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Multiple positive solutions for a class of Neumann problems

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Abstract. We study the existence of multiple positive solutions of the Neumann problem

$$\begin{aligned} -u''(x) &= \lambda f(u(x)), \qquad x \in (0,1), \\ u'(0) &= 0 = u'(1), \end{aligned}$$

where λ is a positive parameter, $f \in C([0,\infty), \mathbb{R})$ and for some $\beta > 0$ such that $f(0) = 0$, $f(s) > 0$ for $s \in (\beta, \infty)$, $f(s) < 0$ for $s \in (0, \beta)$, and θ ($> \beta$) is the unique positive zero of $F(s) = \int_0^s f(t)\,dt$. In particular, we prove that there exist at least $2n+1$ positive solutions for $\lambda \in \left(\frac{n^2\pi^2}{f'(\beta)}, \infty\right)$, where $n \in \mathbb{N}$. The proof of our main result is based upon the bifurcation and continuation methods.

Keywords: multiple positive solutions, Neumann problem, bifurcation method, continuation method.

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1 Introduction

In this paper, we are concerned with the existence of multiple positive solutions to the Neumann problem

$$\begin{aligned} -u''(x) &= \lambda f(u(x)), \qquad x \in (0,1), \\ u'(0) &= 0 = u'(1), \end{aligned} \tag{1.1}$$

where λ is a positive parameter, $f \in C([0,\infty), \mathbb{R})$ and for some $\beta > 0$ such that $f(0) = 0$, $f(s) > 0$ for $s \in (\beta, +\infty)$, $f(s) < 0$ for $s \in (0, \beta)$, and $\theta\,(> \beta)$ is the unique positive zero of $F(s) = \int_0^s f(t)\,dt$.

The Neumann problems have played a significant role in mathematical physics (for example, equilibrium problems concerning beams, columns, or strings and so on), and hence have attracted the attention of many researchers over the last two decades, see [3, 9, 11] and

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the references therein. The existence and multiplicity of positive solutions for the Neumann boundary value problems were investigated in connection with various configurations of f by the fixed point theorems in [3, 11] and by a detailed analysis of time-map associated with (1.1) in [9]. In [8], Maya and Shivaji obtained multiple positive solutions for a class of semi-linear elliptic boundary value problems by using sub-super solutions arguments when $f \in C^1$ satisfies the following conditions:

(f1) $f(0) = 0$;

(f2) $f'(0) < 0$;

(f3) there exists $\beta > 0$ such that $f(u) < 0$ for $u \in (0, \beta)$ and $f(u) > 0$ for $u > \beta$;

(f4) f is eventually increasing and $\lim_{u\to\infty} \frac{f(u)}{u} = 0$.

Recently, Ma [5] studied the global behavior of the components of nodal solutions of asymptotically linear eigenvalue problems by using global bifurcation techniques. For the other results related to the existence of nodal solutions, see [6, 7] and the references therein.

Motivated by the above papers, in this paper, we investigate the existence of multiple positive solutions of (1.1) by applying the bifurcation and continuation methods. In fact, we will transform the Neumann problem into the Dirichlet problem by virtue of the continuation methods, and then we are concerned with determining values of λ, for which there exist nodal solutions of the Dirichlet boundary value problem by means of bifurcation techniques. Consequently, we give the existence results of positive solutions of the problem (1.1), under the following assumptions.

(H1) $f \in C([0, \infty), \mathbb{R})$ and for some $\beta > 0$ such that $f(0) = 0$, $f(s) > 0$ for $s \in (\beta, +\infty)$, $f(s) < 0$ for $s \in (0, \beta)$, and there exists $\theta\ (> \beta)$ a (unique) positive zero of $F(s) = \int_0^s f(t)\, dt$.

(H2) $f'(\beta) = \lim_{s\to 0^+} \frac{f(s+\beta)}{s} > 0$.

(H3) f satisfies the Lipschitz condition in $[0, \beta]$.

We will establish the following theorem.

Theorem 1.1. *Let* (H1)–(H3) *hold and* $n \in \mathbb{N}$. *Then there exist at least* $2n + 1$ *positive solutions of* (1.1) *for* $\lambda \in \left(\frac{n^2\pi^2}{f'(\beta)}, \infty\right)$.

Now to illustrate Theorem 1.1, let us consider the simple example $f(s) = s^2(s - 1)$ for $s \geq 0$. Hence $\beta = 1$ and $f'(\beta) = 1$. Thus, given $n \in \mathbb{N}$, problem (1.1) has at least $2n + 1$ positive solutions for all $\lambda \in (n^2\pi^2, \infty)$.

Remark 1.2. Compared with the configurations of f in [8, 9], we only demand $f \in C[0, \infty)$, so that the quadrature technique does not apply to (1.1). Even if $f \in C^1$ satisfies (H1), it seems rather difficult to make a detailed analysis of the so-called time map to trace down the positive solution of (1.1).

Remark 1.3. Maya and Shivaji [8] obtained multiple positive solutions for a class of semilinear elliptic boundary value problems when f satisfies (f1)–(f4). Notice that $f \in C^1$ implies that f is Lipschitz continuous in $[0, \beta]$, and so (H3) is satisfied. It is worth to be pointed out that in Theorem 1.1 neither $f \in C^1$, nor a growth condition at infinity is required.

The paper is organized as follows. In Section 2 we introduce some notations and auxiliary results and in Section 3 we prove our main result.

2 Some notations and auxiliary results

Let $Y = C[0,2]$ and $E = \{w \in C^1[0,2] \mid w(0) = w(2) = 0\}$ with the norms $\|w\|_\infty = \max_{t\in[0,2]} |w(t)|$ and $\|w\| = \max\{\|w\|_\infty, \|w'\|_\infty\}$ respectively.

The following results are somewhat scattered in Miciano and Shivaji [9].

Lemma 2.1 ([9, Lemma 2.1])**.** *If $u(x)$ is a solution of* (1.1)*, then $u(1-x)$ is also a solution of* (1.1)*.*

Lemma 2.2 ([9, Lemma 2.2])**.** *If $u(x)$ is any solution of* (1.1)*, then $u(x)$ is symmetric with respect to any point $x_0 \in [0,1]$ such that $u'(x_0) = 0$, i.e. $u(x_0 - z) = u(x_0 + z)$ for all $z \in [0, \min\{x_0, 1 - x_0\}]$.*

Remark 2.3. Any zero of f is a solution of (1.1).

Remark 2.4. Let $u(x)$ be a positive solution of (1.1) such that $u(0) = \alpha, u(1) = \gamma, 0 \le \alpha < \beta < \gamma$, and $u'' > 0$ on $(0, t_0)$ and $u'' < 0$ on $(t_0, 1)$, where $t_0 \in (0, 1)$ satisfying $u(t_0) = \beta$. To study positive solution $u(x)$ of (1.1) which has $n - 1$ interior critical points at k/n, $(k = 1, 2, \ldots, n - 1)$, it suffices by Lemma 2.2, to study solution $v_n(x) = u(nx)$ for $x \in [0, 1/n]$. Thus we only need to study the form of positive solution u.

3 Proof of the main result

Proof of Theorem 1.1. We first prove the case $n = 1$. It is divided into three steps.

Step 1. Let $v(x) = u(x) - \beta$. Then the problem (1.1) becomes to

$$\begin{aligned} -v''(x) &= \lambda f\big(v(x) + \beta\big), \qquad x \in (0, 1), \\ v'(0) &= 0 = v'(1). \end{aligned} \tag{3.1}$$

It is easy to see that the solution $u > 0$ of (1.1) is equivalent to the solution v of (3.1) with $v > -\beta$.

To study solutions of (3.1), we consider the auxiliary problem

$$\begin{aligned} -w''(x) &= \lambda f\big(w(x) + \beta\big), \qquad x \in (-t_0, 2 - t_0), \\ w(-t_0) &= 0 = w(2 - t_0), \end{aligned} \tag{3.2}$$

where

$$w(x) = \begin{cases} v(-x), & x \in [-t_0, 0), \\ v(x), & x \in [0, 1], \\ v(2 - x), & x \in (1, 2 - t_0]. \end{cases} \tag{3.3}$$

Since (3.2) is an autonomous equation, we may consider the Dirichlet problem

$$\begin{aligned} -w''(x) &= \lambda f\big(w(x) + \beta\big), \qquad x \in (0, 2), \\ w(0) &= 0 = w(2). \end{aligned} \tag{3.4}$$

Note that any solution $w(x)$ of (3.4) is symmetric with respect to any point $x_0 \in (0, 2)$ such that $w'(x_0) = 0$. In order to find a positive solution of (1.1), it is enough to find such solution of (3.4), which have exactly one simple zero in $(0, 2)$ and is negative near $x = 0$.

Step 2. Let $g(w) = f(w+\beta)$. Then g satisfies

(H1)′ $g \in C\big([-\beta,\infty),\mathbb{R}\big)$, $g(-\beta) = 0$, $g(w) > 0$ for $w \in (0,+\infty)$, $g(w) < 0$ for $w \in (-\beta,0)$ and there exists $\theta - \beta (> 0)$ a (unique) positive zero of $G(s) = \int_{-\beta}^{s} g(t)\,dt$;

(H2)′ $f'(\beta) = \lim\limits_{|s|\to 0} \frac{g(s)}{s} > 0$;

(H3)′ g satisfies the Lipschitz condition in $[-\beta,0]$.

According to [1], we extend the function g to a continuous function $\tilde{g}$ defined on $\mathbb{R}$ in such a way that $\tilde{g}(s) > 0$ for all $s < -\beta$. In the sequel of the proof we shall replace g with $\tilde{g}$, however, for the sake of simplicity, the modified function $\tilde{g}$ will still be denoted by g.

For $\lambda > 0$, we claim $w \ge -\beta$, where w is a solution of the problem

$$\begin{aligned} -w''(x) &= \lambda g(w(x)), \qquad x \in (0,2),\\ w(0) &= 0 = w(2). \end{aligned} \tag{3.5}$$

Suppose that there exists some $x_0 \in (0,2)$ such that $\min_{x\in[0,2]} w(x) = w(x_0) < -\beta$. This implies $w''(x_0) \ge 0$. On the other hand, $-w''(x_0) = \lambda g(w(x_0)) > 0$. This is a contradiction. Hence, $w \ge -\beta$.

Define $L\colon D(L) \subset E \to Y$ by setting

$$Lw := -w'', \qquad w \in D(L)$$

with

$$D(L) = \{w \in C^2[0,2] \mid w(0) = w(2) = 0\}.$$

Then $L^{-1}\colon Y \to E$ is completely continuous. Let $\zeta \in C(\mathbb{R},\mathbb{R})$ be such that

$$g(w) = f'(\beta)w + \zeta(w).$$

Clearly, $\lim_{|w|\to 0} \frac{\zeta(w)}{w} = 0$. Let us consider

$$Lw - \lambda f'(\beta) w = \lambda \zeta(w) \tag{3.6}$$

as a bifurcation problem from the trivial solution $w = 0$. Note that (3.6) is equivalent to (3.5).

By the Krasnoselskii–Rabinowitz bifurcation theorem (see [2, Theorem 22.8]), the following result holds.

Lemma 3.1. *λ_k is a bifurcation point of (3.6) and the associated bifurcation branch $\mathcal{C}_k$ in $\mathbb{R}\times E$ whose closure contains $(\lambda_k, 0)$ is either unbounded or contains a pair $(\lambda_j, 0)$ and $j \ne k$, where $\lambda_k = \frac{k^2\pi^2}{4f'(\beta)}$ is the kth eigenvalue of*

$$\begin{aligned} -\varphi''(x) &= \lambda f'(\beta)\varphi, \qquad x \in (0,2),\\ \varphi(0) &= 0 = \varphi(2). \end{aligned}$$

Let $\mathbb{E} = \mathbb{R} \times E$ under the product topology. Let S_k^+ denote the set of functions in E which have exactly $k-1$ simple zeros in $(0,2)$ and are positive near $t = 0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. They are disjoint and open in E. Finally, let $\Phi_k^{\pm} = \mathbb{R} \times S_k^{\pm}$ and $\Phi_k = \mathbb{R} \times S_k$.

Let $\tilde{\zeta}(w) = \max_{0\le|s|\le w} |\zeta(s)|$, then $\tilde{\zeta}$ is nondecreasing with respect to w and

$$\lim_{w\to 0^+} \frac{\tilde{\zeta}(w)}{w} = 0. \tag{3.7}$$

Further it follows from (3.7) that

$$\frac{\zeta(w)}{\|w\|} \leq \frac{\tilde{\zeta}(|w|)}{\|w\|} \leq \frac{\tilde{\zeta}(\|w\|_\infty)}{\|w\|} \leq \frac{\tilde{\zeta}(\|w\|)}{\|w\|} \to 0, \quad \text{as } \|w\| \to 0. \tag{3.8}$$

Lemma 3.2. *The last alternative of Lemma 3.1 is impossible if $\mathcal{C}_k \subset \Phi_k \cup \{(\lambda_k, 0)\}$.*

Proof. Suppose on the contrary, if there exists $(\lambda_m, w_m) \to (\lambda_j, 0)$ when $m \to +\infty$ with $(\lambda_m, w_m) \in \mathcal{C}_k, w_m \not\equiv 0$ and $j \neq k$. Let $y_m := \frac{w_m}{\|w_m\|}$, then y_m should be a solution of the problem

$$y_m = L^{-1}\left(\lambda_m f'(\beta) y_m + \frac{\lambda_m \zeta(w_m)}{\|w_m\|}\right). \tag{3.9}$$

By (3.8), (3.9) and the compactness of L^{-1}, we obtain that for some convenient subsequence $y_m \to y_0 \neq 0$ as $m \to +\infty$. Now y_0 verifies the equation

$$-y_0'' = \lambda_j f'(\beta) y_0$$

and $\|y_0\| = 1$. Hence $y_0 \in S_j$ which is an open set in E, and as a consequence for some m large enough, $y_m \in S_j$, and this is a contradiction. □

Lemma 3.3. *From each $(\lambda_k, 0)$ it bifurcates an unbounded continuum $\mathcal{C}_k$ of solutions to problems (3.6) with exactly $k-1$ simple zeros.*

Proof. Taking into account Lemma 3.1 and Lemma 3.2, we only need to prove that $\mathcal{C}_k \subset \Phi_k \cup \{(\lambda_k, 0)\}$.

Suppose $\mathcal{C}_k \not\subset \Phi_k \cup \{(\lambda_k, 0)\}$. Then there exists $(\lambda, w) \in \mathcal{C}_k \cap (\mathbb{R} \times \partial S_k)$ such that $(\lambda, w) \neq (\lambda_k, 0), w \notin S_k$, and $(\lambda_n, w_n) \to (\lambda, w)$ with $(\lambda_n, w_n) \in \mathcal{C}_k \cap (\mathbb{R} \times S_k)$. Since $w \in \partial S_k$, $w \equiv 0$. Let $u_n := \frac{w_n}{\|w_n\|}$, then u_n should be a solution of the problem

$$u_n = L^{-1}\left(\lambda_n f'(\beta) u_n + \frac{\lambda_n \zeta(w_n)}{\|w_n\|}\right). \tag{3.10}$$

By (3.8), (3.10) and the compactness of L^{-1} we obtain that for some convenient subsequence $u_n \to u_0 \neq 0$ as $n \to +\infty$. Now u_0 verifies the equation

$$-u_0'' = \lambda f'(\beta) u_0$$

and $\|u_0\| = 1$. Hence $\lambda = \lambda_j$, for some $j \neq k$. Therefore, $(\lambda_n, w_n) \to (\lambda_j, 0)$ with $(\lambda_n, w_n) \in \mathcal{C}_k \cap (\mathbb{R} \times S_k)$. This contradicts Lemma 3.2. □

By the definition of $\mathcal{C}_k^\nu$ in [4, 10], $\mathcal{C}_k^\nu$ is connected, where $\nu \in \{+, -\}$, and $\mathcal{C}_k = \mathcal{C}_k^+ \cup \mathcal{C}_k^-$. According to the Dancer unilateral global bifurcation result [4, Theorem 2], the following result holds.

Lemma 3.4. *Either $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ are both unbounded, or else $\mathcal{C}_k^+ \cap \mathcal{C}_k^- \neq \{(\lambda_k, 0)\}$.*

Connecting Lemma 3.3 with Lemma 3.4, we can easily deduce the following unilateral global bifurcation results.

Lemma 3.5. *Let $\nu \in \{+, -\}$. Then $\mathcal{C}_k^\nu$ is unbounded in $\mathbb{R} \times E$ and*

$$\mathcal{C}_k^\nu \subset \{(\lambda_k, 0)\} \cap (\mathbb{R} \times S_k^\nu) \quad \textit{or} \quad \mathcal{C}_k^\nu \subset \{(\lambda_k, 0)\} \cap (\mathbb{R} \times S_k^{-\nu}). \tag{3.11}$$

Proof. By Lemma 3.3, we can get (3.11) easily. So we only need to prove that both $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ are unbounded. Suppose on the contrary, without loss of generality, we may suppose that $\mathcal{C}_k^-$ is bounded. By Lemma 3.4, we know that $(\mathcal{C}_k^- \cap \mathcal{C}_k^+) \setminus \{(\lambda_k, 0)\} \neq \emptyset$. Therefore, in view of (3.11), there exists $(\lambda_*, w_*) \in \mathcal{C}_k^- \cap \mathcal{C}_k^+$ such that $(\lambda_*, w_*) \neq (\lambda_k, 0)$ and $w_* \in S_k^+ \cap S_k^-$. This contradicts the definitions of S_k^+ and S_k^-. □

From (H1)′–(H3)′, by a proof similar to that of Theorem 2.1 of [5], for any $(\lambda, w) \in \mathcal{C}_k^+ \cup \mathcal{C}_k^-$, $w(x) > -\beta, x \in [0,2]$.

If $w \in \mathcal{C}_2^-$, there exists $0 < x_1 < x_2 < 2$ such that $w'(x_1) = w'(x_2) = 0, \min_{x\in[0,2]} w(x) = w(x_1) > -\beta, \max_{x\in[0,2]} w(x) = w(x_2)$. Multiplying (3.5) by $w'(x)$ and then integrating from x_1 to x_2, we have

$$\int_{w(x_1)}^{w(x_2)} g(s)\, ds = 0.$$

It follows from $(H1)'$ and $w(x_1) > -\beta$ that $-\beta < w(x) < \theta - \beta$.

From Lemma 3.5 and $w \in C^1[0,2]$ is bounded, and so $\mathcal{C}_2^-$ is unbounded in the direction of λ.

Step 3. Let $w \in \mathcal{C}_2^-$, $x_0 \in (0,2)$ such that $w(x_0) = 0$. Since (3.5) is autonomous equation, $w(x)$ is symmetric about $\frac{x_0}{2}$ and $\frac{2+x_0}{2}$. Moreover, $\beta > 0$ is the unique positive zero of f and $u(t_0) = \beta$, which combine with (1.1), (3.1)–(3.3) and (3.5) imply that $x_0 = 2t_0$. So $w \in \mathcal{C}_2^-$, $x \in \left(\frac{x_0}{2}, \frac{2+x_0}{2}\right)$ corresponds to $u(t) > 0, t \in [0,1]$, which is a positive solution of (1.1).

By Lemma 2.1, there exist at least three positive solution of (1.1) for $\lambda \in \left(\frac{\pi^2}{f'(\beta)}, \infty\right)$.

Next, we prove the case $n > 1$.

Consider positive solutions with $n-1$ interior critical points. By Lemma 2.2, the analysis of these types of solutions is achieved by studying nondecreasing positive solutions on the interval $[0, 1/n]$, i.e. it is enough to study the positive solutions of

$$\begin{aligned} -u''(x) &= \lambda f(u(x)), \qquad x \in \left(0, \tfrac{1}{n}\right), \\ u'(0) &= 0 = u'\left(\tfrac{1}{n}\right). \end{aligned} \tag{3.12}$$

The change of variables

$$u(x) = \omega(y), \qquad y = xn, \qquad 0 \le x \le \frac{1}{n} \tag{3.13}$$

transforms (3.12) into

$$\begin{aligned} -\omega''(y) &= \tilde{\lambda} f(\omega(y)), \qquad y \in (0,1), \\ \omega'(0) &= 0 = \omega'(1), \end{aligned} \tag{3.14}$$

where $\tilde{\lambda} := \lambda / n^2$.

As (3.14) is of the same type as (1.1), by the analysis already done in the case $n = 1$, it becomes apparent that (3.14) possesses a nondecreasing positive solution if $\tilde{\lambda} \in \left(\frac{\pi^2}{f'(\beta)}, \infty\right)$.

In fact, $u(1-x)$ is also a solution (see Lemma 2.1) with $n-1$ critical interior points.

By Theorem 1.1 in case $n = 1$ and Lemma 2.1, for each $m = 1, 2, \ldots, n$, we obtain two solutions with $m-1$ interior critical points. These solutions along with the solution $u \equiv \beta$, which implies that the problem (1.1) has at least $2n+1$ positive solutions for $\lambda \in \left(\frac{n^2\pi^2}{f'(\beta)}, \infty\right)$. This completes the proof of Theorem 1.1. □

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