Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **38**, 1–11; doi: 10.14232/ejqtde.2015.1.38 http://www.math.u-szeged.hu/ejqtde/

An extended Halanay inequality of integral type on time scales

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Received 30 January 2015, appeared 12 July 2015
Communicated by Paul Eloe

Abstract. In this paper, we obtain a Halanay-type inequality of integral type on time scales which improves and extends some earlier results for both the continuous and discrete cases. Several illustrative examples are also given.

Keywords: time scale, delay dynamic equation, comparison theorem.

2010 Mathematics Subject Classification: 34K11, 39A10, 39A99.

1 Introduction and preliminaries

As is well-known, Halanay-type differential inequalities have been very useful in the stability analysis of time-delay systems and these have led to some interesting new stability conditions. In [5], Halanay proved the following result.

Basic Halanay lemma. *If*

$$f'(t) \le -\alpha f(t) + \beta \sup_{s\in[t-\tau,t]} f(s), \quad \textit{for } t \ge t_0$$

and $\alpha > \beta > 0$, *then there exist* $\gamma > 0$ *and* $K > 0$ *such that*

$$f(t) \le Ke^{-\gamma(t-t_0)}, \quad \textit{for } t \ge t_0.$$

In 2000, S. Mohamad and K. Gopalsamy established the following result.

Lemma 1.1 ([8]). *Let* $x(\cdot)$ *be a nonnegative differentiable function satisfying*

$$x'(t) \le -a(t)x(t) + b(t) \sup_{s\in[t-\tau(t),t]} x(s), \qquad t > t_0, \tag{1.1}$$

$$x(s) = |\varphi(s)| \quad \textit{for } s \in [t_0 - \tau^*, t_0], \tag{1.2}$$

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where $\varphi(s)$ is defined for $s \in [t_0 - \tau^*, t_0]$ and is continuous and bounded. We assume further that $\tau(t)$, $a(t)$, $b(t)$ are defined for $t \in \mathbb{R}$ and are nonnegative, continuous bounded functions; and $\sup_{t\in\mathbb{R}} \tau(t) = \tau^*$. *Suppose*

$$a(t) - b(t) \geq \sigma^* > 0, \tag{1.3}$$

where $\sigma^* = \inf_{t\in\mathbb{R}} (a(t) - b(t)) > 0$. *Then there exists a positive number $\widetilde{\mu}$ such that*

$$x(t) \leq \left(\sup_{s\in[t_0-\tau^*,t_0]} x(s) \right) e^{-\widetilde{\mu}(t-t_0)}, \qquad t > t_0. \tag{1.4}$$

In many of the results, the condition $a(t) - b(t) > \delta$, $\delta > 0$ for all t is assumed to hold. In [7], the authors replace this point-wise inequality by an inequality of integral type of the form $\int_{t_0+nT}^{t_0+(n+1)T} [a(t) - b^+(t)]\, dt \geq \delta > 0$. Obviously this inequality is more general since the inequality $a(t) - b(t) > \delta$, $\delta > 0$ only needs to hold "on average", or in the mean sense.

In 2012, Bo Liu proved the following lemma (in what follows, for any function $u = u(t)$ we use the notation $u^+(t) = \max\{0, u(t)\}$).

Lemma 1.2 ([7]). *Let $x(\cdot)$ be a nonnegative function satisfying*

(H_1)

$$x'(t) \leq -a(t)x(t) + b(t) \sup_{0\leq s\leq\overline{\tau}} x(t-s), \qquad t \geq 0, \tag{1.5}$$

$$x(s) = \phi(s), \qquad s \in [-\overline{\tau}, 0], \tag{1.6}$$

where $\overline{\tau} > 0$ is a constant and $\phi(s)$ is a nonnegative continuous function defined for $s \in [-\overline{\tau}, 0]$.

(H_2) *$a(\cdot)$, $b(\cdot)$ are defined in $\mathbb{R}$ and are continuous bounded functions and we define $M_a; M_b > 0$, by*

$$\max\{|a(t)|\} = M_a \quad \text{and} \quad \max\{|b(t)|\} = M_b.$$

(H_3) $\exists t_0 > 0$, $T > 0$ *and* $\delta > 0$, $\forall n \in N$

$$\int_{t_0+nT}^{t_0+(n+1)T} [a(t) - b^+(t)]\, dt > \delta > 0. \tag{1.7}$$

Then for each $\overline{\tau} < \frac{1}{M_a} \ln(1 + \frac{\delta}{M_b^+ T})$, where $M_b^+ = \sup_{t\in[-\overline{\tau},\infty)} b^+(t)$; it follows that $x(t)$ is exponentially stable, i.e., there exists $C > 0$ (which may depend on the initial value) and $\alpha > 0$ such that

$$x(t) \leq Ce^{-\alpha t}, \qquad t \in [0, \infty). \tag{1.8}$$

As a consequence of Lemma 1.2, we note that the condition (1.7) can be viewed as a relaxation of the condition (1.3). This means that for asymptotic stability of the system, we do not need the inequality (1.3) to hold at every time t, but only require it to hold in an average sense. Often, it is easier to investigate the time average system, so this lemma provides an average system-based approach for the study of delayed dynamical systems.

To study such problems more generally in the time-scale setting, the authors in [1] introduced the notion of shift operators, $\delta_-(s, t), \delta_+(s, t)$ and obtained the following lemma.

Lemma 1.3 ([1]). *If*

$$\omega^{\triangle}(t) \leq f\big(t, \omega(t), g\big(\omega(\delta_-(h_1, t)), \omega(\delta_-(h_2, t)), \ldots, \omega(\delta_-(h_r, t))\big)\big)$$

for $t \in [s_0, \delta_+(\alpha, s_0))_{\mathbb{T}}$ and $y(t; s_0, \omega)$ is a solution of the equation

$$y^{\triangle}(t) = f\big(t, y(t), g\big(y(\delta_-(h_1, t)), y(\delta_-(h_2, t)), \ldots, y(\delta_-(h_r, t))\big)\big),$$

which coincides with ω in $[\delta_-(h_r, s_0), s_0]_{\mathbb{T}}$, then, supposing that this solution is defined in $[s_0, \delta_+(\alpha, s_0))_{\mathbb{T}}$, it follows that $\omega(t) \leq y(t; s_0, \omega)$ for $t \in [s_0, \delta_+(\alpha, s_0))_{\mathbb{T}}$.

For completeness, we recall the following concepts related to the notion of time scales. We refer to [3] and [4] for additional details concerning the calculus on time scales.

Definition 1.4. Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. The forward jump operator is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \tag{1.9}$$

and the backward jump operator is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \tag{1.10}$$

where $\sup \emptyset := \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$ we say t is left-dense. Given a time scale interval $[c, d]_{\mathbb{T}} := \{t \in \mathbb{T} : c \leq t \leq d\}$ in $\mathbb{T}$ the notation $[c, d]^{\kappa}_{\mathbb{T}}$ denotes the interval $[c, d]_{\mathbb{T}}$ in case $\rho(d) = d$ and denotes the interval $[c, d)_{\mathbb{T}}$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f\colon \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. We also recall that the notation C_{rd} denotes the set of all functions which are continuous at all right-dense points and have finite left-sided limits at left-dense points. We say that $x\colon \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}$ provided

$$x^{\Delta}(t) := \lim_{s \to t} \frac{x(t) - x(s)}{t - s}, \tag{1.11}$$

exists when $\sigma(t) = t$ (here by $s \to t$ it is understood that s approaches t in the time scale) and when x is continuous at t and $\sigma(t) > t$

$$x^{\triangle}(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}, \tag{1.12}$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$ the delta derivative is just the forward difference operator. Hence our results contain the discrete and continuous cases as special cases and generalize these results to time scales with bounded graininess.

Definition 1.5. A function $h\colon \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)h(t) \neq 0$ for all $t \in \mathbb{T}$, where $\mu(t) = \sigma(t) - t$. The set of all regressive rd-continuous functions $\varphi : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathfrak{R}$ while the set $\mathfrak{R}^+$ is given by $\mathfrak{R}^+ = \{\varphi \in \mathfrak{R} : 1 + \mu(t)\varphi(t) > 0 \text{ for all } t \in \mathbb{T}\}$. Let $\varphi \in \mathfrak{R}$. The exponential function on $\mathbb{T}$ is defined by $e_{\varphi}(t, s) = \exp\big(\int_s^t \xi_{\mu(r)}(\varphi(r))\Delta r\big)$.

Here $\xi_{\mu(s)}$ is the cylinder transformation given by

$$\xi_{\mu(r)}(\varphi(r)) := \begin{cases} \frac{1}{\mu(r)} \operatorname{Log}\Big(1 + \mu(r)\varphi(r)\Big), & \mu(r) > 0, \\ \varphi(r), & \mu(r) = 0. \end{cases}$$

It is well known that (see [3, Theorem 2.48]) if $p \in \mathfrak{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the **unique** solution to the initial value problem $y^\Delta = p(t)y,\ y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 1.6 ([1, 3]). *Let $p, q \in \mathfrak{R}$. Then*

$$
\begin{aligned}
&(i)\ \ e_0(s,t) \equiv 1 \quad \textit{and} \quad e_p(t,t) \equiv 1,\\
&(ii)\ \ e_p(\sigma(t),s) = \Big(1+\mu(t)p(t)\Big)e_p(t,s),\\
&(iii)\ \ \frac{1}{e_p(t,s)} = e_{\ominus p}(t,s) \quad \textit{where} \quad \ominus\, p(t) = -\frac{p(t)}{1+\mu(t)p(t)},\\
&(iv)\ \ e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t),\\
&(v)\ \ e_p(t,s)e_p(s,r) = e_p(t,r),\\
&(vi)\ \ \Big(\frac{1}{e_p(\cdot,s)}\Big)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}.
\end{aligned}
\tag{1.13}
$$

Lemma 1.7 ([1]). *For a nonnegative function φ with $-\varphi \in \mathfrak{R}^+$, we have the inequalities*

$$
1 - \int_s^t \varphi(u) \le e_{-\varphi}(t,s) \le \exp\left\{-\int_s^t \varphi(u)\right\} \quad \textit{for all } t \ge s. \tag{1.14}
$$

If φ is rd-continuous and nonnegative, then

$$
1 + \int_s^t \varphi(u) \le e_{\varphi}(t,s) \le \exp\left\{\int_s^t \varphi(u)\right\} \quad \textit{for all } t \ge s. \tag{1.15}
$$

Remark 1.8. If $p \in \mathfrak{R}^+$ and $p(r) > 0$ for all $r \in [s,t]_\mathbb{T}$, then

$$
e_p(t,r) \le e_p(t,s), \qquad e_p(a,b) < 1 \quad \text{and} \quad e_{-p}(b,a) < 1 \quad \text{for } s \le a < b \le t. \tag{1.16}
$$

In this paper, in the time scale setting, we establish a new Halanay-type inequality, using differential and integral inequality comparison methods and obtain thereby an improvement of the results in [7] and several other references.

2 Main theorem

Theorem 2.1. *Let $x(\cdot)$ be a nonnegative function satisfying:*

$$
\begin{cases}
x^\triangle(t) \le -a(t)x(t) + b(t) \sup\limits_{0 \le s \le \tau(t)} x(t-s) + c(t)\displaystyle\int_0^\infty K(t,s)x(t-s)\Delta s; \quad t \ge t_0,\\
x(s) = \phi(s); \quad s \in (-\infty, t_0]_\mathbb{T}.
\end{cases}
\tag{2.1}
$$

where $\tau(t)$ denotes a nonnegative, continuous and bounded function defined for $t \in \mathbb{T}$ and $\overline{\tau} := \sup_{t\in\mathbb{T}} \tau(t)$; $\phi(s)$ is a nonnegative continuous function defined for $s \in (-\infty, t_0]_\mathbb{T}$. We suppose further that $1 - a(t)\mu(t) > 0$, and that $a(t),\ b(t);\ c(t);\ \frac{a(t)}{1-\mu(t)a(t)}$ are continuous and bounded functions for $t \in [t_0, +\infty)_\mathbb{T}$.

We also assume $\sup_{t\in\mathbb{T}} \left\{|a(t)|,\ |b(t)|,\ |c(t)|,\ \frac{|a(t)|}{1-\mu(t)a(t)}\right\} =: A$ and that the delay kernel $K(t,s)$ is nonnegative and continuous for $(t,s) \in \mathbb{T} \times [0,\infty)_\mathbb{T}$ and satisfies the following properties:

(I)

$$\int_0^\infty K(t,s)e_A(t,t-s)\Delta s \quad \textit{is uniformly bounded for } t \in \mathbb{T}; \tag{2.2}$$

(II) *There exists* $\widetilde{t} > t_0$, $T > 0$ *and* $\delta > 0$ *such that for each* $n \in \mathbb{N}$, *we have*

$$\int_{\widetilde{t}+nT}^{\widetilde{t}+(n+1)T} \left[a(t) - b^+(t) - c^+(t)\int_0^\infty K(t,s)\,\Delta s\right]\Delta t > \delta. \tag{2.3}$$

Then for each $\overline{\tau} < \frac{1}{A}\ln(1 - B + \frac{\delta}{AT})$ *where* $\sup_{t\in\mathbb{T}}\int_0^\infty K(t,s)\,(e_A(t,t-s)-1)\,\Delta s =: B < \frac{\delta}{AT}$; $x(t)$ *is exponentially stable, i.e., there exists* $M > 0$ *(which may depend on the initial value), and* $\alpha > 0$ *such that*

$$x(t) \le Me_{\ominus\alpha}(t,t_0), \qquad t \in [t_0,\infty)_{\mathbb{T}}. \tag{2.4}$$

Proof. From condition (2.3) and $0 < K(t,s) < K(t,s)e_A(t,t-s)$; $A > 0$; $s > 0$, it follows that

$$\int_0^\infty K(t,s)\,\Delta s \quad \text{and} \quad \int_0^\infty K(t,s)\,(e_A(t,t-s)-1)\,\Delta s$$

are uniformly bounded for $t \in \mathbb{T}$.

That is, there exist real numbers B, $C > 0$ with

$$\sup_{t\in\mathbb{T}}\int_0^\infty K(t,s)\,(e_A(t,t-s)-1)\,\Delta s = B, \quad \text{and} \quad \sup_{t\in\mathbb{T}}\int_0^\infty K(t,s)\,\Delta s =: C.$$

From (2.1), we have

$$\begin{cases} x^{\triangle}(t) \le -a(t)x(t) + b^+(t)\sup\limits_{0\le s\le\overline{\tau}} x(t-s) + c^+(t)\displaystyle\int_0^\infty K(t,s)x(t-s)\Delta s; \quad t \ge t_0, \\ x(s) = \phi(s); \quad s \in (-\infty,t_0]_{\mathbb{T}}. \end{cases} \tag{2.5}$$

Suppose now that $y(t)$ is a nonnegative function satisfying the autonomous system:

$$\begin{cases} y^{\triangle}(t) = -a(t)y(t) + b^+(t)\sup\limits_{0\le s\le\overline{\tau}} y(t-s) + c^+(t)\displaystyle\int_0^\infty K(t,s)y(t-s)\Delta s; \quad t \ge t_0, \\ y(s) = \phi(s); \quad s \in (-\infty,t_0]_{\mathbb{T}}. \end{cases} \tag{2.6}$$

We first note that the following inequality holds:

$$x(t) \le y(t); \qquad t \in [t_0,\infty)_{\mathbb{T}}. \tag{2.7}$$

To see this, we can directly apply the results in Lemma 1.3, which gives $x(t) \le y(t)$ for all $t \in [t_0,\infty)_{\mathbb{T}}$.

We next will prove the following inequality:

$$y(t) \le Me_{\ominus\alpha}(t,t_0), \qquad t \in [t_0,\infty)_{\mathbb{T}}. \tag{2.8}$$

To establish (2.8), we prove the following preliminary inequality:

$$\text{for all } t_1,t_2\text{: } t_2 > t_1 \ge t_0 \quad \text{we have} \quad y(t_1) \le y(t_2)e_A(t_2,t_1). \tag{2.9}$$

Since $\forall t \ge t_0$, $y(t) \ge 0$, $b^+(t)\sup_{0\le s\le\hat{\tau}} y(t-s) \ge 0$ and $c^+(t)\int_0^\infty K(t,s)y(t-s)\,\Delta s \ge 0$, we have

$$y^{\triangle}(t) = -a(t)y(t) + b^+(t)\sup_{0\le s\le\overline{\tau}} y(t-s) + c^+(t)\int_0^\infty K(t,s)y(t-s)\,\Delta s \ge -a(t)y(t)$$

and

$$\left(\frac{y(t)}{e_{-a}(t,t_1)}\right)^{\Delta} = \frac{y^{\triangle}(t)+a(t)y(t)}{e_{-a}(\sigma(t),t_1)} \geq 0. \tag{2.10}$$

Since $\frac{y(t)}{e_{-a}(t,t_1)}$ is a monotonically increasing function for $t \geq t_1$, we see that

$$y(t) \geq y(t_1)e_{-a}(t,t_1) \quad \text{for } t > t_1; \tag{2.11}$$

and so

$$\forall t_1, t_2\colon t_2 > t_1 \geq t_0, \qquad y(t_1) \leq y(t_2)e_{\ominus(-a)}(t_2,t_1) \leq y(t_2)e_A(t_2,t_1); \tag{2.12}$$

where $\ominus(-a(t)) = \frac{a(t)}{1-\mu(t)a(t)} \Longrightarrow |\ominus(-a(t))| \leq A$, which implies that (2.9) holds.

Therefore we have

$$\begin{aligned} \sup_{0\leq s\leq\overline{\tau}} y(t-s) &\leq \sup_{0\leq s\leq\overline{\tau}} y(t)e_A(t,t-s) \leq y(t)e_A(t,t-\overline{\tau}); \\ \int_0^\infty K(t,s)y(t-s)\Delta s &\leq y(t)\int_0^\infty K(t,s)e_A(t,t-s)\Delta s. \end{aligned} \tag{2.13}$$

We put (2.13) into (2.6), and so we obtain

$$\begin{aligned} y^{\triangle}(t) &\leq -a(t)y(t) + b^+(t)y(t)e_A(t,t-\overline{\tau}) + c^+(t)y(t)\int_0^\infty K(t,s)e_A(t,t-s)\,\Delta s \\ &= \left(-a(t) + b^+(t)e_A(t,t-\overline{\tau}) + c^+(t)\int_0^\infty K(t,s)e_A(t,t-s)\,\Delta s\right)y(t). \end{aligned} \tag{2.14}$$

Let

$$p(t) = -a(t) + b^+(t)e_A(t,t-\overline{\tau}) + c^+(t)\int_0^\infty K(t,s)e_A(t,t-s)\,\Delta s. \tag{2.15}$$

From (2.14), (2.15), we have

$$\forall t_1, t_2\colon t_2 > t_1 \geq t_0, \qquad y(t_2) \leq y(t_1)e_p(t_2,t_1). \tag{2.16}$$

By the assumptions (2.3), (2.15), we have

$$\begin{aligned} \int_{\widetilde{t}+nT}^{\widetilde{t}+(n+1)T} p(t)\,\Delta t &= \int_{\widetilde{t}+nT}^{\widetilde{t}+(n+1)T}\left[-a(t) + b^+(t)e_A(t,t-\overline{\tau}) + c^+(t)\int_0^\infty K(t,s)e_A(t,t-s)\Delta s\right]\Delta t \\ &= \int_{\widetilde{t}+nT}^{\widetilde{t}+(n+1)T}\left[-a(t) + b^+(t) + c^+(t)\int_0^\infty K(t,s)\Delta s\right]\Delta t \\ &\quad + \int_{\widetilde{t}+nT}^{\widetilde{t}+(n+1)T} b^+(t)\left(e_A(t,t-\overline{\tau}) - 1\right)\Delta t \\ &\quad + \int_{\widetilde{t}+nT}^{\widetilde{t}+(n+1)T} c^+(t)\int_0^\infty K(t,s)\left[e_A(t,t-s) - 1\right]\Delta s\,\Delta t \\ &< -\delta + AT(\exp(A\overline{\tau}) + B - 1). \end{aligned} \tag{2.17}$$

Since $\overline{\tau} < \frac{1}{A}\ln(1 - B + \frac{\delta}{AT})$, we have that $-\delta + AT(\exp(A\overline{\tau}) + B - 1) < 0$.

Denote

$$\alpha = \frac{\delta - AT(\exp(A\overline{\tau}) + B - 1)}{T} > 0. \tag{2.18}$$

By (2.16), (2.18), then for each $n \in N$

$$\begin{aligned} y(\tilde{t}+nT) &\le y(\tilde{t}+(n-1)T)e_p(\tilde{t}+nT,\tilde{t}+(n-1)T) \\ &\le y(\tilde{t}+(n-1)T)\exp\left(\int_{\tilde{t}+(n-1)T}^{\tilde{t}+nT} p(t)\,dt\right) < y(\tilde{t}+(n-1)T)\exp(-\alpha T) \\ &< y(\tilde{t})\exp(-\alpha nT). \end{aligned} \tag{2.19}$$

Therefore, for $t > \tilde{t} > t_0$, $\exists n \in \mathbb{N} \Rightarrow t \in \big(\tilde{t}+nT,\ \tilde{t}+(n+1)T\big]_{\mathbb{T}}$, and

$$\begin{aligned} t-(\tilde{t}+nT) &< \tilde{t}+(n+1)T-(\tilde{t}+nT) = T, \\ p(t) &= -a(t)+b^{+}(t)e_A(t,t-\overline{\tau})+c^{+}(t)\int_0^{\infty} K(t,s)e_A(t,t-s)\Delta s \\ &\le A+A\exp(A\overline{\tau})+A(B+C) = A[\exp(A\overline{\tau})+B+C+1] := \theta. \end{aligned} \tag{2.20}$$

By (2.19), (2.20) and (1.16)

$$\begin{aligned} y(t)e_\alpha(t,t_0) &\le y(\tilde{t}+nT)e_p(t,\tilde{t}+nT)e_\alpha(t,t_0) \\ &\le y(\tilde{t}+nT)e_\theta(t,\tilde{t}+nT)\cdot\exp\left(\int_{t_0}^{t}\alpha\,du\right) \\ &\le y(\tilde{t})\exp\Big[\theta(t-\tilde{t}-nT)-\alpha nT+\alpha t-\alpha t_0\Big] \\ &\le y(\tilde{t})\exp\Big[(\theta+\alpha)(t-\tilde{t}-nT)+\alpha\tilde{t}-\alpha t_0\Big] \\ &\le y(\tilde{t})\exp\Big[(\theta+\alpha)T+\alpha(\tilde{t}-t_0)\Big]; \end{aligned} \tag{2.21}$$

when $t_0 \le t \le \tilde{t}$,

$$y(t)e_\alpha(t,t_0) \le \sup_{s\le\tilde{t}}\Big(y(s)e_\alpha(s,t_0)\Big). \tag{2.22}$$

Let

$$M = \max\left\{\sup_{s\le\tilde{t}}\Big(y(s)e_\alpha(s,t_0)\Big),\ y(\tilde{t})\exp\Big[(\theta+\alpha)T+\alpha(\tilde{t}-t_0)\Big]\right\}, \tag{2.23}$$

where $\theta = A[\exp(A\overline{\tau})+B+C+1]$.

By (2.21), (2.22), (2.23) and (2.7), we have

$$x(t) \le y(t) \le Me_{\ominus\alpha}(t,t_0), \qquad t \in [t_0,\infty)_{\mathbb{T}}. \tag{2.24}$$

This completes the proof. □

When $c(t) = 0$, we can obtain the following corollary, which can be regarded as an extension of Theorem 1 of [7].

Corollary 2.2. *Let $x(\cdot)$ be a nonnegative function satisfying*

$$x^{\triangle}(t) \le -a(t)x(t)+b(t)\sup_{0\le s\le\tau(t)} x(t-s); \qquad t \ge t_0; \tag{2.25}$$

$$x(s) = \phi(s); \qquad s \in (-\infty,t_0]_{\mathbb{T}}. \tag{2.26}$$

where $\tau(t)$ denotes a nonnegative, continuous and bounded function defined for $t \in \mathbb{T}$ and $\overline{\tau} = \sup_{t\in\mathbb{T}}\tau(t)$; $\phi(s)$ is a nonnegative continuous function defined for $s \in (-\infty,t_0]_{\mathbb{T}}$. We also assume

$a(t)$, $b(t)$, $\frac{a(t)}{1-\mu(t)a(t)}$ are continuous bounded functions fort $\in \mathbb{T}$ with $1-a(t)\mu(t) > 0$ and define $A > 0$ by

$$A := \sup_{t\in\mathbb{T}} \left\{ |a(t)|,\ |b(t)|,\ \frac{|a(t)|}{1-\mu(t)a(t)} \right\}.$$

Suppose also that there exist $\widetilde{t} > t_0, T > 0$ and $\delta > 0$ such that for each $n \in \mathbb{N}$.

$$\int_{\widetilde{t}+nT}^{\widetilde{t}+(n+1)T} \Big[a(t) - b^+(t) \Big] \Delta t > \delta. \tag{2.27}$$

Then for each $\overline{\tau} < \frac{1}{A}\ln(1+\frac{\delta}{AT})$, $x(t)$ is exponentially stable, i.e., there exists $M > 0$ (which may depend on the initial value), and $\alpha > 0$ such that

$$x(t) \leq M e_{\ominus\alpha}(t,t_0), \qquad t \in [t_0,\infty)_{\mathbb{T}}. \tag{2.28}$$

3 Examples

Suppose $x(\cdot)$ is a nonnegative function satisfying the following delay dynamic equation:

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + b(t)x(t-\tau) + c(t)\int_0^{\infty} K(t,s)x(t-s)\,\Delta s, \qquad t \in [t_0,+\infty)_{\mathbb{T}} \tag{3.1}$$

where $x(s) = \varphi(s)$, for $s \in (-\infty,t_0]_{\mathbb{T}}$, and where φ is rd-continuous, nonnegative and bounded, and τ is a constant. Let $a(t)$, $b(t)$; $c(t)$; $\frac{a(t)}{1-\mu(t)a(t)}$ denote rd-continuous and bounded functions for $t \in [t_0,+\infty)_{\mathbb{T}}$; $1-a(t)\mu(t) > 0$. Namely

$$\exists A > 0, \quad \forall t \in \mathbb{T}, \quad \text{s.t.} \sup_{t\in\mathbb{T}} \left\{ |a(t)|,\ |b(t)|,\ |c(t)|,\ \frac{|a(t)|}{1-\mu(t)a(t)} \right\} = A; \tag{3.2}$$

Assume further that the delay kernel $K(t,s)$ is nonnegative and continuous for $(t,s) \in \mathbb{T}\times[t_0,\infty)_{\mathbb{T}}$ and satisfies the following properties.

$$\int_0^{\infty} K(t,s)e_A(t,t-s)\,\Delta s \quad \text{is uniformly bounded for } t \in \mathbb{T}; \tag{3.3}$$

$\exists\widetilde{t} > t_0, T > 0$ and $\delta > 0$ such that for each $n \in \mathbb{N}$

$$\int_{\widetilde{t}+nT}^{\widetilde{t}+(n+1)T} \left[a(t) - b^+(t) - c^+(t)\int_0^{\infty} K(t,s)\,\Delta s \right] \Delta t > \delta. \tag{3.4}$$

From (3.1), we have

$$x(t) = x(t_0)e_{-a}(t,t_0) + \int_{t_0}^{t} e_{-a}(t,\sigma(s)) \left[b(s)x(s-\tau) + c(s)\int_0^{\infty} K(s,v)x(s-v)\,\Delta v \right] \Delta s. \tag{3.5}$$

Let the function $y(t)$ be defined as follows: $y(t) = x(t)$, for $t \in (-\infty,t_0]_{\mathbb{T}}$ and

$$\begin{aligned} y(t) &= x(t_0)e_{-a}(t,t_0) \\ &\quad + \int_{t_0}^{t} e_{-a}(t,\sigma(s)) \left[b(s)\sup_{0\leq\theta\leq\tau} x(s-\theta) + c(s)\int_0^{\infty} K(s,v)x(s-v)\,\Delta v \right] \Delta s \end{aligned} \tag{3.6}$$

for $t > t_0$. Then we have $x(t) \leq y(t)$, for all $t \in (-\infty,+\infty)_{\mathbb{T}}$.

By [3, Theorem 5.37], we get that

$$\begin{aligned} y^{\Delta}(t) = & -a(t)\Bigg\{x(t_0)e_{-a}(t,t_0) \\ & + \int_{t_0}^{t} e_{-a}(t,\sigma(s)) \left[b(s) \sup_{0\le\theta\le\tau} x(s-\theta) + c(s)\int_0^\infty K(s,v)x(s-v)\,\Delta v \right] \Delta s \Bigg\} \\ & + b(t) \sup_{0\le\theta\le\tau} y(t-\theta) + c(t)\int_0^\infty K(t,v)y(t-v)\,\Delta v \\ \le & -a(t)y(t) + b(t) \sup_{0\le\theta\le\tau} y(t-\theta) + c(t)\int_0^\infty K(t,v)y(t-v)\,\Delta v \end{aligned} \tag{3.7}$$

for all $t \in [t_0, \infty)$.

Therefore, it follows from the main theorem that for each $\overline{\tau} < \frac{1}{A}\ln(1 - B + \frac{\delta}{AT})$ with $\sup_{t\in\mathbb{T}} \int_0^\infty K(t,s)\,(e_A(t,t-s)-1)\,\Delta s = B$; $x(t)$ is exponentially stable, i.e., there exists $M > 0$ (which may depend on the initial value), $\alpha > 0$ such that

$$x(t) \le y(t) \le Me_{\ominus\alpha}(t,t_0), \qquad t \in [t_0,\infty)_{\mathbb{T}}. \tag{3.8}$$

In the following, we let $\mathbb{T} = \mathbb{Z}$ and will choose some explicit functions for $a(t)$, $b(t)$, $c(t)$, $K(t,s)$.

Let $x(\cdot)$ be a nonnegative function satisfying (here we have $\mu(t) = 1$)

$$\Delta x(n) \le -a(n)x(n) + b(n) \sup_{0\le j\le\tau} x(n-j) + c(n)\sum_{j=0}^{\infty} K(n,j)x(n-j), \qquad n \ge 1$$

where

$$\begin{gathered} a(4n) = \frac{11}{16}, \qquad a(4n+1) = -\frac{1}{8}, \qquad a(4n+2) = \frac{5}{8}, \\ a(4n+3) = \frac{3}{4}; \qquad b(n) = \frac{1}{16}; \qquad c(n) = \frac{4n+3}{4n+4}; \\ K(n,j) = \frac{1}{16(4n+4)^j}; \qquad \forall n \ge 1; \end{gathered}$$

and

$$\begin{aligned} & \sup_{n\ge1}\left\{|a(n)|,\ |b(n)|,\ |c(n)|,\ \frac{|a(n)|}{1-a(n)}\right\} = A = 3, \\ \forall n \ge 1, \qquad & \sum_{j=0}^{\infty} K(n,j)e_A(n,n-j) = \sum_{j=0}^{\infty} \frac{1}{16(4n+4)^j}(1+A)^j e_A(n-j,n-j); \\ & = \sum_{j=0}^{\infty} \frac{1}{16}\Big(\frac{1}{n+1}\Big)^j e_A(n-j,n-j) = \frac{n+1}{16n}; \\ B = & \sup_{n\ge1}\sum_{j=0}^{\infty} K(n,j)\Big(e_A(n,n-j)-1\Big) = \sup_{n\ge1}\frac{3n+3}{16n(4n+3)} = \frac{3}{56}, \end{aligned}$$

where $e_A(n-j,n-j)=1$. Thus

$$a(n)-b^+(n)-c^+(n)\sum_{j=0}^{\infty}K(n,j)=a(n)-\frac{1}{8};$$

$$\text{If } n=4k, \text{ then } a(4k)-b^+(4k)-c^+(4k)\sum_{j=0}^{\infty}K(4k,j)=\frac{9}{16}>0.$$

$$\text{If } n=4k+1, \text{ then } a(4k+1)-b^+(4k+1)-c^+(4k+1)\sum_{j=0}^{\infty}K(4k+1,j)=-\frac{1}{4}<0.$$

$$\text{If } n=4k+2, \text{ then } a(4k+2)-b^+(4k+2)-c^+(4k+2)\sum_{j=0}^{\infty}K(4k+2,j)=\frac{1}{2}>0.$$

$$\text{If } n=4k+3, \text{ then } a(4k+3)-b^+(4k+3)-c^+(4k+3)\sum_{j=0}^{\infty}K(4k+3,j)=\frac{5}{8}>0.$$

This shows that the point-wise Halanay inequality does not apply to this example. However, since we have that there exists $T=4$ and $\delta=\frac{7}{5}$ such that

$$\sum_{k=nT}^{(n+1)T-1}\left(a(k)-b^+(k)-c^+(k)\sum_{j=0}^{\infty}K(k,j)\right)=\sum_{k=4n}^{4(n+1)-1}\left(a(k)-\frac{1}{8}\right)=\frac{23}{16}>\frac{7}{5}=\delta. \tag{3.9}$$

Therefore, it follows that if $\tau<\frac{1}{A}\ln(1-B+\frac{\delta}{AT})=\frac{1}{3}\ln\frac{893}{840}\approx 0.0204$, and $\frac{3}{56}=B<\frac{\delta}{AT}=\frac{7}{60}$, then $x(t)$ is exponentially stable, i.e., there exists $M>0$ (which may depend on the initial value), and

$$\alpha=\frac{\delta-AT(\exp(A\tau)+B-1)}{T}=\frac{\frac{7}{5}-12(e^{0.03}+\frac{3}{56}-1)}{4}\approx 0.0979 \qquad (\text{let } \tau=0.01), \tag{3.10}$$

such that

$$x(t)\le Me_{\ominus 0.0979}(n,1), \qquad n\ge 1.$$

Acknowledgements

The second author was supported by The National Natural Science Foundation of China (No.11271380).

References

[1] M. Adivar, E. A. Bohner, Halanay type inequalities on time scales with applications, *Nonlinear Anal.* **74**(2011), 7519–7531. MR2833732

[2] R. P. Agarwal, *Difference equations and inequalities. Theory, methods and applications*, Marcel Dekker Inc., New York, 1992. MR1155840

[3] M. Bohner, A. Peterson, *Dynamic equations on time scales. An introduction with applications*, Birkhäuser Boston Inc., Boston, MA, 2001. MR1843232

[4] M. Bohner, A. Peterson, (Eds.), *Advances in dynamic equations on time scales*, Birkhäuser Boston Inc., Boston, MA, 2003. MR1962542; url

[5] A. Halanay, *Differential equations: Stability, oscillations, time lags*, Academic Press, New York, NY USA, 1966. MR0216103

[6] B. G. Jia, L. Erbe, R. Mert, A Halanay-type inequality on time scales in higher dimensional spaces, *Math. Inequal. Appl.* **17**(2014), 813–821. MR3224836

[7] B. Liu, W. L. Lu, T. P. Chen, Stability analysis of some delay differential inequalities with small time delays and its applications, *Neural Netw.* **33**(2012) 1–6. url

[8] S. Mohamad, K. Gopalsamy, Continuous and discrete Halanay-type inequalities, *Bull. Aust. Math. Soc.* **61**(2000), 371–385. MR1762635