{aligned}$$

and similarly

$$e^{-t}\left|\frac{d^k(Fx)}{dt^k}(t) - \frac{d^k(Fy)}{dt^k}(t)\right| \le \lambda \max_{0\le i\le k} \operatorname{diam}(X^{(i)}) + \zeta_k(t),$$

where

$$\begin{aligned}
\zeta_0(t) &= \sup\left\{e^{-t}\left|\int_0^t \left[g(t,s,x_0(s),\ldots,x_{n+1}(s)) - g(t,s,y_0(s),\ldots,y_{n+1}(s))\right]ds\right| :\right. \\
&\qquad \left. x_i, y_i \in BC(\mathbb{R}_+)\right\}, \\
\zeta_k(t) &= \sup\left\{e^{-t}\left|\int_0^t \left[\frac{d^k g}{dt^k}(t,s,x_0(s),\ldots,x_{n+1}(s)) - \frac{d^k g}{dt^k}(t,s,y_0(s),\ldots,y_{n+1}(s))\right]ds\right| :\right. \\
&\qquad \left. x_i, y_i \in BC(\mathbb{R}_+), 1 \le k \le n\right\}.
\end{aligned}$$

Thus

$$\operatorname{diam}([FX]^{(k)}) \le \lambda \max_{0\le i\le k} \operatorname{diam}(X^{(i)}) + \zeta_k(t). \tag{4.13}$$

Taking $t \longrightarrow \infty$ in the inequality (4.13), then using (iii) we arrive at

$$\max_{0\le k\le n} \limsup_{t\longrightarrow\infty} \operatorname{diam}([FX]^{(k)}) \le \lambda \max_{0\le k\le n} \limsup_{t\longrightarrow\infty} \operatorname{diam}(X^{(k)}). \tag{4.14}$$

Further, combining (4.12) and (4.14) we get

$$\begin{aligned}
&\max_{0\le k\le n}\left\{\omega_0([F(X)]^{(k)}) + \limsup_{t\longrightarrow\infty} \operatorname{diam}([FX]^{(k)})\right\} \\
&\qquad \le \lambda \max_{0\le k\le n}\left\{\omega_0(X^{(k)}) + \limsup_{t\longrightarrow\infty} \operatorname{diam}(X^{(k)})\right\},
\end{aligned} \tag{4.15}$$

or, equivalently
$$\mu(FX) \leq \lambda\mu(X),$$
where $\lambda \in [0,1)$. From Theorem 2.2 we obtain that the operator F has a fixed point x in $\bar{B}_{r_0}$ and thus the functional integral-differential equation (1.2) has at least a solution in $BC^n(\mathbb{R}_+)$. □

Corollary 4.2. *Assume that the following conditions are satisfied:*

(i) $k\colon \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ *and* $\xi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *are continuous functions such that*
$$\sup\left\{e^{-t}\left|\int_0^\infty k(t,s)e^s\,ds\right| : t \in \mathbb{R}_+\right\} \leq 1;$$
and
$$\lim_{t\longrightarrow\infty} e^{-t}\left|\int_0^\infty k(t,s)e^s\,ds\right| = 0.$$
Moreover, $x_0, x_1, \ldots, x_{n-1} \in \mathbb{R}_+$.

(ii) $f_1 : \mathbb{R}_+ \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ *is continuous and there exist continuous functions* $a, b\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *such that*
$$|f_1(t, x_0, x_1, \ldots, x_n)| \leq a(t)b(\max_{0\leq i\leq n} |x_i|). \tag{4.16}$$
Moreover, there exists a positive constant D *such that*
$$\sup\left\{e^{-t}\left|\int_0^t a(s)(t-s)^k\,ds\right| : t \in \mathbb{R}_+,\ 0 \leq k \leq n-1\right\} \leq D,$$
and
$$\lim_{t\longrightarrow\infty} \sup\left\{e^{-t}\left|\int_0^t a(s)(t-s)^k\,ds\right| : 0 \leq k \leq n-1\right\} = 0.$$

(iii) $f_2 : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *is continuous such that*
$$|f_2(t, x_0, x_1, \ldots, x_{n-1})| \leq \max_{0\leq i\leq n-1} |x_i|. \tag{4.17}$$

(iv) *There exists a positive solution* r_0 *to the inequality*
$$Db(r) \leq r.$$

Then the functional integro-differential equation (1.3) *has at least a solution in the space* $BC^n(\mathbb{R}_+)$.

Proof. It is easy to see that Eq. (1.3) has at least one solution in the space $BC^n(\mathbb{R}_+)$ if and only if equation
$$\begin{aligned} x(t) = {} & x_0 + x_1 t + \cdots + \frac{x_{n-1}}{(n-1)!}t^{n-1} \\ & + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f_1\Big(s, x(\xi(s)), x'(\xi(s)), \ldots, x^{(n-1)}(\xi(s)), \\ & \qquad \int_0^\infty k(s,v) f_2(v, x(v), x'(v), \ldots, x^{(n-1)}(v))dv\Big)ds \end{aligned} \tag{4.18}$$

has at least a solution in the space $BC^{n-1}(\mathbb{R}_+)$. Eq. (4.18) is a special case of Eq. (1.2) where

$$\begin{aligned} g(t,s,x_0,x_1,x_2,\ldots,x_n) &= \frac{(t-s)^{n-1}}{(n-1)!}f_1(s,x_0,x_1,x_2,\ldots,x_n),\\ Tx(t) &= \int_0^\infty k(t,s)f_2(s,x(s),x'(s),\ldots,x^{(n-1)}(s))\,ds.\\ p(t) &= x_0+x_1t+\ldots+\frac{x_{n-1}}{(n-1)!}t^{n-1},\\ q(t) &= 0 \end{aligned}$$

From the definitions of p, q and ξ, hypotheses (i) and (ii) of Theorem 4.1 obviously are satisfied with $\lambda=0$. Also we have

$$\begin{aligned} \sup\{|g(t,t,x_0,\ldots,x_n)| : t\in\mathbb{R}_+, x_i\in\mathbb{R}\} &= \sup\left\{\left|\frac{(t-t)^{n-1}}{(n-1)!}f_1(t,x_0,\ldots,x_n)\right| : t\in\mathbb{R}_+,\ x_i\in\mathbb{R}\right\}\\ &= 0, \end{aligned}$$

and similarly

$$\sup\left\{\left|\frac{\partial^k g}{\partial t^k}(t,t,x_0,x_1,\ldots,x_n)\right| : t\in\mathbb{R}_+,\ x_i\in\mathbb{R},\ 1\le k\le n\right\}=0.$$

Now, assume that $r>0$ is an arbitrary constant. Let $x_i\in\mathbb{R}_+$ with $\|x_i\|_u\le r$, $\psi(r)=Db(r)$, we obtain

$$\begin{aligned} e^{-t}\left|\int_0^t g\big(t,s,x_0(\xi(s)),\ldots,x_n(\xi(s))\big)\,ds\right| &\le e^{-t}\left|\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}a(s)b\Big(\max_{0\le i\le n}\{|x_i(\xi(s))|\}\Big)\,ds\right|\\ &\le \frac{D}{(n-1)!}b(r)\le\psi(r), \end{aligned}$$

and similarly

$$e^{-t}\left|\int_0^t \frac{\partial^k g}{\partial t^k}(t,s,x_0(\xi(s)),\ldots,x_n(\xi(s)))\,ds\right|\le\frac{D}{(n-k-1)!}b(r)\le\psi(r).$$

Thus, we conclude that the function g satisfies condition (4.3). Moreover, for any $r\in\mathbb{R}_+$

$$\begin{aligned} &\lim_{t\to\infty}e^{-t}\left|\int_0^t \big[g\big(t,s,x_0(\xi(s)),\ldots,x_n(\xi(s))\big)-g\big(t,s,y_0(\xi(s)),\ldots,y_n(\xi(s))\big)\big]ds\right|\\ &\le\lim_{t\to\infty}e^{-t}\int_0^t\big[\big|g\big(t,s,x_0(\xi(s)),\ldots,x_n(\xi(s))\big)\big|+\big|g\big(t,s,y_0(\xi(s)),\ldots,y_n(\xi(s))\big)\big|\big]ds\\ &\le\lim_{t\to\infty}e^{-t}\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}a(s)\Big[b\Big(\max_{0\le i\le n}\{|x_i(\xi(s))|\}\Big)+b\Big(\max_{0\le i\le n}\{|y_i(\xi(s))|\}\Big)\Big]\,ds\\ &\le\lim_{t\to\infty}2b(r)e^{-t}\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}a(s)\,ds\\ &=0, \end{aligned}$$

and similarly

$$\lim_{t\to\infty}e^{-t}\left|\int_0^t\left[\frac{\partial^k g}{\partial t^k}(t,s,x_0(\xi(s)),\ldots,x_n(\xi(s)))-\frac{\partial^k g}{\partial t^k}(t,s,y_0(\xi(s)),\ldots,y_n(\xi(s)))\right]ds\right|=0$$

uniformly with respect to $x_i, y_i \in \bar{B}_r$. Hence, g satisfies the condition (4.4) and hypothesis (iii) of Theorem 4.1. Next, hypothesis (iv) implies that condition (v) of Theorem 4.1 holds. To finish the proof we only need to verify that T is continuous. For this, take $x \in BC^n(\mathbb{R}_+)$ and $\varepsilon > 0$ arbitrarily, and consider $y \in BC^n(\mathbb{R}_+)$ with $\|x - y\|_{BC^n(\mathbb{R}_+)} < \varepsilon$. Then, considering condition (i), there exists $T > 0$ such that for $t > T$ we obtain

$$\begin{aligned} e^{-t}&|Tx(t) - Ty(t)| \\ &\leq e^{-t} \int_0^\infty k(t,s) \Big| f_2(s, x(s), x'(s), \ldots, x^{(n-1)}(s)) - f_2(s, y(s), y'(s), \ldots, y^{(n-1)}(s)) \Big| \, ds \\ &\leq e^{-t} \int_0^\infty k(t,s) \Big[|f_2(s, x(s), x'(s), \ldots, x^{(n-1)}(s))| \\ &\qquad\qquad + |f_2(s, y(s), y'(s), \ldots, y^{(n-1)}(s))| \Big] \, ds \\ &\leq 2(\|x\|_{BC^{n-1}(\mathbb{R}_+)} + \varepsilon) e^{-t} \int_0^\infty k(t,s) e^s \, ds \\ &\leq 2(\|x\|_{BC^{n-1}(\mathbb{R}_+)} + \varepsilon)\varepsilon \end{aligned} \tag{4.19}$$

Also if $t \in [0, T]$, then the first inequality in (4.19) follows that

$$e^{-t}|Tx(t) - Ty(t)| \leq \vartheta(\varepsilon), \tag{4.20}$$

where

$$\begin{aligned} \vartheta(\varepsilon) = \Big\{ &|f_2(t, x_0, x_1, \ldots, x_{n-1}) - f_2(t, y_0, y_1, \ldots, y_{n-1})| : t \in [0, T], \\ &\qquad\qquad\qquad\qquad x_i, y_i \in [-q_x, q_x], |x_i - y_i| \leq \varepsilon \Big\}, \end{aligned}$$

with $q_x = e^T(\|x\| + \varepsilon)$. By using the continuity of g on the compact set $[0, T] \times [-q_x, q_x]^n$, we have $\vartheta(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Thus from (4.20) we infer that T is a continuous function. Moreover by hypothesis (i) we easily obtain that

$$\|T(x)\|_{BC^0(\mathbb{R}_+)} \leq \|x\|_{BC^n(\mathbb{R}_+)},$$

and complete the proof. □

Corollary 4.3. *Assume that the following conditions are satisfied:*

(i) $k\colon \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *and* $\xi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *are continuous functions such that*

$$\sup \left\{ e^{-t} \left| \int_0^t k(t,s) e^s \, ds \right| : t \in \mathbb{R}_+ \right\} \leq 1;$$

and

$$\lim_{t \longrightarrow \infty} e^{-t} \left| \int_0^\infty k(t,s) e^s \, ds \right| = 0.$$

Moreover, $x_0, x_1, \ldots, x_{n-1} \in \mathbb{R}_+$

(ii) $f_1\colon \mathbb{R}_+ \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ *is continuous and there exist continuous functions* $a, b\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *such that*

$$|f_1(t, x_0, x_1, \ldots, x_n)| \leq a(t) b\Big(\max_{0 \leq i \leq n} |x_i| \Big).$$

Moreover, there exists a positive constant D such that

$$\sup\left\{e^{-t}\left|\int_0^t a(s)(t-s)^k\,ds\right| : t\in\mathbb{R}_+, 0\le k\le n-1\right\}\le D$$

and

$$\lim_{t\longrightarrow\infty}\left\{e^{-t}\left|\int_0^t a(s)(t-s)^k\,ds\right| : 0\le k\le n-1\right\}=0.$$

(iii) $f_2\colon \mathbb{R}_+\times\mathbb{R}^n\longrightarrow\mathbb{R}$ *is continuous such that*

$$|f_2(t,x_0,x_1,\ldots,x_{n-1})|\le\max_{0\le i\le n-1}|x_i|.$$

(iv) *There exists a positive solution r_0 to the inequality*

$$Db(r)\le r.$$

Then the functional integro-differential equation (1.4) *has at least a solution in the space $BC^n(\mathbb{R}_+)$.*

Proof. Since the proof of this Corollary can be completed essentially on the line of the proof of Corollary 4.2, hence details are omitted. □

Example 4.4. Consider the following functional integral-differential equation

$$x(t)=e^{-t-3}x(t)+\int_0^t\frac{(t-s)^3\ln(|x(\sqrt{s})|+1)+\sin((t-s)^7)\tanh(x'(\sqrt{s}))+(t-s)^3\sin(x''(\sqrt{s}))}{2+\sin(x^{(3)}(\sqrt{s}))}\,ds. \tag{4.21}$$

Eq. (4.21) is a special case of Eq. (1.2) with

$$p(t)=0,\qquad q(t)=e^{-t-3},\qquad \xi(t)=\sqrt{t},\qquad T=0,$$
$$g(t,s,x_0,x_1,x_2,x_3,x_4)=\frac{(t-s)^3\ln(|x_0|+1)+\sin((t-s)^7)\tanh(x_1)+(t-s)^3\sin(x_2)}{2+\sin(x_3)}.$$

It is easy to see that $p,q\in BC^3(\mathbb{R}_+)$ and $\lambda=\frac{8}{e^3}$. Also, g is continuous and has a continuous derivative of order 3 with respect to the first argument such that condition (4.2) is satified. By simple calculation we obtain that

$$e^{-t}|g(t,s,x_0,x_1,x_2,x_3,x_4)|\le e^{-t}(2t^3+1)\phi(\max\{x_i\}),$$
$$e^{-t}\left|\frac{\partial g}{\partial t}(t,s,x_0,x_1,x_2,x_3,x_4)\right|\le e^{-t}(6t^2+7t^6)\phi(\max\{x_i\}),$$
$$e^{-t}\left|\frac{\partial^2 g}{\partial t^2}(t,s,x_0,x_1,x_2,x_3,x_4)\right|\le e^{-t}(12t+42t^5+49t^{12})\phi(\max\{x_i\}),$$
$$e^{-t}\left|\frac{\partial^3 g}{\partial t^3}(t,s,x_0,x_1,x_2,x_3,x_4)\right|\le e^{-t}(12+210t^4+294t^{12}+343t^{18}+588t^{11})\phi(\max\{x_i\}),$$

where $\phi(t)=\max\{\ln(t+1),\sin t,\tanh t\}$. If we define $\psi(t)=1447\phi(t)$. Then, conditions (4.3) and (4.4) hold. On the other hand, since $T=0$, then condition (iv) of Theorem 4.1 is correct.

Also, it is easy to see that each number $r \geq 25000$ satisfies the inequality in condition (v) of Theorem 4.1, i.e.,

$$\|p\| + \lambda r + \psi(r) \leq 0 + \frac{8}{e^3} r + \psi(r) \leq r.$$

Thus, as the number r_0 we can take $r_0 = 25000$. Consequently, all the conditions of Theorem 4.1 are satisfied. Hence the functional integral-differential equation (4.21) has at least a solution which belongs to the space $BC^3(\mathbb{R}_+)$.

Example 4.5. Consider the following functional integro-differential equation

$$\begin{aligned} x^{(5)}(t) &= \frac{1}{2} e^{-2t} x(t^2) + \frac{1}{2} \sum_{k=1}^{4} e^{-(k+1)t} x^{(k)}(t^2) \\ &\quad + \frac{1}{10} e^{-2t} \int_0^\infty \frac{e^{-s}(t^2+1)}{s^2+4} \left(x(s) \tanh(s) + \sum_{i=1}^{4} \frac{x^{(i)}}{s^i + 4} \right) ds, \\ u(0) &= 0,\ u'(0) = 1,\ u''(0) = 2,\ u^{(3)}(0) = 3,\ u^{(4)}(0) = 4. \end{aligned} \tag{4.22}$$

Eq. (4.22) is a special case of Eq. (1.3) with

$$\begin{aligned} f_1(t, x_0, x_1, x_2, \ldots, x_5) &= \frac{1}{2} e^{-2t} x_0 + \frac{1}{2} \sum_{k=1}^{4} e^{-(k+1)t} x_k + \frac{1}{2} e^{-2t} x_5, \\ f_2(t, x_0, x_1, x_2, \ldots, x_4) &= \frac{1}{5} \left[x_0 \tanh(t) + \sum_{i=1}^{4} \frac{x_i}{t^i + 1} \right], \\ k(t,s) &= \frac{e^{-s}(t^2+1)}{s^2+4}, \qquad \xi(t) = t^2. \end{aligned}$$

It is easy to see that k and ξ are continuous,

$$\sup \left\{ e^{-t} \left| \int_0^\infty k(t,s) e^s \, ds \right| : t \in \mathbb{R}_+ \right\} \leq \sup \left\{ e^{-t} \left| \int_0^\infty \frac{t^2+1}{s^2+4} \, ds \right| : t \in \mathbb{R}_+ \right\} \leq \frac{\pi}{4} \leq 1$$

and

$$\lim_{t \longrightarrow \infty} e^{-t} \left| \int_0^\infty k(t,s) e^s \, ds \right| \leq \lim_{t \longrightarrow \infty} e^{-t} \left| \int_0^\infty \frac{t^2+1}{s^2+4} \, ds \right| = 0.$$

Then, condition (i) of Corollary 4.2 is valid. Also, f_1 is continuous and satisfying the inequality (4.3) of Corollary 4.2, where $a(t) = e^{-2t}$ and $b(t) = \frac{1}{2} t$. Moreover, we deduce

$$\begin{aligned} &\sup \left\{ e^{-t} \left| \int_0^t a(s)(t-s)^k \, ds \right| : t \in \mathbb{R}_+,\ 0 \leq k \leq 4 \right\} \\ &\leq \sup\{ e^{-t}(t^4 + t^3 + t^2 + t + 1 + e^{-2t}) : t \in \mathbb{R}_+ \} \\ &\leq 2 \end{aligned}$$

and

$$\lim_{t \longrightarrow \infty} \sup \left\{ e^{-t} \left| \int_0^t a(s)(t-s)^k \, ds \right| : 0 \leq k \leq 4 \right\} = \lim_{t \longrightarrow \infty} e^{-t}(t^4 + t^3 + t^2 + t + 1 + e^{-2t}) = 0.$$

Now with choosing $D = 2$ the condition (ii) of Corollary 4.2 is satisfied. Next, f_2 is continuous and

$$f_2(t, x_0, x_1, x_2, \ldots, x_4) = \frac{1}{5} \left[x_0 \tanh(t) + \sum_{i=1}^{4} \frac{x_i}{t^i + 1} \right] \leq \max_{0 \leq i \leq 4} |x_i|.$$

Hence, the condition (iii) of Corollary 4.2 holds. It is easy to see that each number $r \geq 1$ satisfies the inequality in condition (iv) of Corollary 4.2, i.e.,

$$Db(r) = 2\frac{1}{2}r \leq r.$$

Thus, as the number r_0 we can take $r_0 = 1$. Consequently, all the conditions of Corollary 4.2 are satisfied. This shows that the functional integro-differential equation (1.3) has a solution which belongs to the space $BC^5(\mathbb{R}_+)$.

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