RESTRICTED SKEW-MORPHISMS ON MATRIX ALGEBRAS

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Abstract. In this paper, skew-morphisms, which are extensively studied in graph theory, are considered in the setting of matrix algebras. Different properties of skew-morphisms are obtained and their classification in some specific cases is given.

1. Introduction

Let $n \geq 2$ be an integer and let M_n be the algebra of n-by-n matrices over a field $\mathbb{F}$. We will consider maps $\phi\colon M_n \to M_n$ with the following property: there exists a *power function* $\kappa\colon M_n \to \{0, 1, 2, \ldots\}$ such that

$$\phi(AB) = \phi(A)\phi^{\kappa(A)}(B); \qquad \forall\, A, B \in M_n, \tag{1}$$

where as usual $\phi^0 = \mathrm{id}$, the identity mapping, and $\phi^k(x) = \phi(\phi^{k-1}(x))$. Maps satisfying (1) will be called *restricted skew-morphisms* to distinguish them from skew-morphisms. Skew-morphisms were recently introduced by Jajcay and Širáň [6] as bijective unital maps on groups with property (1) but where $\kappa(A)$ takes the values in $\mathbb{Z}$, the set of all integers. It needs to be said that in [6] they considered only unital bijections on finite groups, which are consequently of finite order and allow one to replace negative powers of ϕ by nonnegative powers, modulo the order of ϕ. Jajcay and Širáň used skew-morphisms in an attempt to give a unified treatment of regular Cayley maps, which by definition are 2-cell embeddings of Cayley graphs into orientable surfaces which preserve a given orientation at each vertex.

Skew-morphisms also arise naturally in studying cyclic extensions of groups. In fact, if a group $G = AC$ is a product of a subgroup A and a finite cyclic subgroup $C = \langle c \rangle$ with $A \cap C = \{1\}$ then each element $g \in G$ can be written uniquely as $g = ac^i$ for some $a \in A$ and some integer i. For $g = ca$ it follows that $ca = \phi(a)c^i$ for a unique element $\phi(a) \in A$ and a unique integer $i \in \{0, \ldots, |c| - 1\}$, where $|c|$ is the order of c. Given $a, b \in A$ we have

$$c(ab) = (ca)b = \phi(a)c^i b = \phi(a)\phi^i(b)c^k$$

2000 *Mathematics Subject Classification.* 20D45 (primary), 15A04, 15A86 (secondary).

Key words and phrases. Matrix algebra, Skew-morphism.

The first and the second author were partially supported by Ministry of Education, Science and Sport of Slovenia. The third and fourth author were supported by the “Lendület” Program (LP2012-46/2012) of the Hungarian Academy of Sciences.

for appropriate integer k. This gives $\phi(ab) = \phi(a)\phi^i(b)$ for some integer i that depends only on a, i.e., ϕ is a (restricted) skew-morphism. We were informed by R. Jajcay that this connection between cyclic extensions of groups and skew-morphisms was observed already in 1938 by Ore [8, p. 805]; see also Conder, Jajcay, and Tucker [1, p. 262–263].

Clearly, skew-morphisms are generalizations of automorphisms. Classification of automorphisms can be quite involved, see for example Dieudonné [3, 4] for a general linear group, so it is not surprising that until now characterizations of skew-morphisms were obtained only for some special cases. For example, Kovács and Nedela [7] studied skew-morphisms on $\mathbb{Z}_n$, the cyclic group of order n, and obtained classification for some specific n. In particular, they showed that every skew-morphism on $\mathbb{Z}_n$ is an automorphism if and only if $n = 4$ or n is relatively prime with $\varphi(n)$, the Euler totient function. Another example of investigation is the result of Zhang [9], who studied skew-morphisms which are automorphisms on a subgroup of index three, and whose power function assume three values, one of which is 1.

It is the aim of this paper to study restricted skew-morphisms on the semigroup M_n, $n \geq 2$. We give a complete classification of linear restricted skew-morphisms, see Theorem 4. In the case of general restricted surjective skew-morphisms we prove that they preserve the rank, see Theorem 8, and describe them when their power functions are constantly equal to one on GL_n, see Theorem 17. We also prove that a surjective restricted skew-morphism has a finite order if its power function takes a value which is greater than one on GL_n, see Theorem 18. The case when the power function takes the value zero on GL_n was treated in [6], see also Lemma 1 below.

Note that the power function κ from (1) can take negative values only in the case when ϕ is a bijective function. Since we will not assume bijectivity of ϕ we have to restrict the codomain of the power function κ to the set of non-negative integers. To simplify the notation we will skip the adjective "restricted" throughout the paper and refer to maps satisfying (1) simply as skew-morphisms.

2. Results

Let us denote by GL_n the general linear group with identity Id, and by SL_n the special linear group. For every matrix M, including the case when M is a zero matrix, we define $M^0 = \mathrm{Id}$. Let E_{ij} be the standard matrix unit which has 1 in its (i, j) entry and 0 elsewhere.

Before proving the first result observe that each homomorphism of M_n is also a skew-morphism with its power function κ constantly equal to 1. Observe also that if ϕ is a unital skew-morphism and its power function κ is a constant, then ϕ is multiplicative. Namely, from $\phi(AB) = \phi(A)\phi^\kappa(B)$, by inserting $A = \mathrm{Id} = \phi(\mathrm{Id})$, we get $\phi(B) = \phi^\kappa(B)$, so $\phi(AB) = \phi(A)\phi^\kappa(B) = \phi(A)\phi(B)$.

2.1. **Preliminary results and characterization of linear skew-morphisms.** Let us start by proving two lemmas which are valid also for possibly non-linear maps. We remark that the first one was already proved in [6, p. 171] for skew-morphisms on groups.

Lemma 1. *Let $\phi\colon M_n \to M_n$ be a skew-morphism. If $\kappa(G_0) = 0$ for some $G_0 \in \mathrm{GL}_n$, then there exists $M \in M_n$ such that $\phi(X) = MX$ for all $X \in M_n$. In addition, if ϕ is surjective, then $M \in \mathrm{GL}_n$.*

Proof. Since $\kappa(G_0) = 0$ for some invertible matrix G_0, then for an arbitrary matrix $X \in M_n$ we obtain

$$\phi(X) = \phi(G_0 G_0^{-1} X) = \phi(G_0)\phi^0(G_0^{-1}X) = \phi(G_0)G_0^{-1}X.$$

If ϕ is surjective, then clearly $M \in \mathrm{GL}_n$, since otherwise the image of ϕ would be contained in the set of singular matrices. □

Remark 2. *Observe that the skew-morphism ϕ from Lemma 1 is linear. Actually, $\phi(X) = MX$ is a special case of $\phi(A) = MAN$, which is the general form of linear skew-morphisms, see Theorem 4 below with $s = 0$, $\lambda = 1$, and $N = \mathrm{Id}$.*

Lemma 3. *Let $\phi\colon M_n \to M_n$ be a surjective skew-morphism. Then ϕ maps 0 to 0, singular matrices surjectively onto singular ones, and GL_n surjectively onto GL_n.*

Proof. Observe that, if $\kappa(G_0) = 0$ for some $G_0 \in \mathrm{GL}_n$, then the conclusion of Lemma 3 holds by Lemma 1. So in the rest of the proof we will assume that $\kappa(S) \geq 1$ for every $S \in \mathrm{GL}_n$.

First we show that ϕ maps GL_n into GL_n. By surjectivity there exists $B \in M_n$ such that $\phi(B) = \mathrm{Id}$. Since ϕ is a skew-morphism, it follows from (1) that

$$\mathrm{Id} = \phi(B) = \phi(\mathrm{Id}\, B) = \phi(\mathrm{Id})\phi^{\kappa(\mathrm{Id})}(B),$$

which shows that $\phi(\mathrm{Id})$ is right-invertible, thus invertible. Let $A \in \mathrm{GL}_n$ be arbitrary. Then

$$\phi(\mathrm{Id}) = \phi(AA^{-1}) = \phi(A)\phi^{\kappa(A)}(A^{-1})$$

and since $\phi(\mathrm{Id})$ is invertible, it follows that $\phi(A)$ is also invertible. So $\phi(\mathrm{GL}_n) \subseteq \mathrm{GL}_n$.

Next we show that ϕ annihilates 0. Observe that $\phi(0) = \phi(0X) = \phi(0)\phi^{\kappa(0)}(X)$ for every $X \in M_n$. By surjectivity of ϕ, hence also of $\phi^{\kappa(0)}$, we can find X_0 such that $\phi^{\kappa(0)}(X_0) = 0$, whence

$$\phi(0) = \phi(0)0 = 0.$$

Now, let $A \in M_n$ be singular and suppose $\phi(A) \in \mathrm{GL}_n$. Let $S, T \in \mathrm{GL}_n$ be arbitrary. Then $\phi(SAT) = \phi(S)\phi^{\kappa(S)}(AT) = \phi(S)\phi^{\kappa(S)-1}(\phi(AT)) = \phi(S)\phi^{\kappa(S)-1}(\phi(A)\phi^{\kappa(A)}(T))$. Since $\phi(A) \in \mathrm{GL}_n$ and GL_n is invariant for ϕ, we see that every matrix with the same rank as A is mapped into an invertible one. In particular, the nilpotent matrix $N = \sum_{i=1}^{\mathrm{rk}\, A} E_{i\,(i+1)}$ with rank equal to $\operatorname{rk} A$, is mapped into an invertible one. Let us show that $\kappa(N) = 0$ is not possible. Otherwise we would have $\phi(NX) = \phi(N)X$ and so $X = \phi(N)^{-1}\phi(NX)$ for each $X \in M_n$. Since N is singular we can choose a nonzero matrix X with $NX = 0$. It follows that for this X we have $0 \neq X = \phi(N)^{-1}\phi(0) = \phi(N)^{-1} \cdot 0 = 0$, a contradiction. Hence $\kappa(N) \geq 1$. But then $\phi(N^2) = \phi(N)\phi^{\kappa(N)}(N)$ is the product of two invertible matrices and therefore invertible. Proceeding in the same way we obtain that $\phi(N^{2^i})$ is invertible for every positive integer i. Hence $\phi(0)$ is invertible, a contradiction.

The conclusions of Lemma 3 now follow by surjectivity of ϕ. □

Theorem 4. *A linear map $\phi\colon M_n \to M_n$ is a skew-morphism, i.e., satisfies equation* (1)*, if and only if there exist a nonnegative integer s, a nonzero scalar λ, and matrices $M, N \in M_n$ with N invertible and $N^{1-s} = NM^s = \lambda\,\mathrm{Id}$, such that ϕ is of the form*

$$\phi(A) = MAN \quad (A \in M_n). \tag{2}$$

Remark 5. *Observe that M does not need to be invertible. In this case, $s = 0$ and we can take $N = \mathrm{Id}$.*

Proof of Theorem 4. If ϕ is of the form (2), then ϕ clearly satisfies equation (1) with the choice $\kappa(A) = s$ for all $A \in M_n$ and hence ϕ is a skew-morphism.

As for the converse, assume first that ϕ is not bijective, i.e., $\phi(A) = 0$ for some nonzero matrix $A \in M_n$. We distinguish two subcases.

(i) Suppose $\kappa(G) \geq 1$ for every invertible matrix $G \in M_n$. Let $R \in M_n$ be an arbitrary rank-one matrix. Then there exist an invertible matrix S and a rank-one matrix T, such that $R = SAT$. Recall that $\phi(A) = 0$ and that by linearity $\phi(0) = 0$. It follows that $\phi(R) = \phi(SAT) = \phi(S)\phi^{\kappa(S)-1}(\phi(AT)) = \phi(S)\phi^{\kappa(S)-1}(\phi(A)\phi^{\kappa(A)}(T)) = \phi(S)\phi^{\kappa(S)-1}(0) = 0$ for every rank-one matrix R, hence by linearity ϕ is a zero map, i.e., ϕ is of the form (2) with $s = 0$, $\lambda = 1$, $M = 0$, and $N = \mathrm{Id}$.

(ii) Suppose $\kappa(G_0) = 0$ for some invertible matrix G_0. Then by Lemma 1, ϕ is of the form (2) with $s = 0$, $\lambda = 1$, and $N = \mathrm{Id}$.

Second, assume that ϕ is bijective. Then by Lemma 3, ϕ maps invertible matrices surjectively onto invertible matrices. Hence, ϕ^{-1} maps singular matrices into singular ones. The structure of such linear maps was determined by Dieudonné [2], they are of the form

$$\phi(A) = MAN \quad (A \in M_n) \tag{3}$$

or of the form

$$\phi(A) = MA^tN \quad (A \in M_n) \tag{4}$$

for some invertible matrices M and N.

Let us show that map (4) is not a skew-morphism. In fact, for rank-one matrices $A = E_{11}$ and $B = E_{12}$ we see that $M^{-1}\phi(AB)N^{-1} = E_{21} \neq E_{11}N \cdot \phi^{\kappa(A)}(E_{12})N^{-1} = M^{-1}\phi(A)\phi^{\kappa(A)}(B)N^{-1}$. So ϕ is of the form (3).

Set $s = \kappa(\mathrm{Id})$. Let x be an arbitrary column vector. If we insert $A = \mathrm{Id}$ and $B = xx^t$ into (1), then the equation $B = \mathrm{Id}\, B$ transforms into

$$Mxx^tN = MNM^s xx^t N^s$$

which yields that

$$xx^t = (NM^s x)((N^{s-1})^t x)^t.$$

In particular, every vector x is an eigenvector of NM^s and of $(N^{s-1})^t$. It follows that there exists a nonzero scalar λ, such that

$$NM^s = \lambda\, \mathrm{Id} \quad \text{and} \quad \lambda N^{s-1} = \mathrm{Id},$$

which completes the proof. □

Recall that ϕ is unital if $\phi(\mathrm{Id}) = \mathrm{Id}$.

Corollary 6. *A unital linear skew-morphism ϕ is of the form*

$$\phi(A) = SAS^{-1} \quad (A \in M_n)$$

for some invertible matrix $S \in M_n$.

As the following example shows, in general skew-morphisms are not linear and therefore are not of the form (2).

Example 7. *By a straightforward computation we see that the map $\phi\colon M_2(\mathbb{Z}_2) \to M_2(\mathbb{Z}_2)$ defined below is a nonlinear bijective unital skew-morphism.*

$$\begin{array}{ll}
\phi\left(\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right)=2\,, & \phi\left(\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)\right)=3\,,\\
\phi\left(\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)\right)=2\,, & \phi\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\right)=1\,,\\
\phi\left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)=1\,, & \phi\left(\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)\right)=3\,,\\
\phi\left(\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\right)=2\,, & \phi\left(\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\right)=1\,,\\
\phi\left(\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)\right)=3\,, & \phi\left(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\right)=0\,,\\
\phi\left(\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)\right)=1\,, & \phi\left(\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)\right)=0\,,\\
\phi\left(\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)\right)=3\,, & \phi\left(\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)\right)=2\,,\\
\phi\left(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\right)=0\,, & \phi\left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\ \kappa\left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right)=1\,.
\end{array}$$

2.2. **Every surjective skew-morphism is a rank preserver.** Note that the skew-morphism ϕ defined in Example 7 maps rank-one matrices onto rank-one matrices. In the sequel we will show that this is true also in the general case. Even more, we will prove that every surjective skew-morphism on $M_n(\mathbb{F})$ maps the set of rank k matrices onto itself for every $k \in \{0, \dots, n\}$.

Observe that by Lemma 1 every surjective skew-morphism ϕ preserves the rank if $\kappa(A) = 0$ for some invertible matrix A. Therefore we will assume from now on that $\phi\colon M_n \to M_n$ is a surjective skew-morphism with

$$\kappa(A) \geq 1 \quad \text{for every } A \in \mathrm{GL}_n\,. \tag{5}$$

The main result in this subsection is the following.

Theorem 8. *Let $\phi\colon M_n \to M_n$ be a surjective skew-morphism. Then* $\operatorname{rk}\phi(A) = \operatorname{rk} A$ *for every $A \in M_n$.*

The proof will be given at the end of this section after a series of preliminary lemmas.

Lemma 9. *If $\phi(A) = 0$ for some A, then $\phi(X) = 0$ for every X with* $\operatorname{rk} X \le \operatorname{rk} A$.

Proof. Let T be invertible. Then $\kappa(T) \ge 1$ by the assumption (5), and therefore, with every matrix S

$$\phi(TAS) = \phi(T)\phi^{\kappa(T)}(AS) = \phi(T)\phi^{\kappa(T)-1}(\phi(A)\phi^{\kappa(A)}(S)) = 0$$

by Lemma 3. Every matrix with rank at most $\operatorname{rk} A$ can be written as TAS for appropriate matrices T, S with T invertible. Thus, ϕ annihilates each matrix with rank at most $\operatorname{rk} A$. □

Lemma 10. $\phi(X) = 0$ *if and only if* $X = 0$.

Proof. By Lemma 3, $\phi(0) = 0$, and ϕ maps the set of singular matrices surjectively onto itself and GL_n surjectively onto GL_n. Assume $\phi(A) = 0$ for some A with $\operatorname{rk} A \ge 1$. Lemma 9 implies that $\phi(X) = 0$ for every X with $\operatorname{rk} X = 1$. This is the starting point of induction to prove that $\phi^k(X) = 0$ for each X with $\operatorname{rk} X \le k$, $k = 1, \ldots, n-1$, which is in contradiction with the fact that ϕ^{n-1} maps singular matrices surjectively onto themselves.

Set $A_1 = A$ and assume $A_2, \ldots, A_k$ are such that $\operatorname{rk} A_i \ge i$ and $\phi(A_i) = A_{i-1}$, $i = 2, \ldots, k$. Since ϕ is surjective on singular matrices, there exists a singular matrix A_{k+1} with $\phi(A_{k+1}) = A_k$. By the inductive step, $\operatorname{rk} A_{k+1} > k$, for otherwise, $A = A_1 = \phi^k(A_{k+1}) = 0$, a contradiction. However, note that

$$\phi^{k+1}(A_{k+1}) = \phi(A) = 0. \tag{6}$$

Let T be invertible and let S be an arbitrary matrix. Then by Jajcay and Širáň [6, Lemma 2]

$$\phi^{k+1}(TA_{k+1}S) = \phi^{k+1}(T)\phi^{\kappa_{k+1}(T)}(A_{k+1}S), \qquad \kappa_{k+1}(T) = \sum_{i=0}^{k} \kappa(\phi^i(T)). \tag{7}$$

Since $\phi^i(T) \in \mathrm{GL}_n$ for $i = 0, \ldots, k$, and as $\kappa(G) \ge 1$ for every invertible matrix G by the assumption (5), we see that $\kappa_{k+1}(T) \ge k+1$. Hence by (6)

$$\phi^{k+1}(TA_{k+1}S) = \phi^{k+1}(T)\phi^{\kappa_{k+1}(T)-k-1}(\phi^{k+1}(A_{k+1})\phi^{\kappa_{k+1}(A_{k+1})}(S)) = 0.$$

Thus, as in the final step in the proof of Lemma 9, ϕ^{k+1} annihilates all matrices X with $\operatorname{rk} X \le \operatorname{rk} A_{k+1}$. □

Lemma 11. *Let $k \in \{0, 1\}$. If $y \in \mathbb{F}^n$ is a nonzero vector, then there exists a rank-one matrix R such that $\phi^k(R) = yg^t$ for some nonzero vector $g \in \mathbb{F}^n$.*

Proof. Observe that for $k = 0$ we can take $R = yy^t$. If $k = 1$, then by surjectivity of ϕ there exists a matrix A such that $\phi(A) = yy^t$. Also, there exists a matrix B with $\operatorname{rk}(AB) = 1$. By Lemma 10 and by definition of skew-morphisms it follows that

$$0 < \operatorname{rk}\phi(AB) = \operatorname{rk}(\phi(A)\phi^{\kappa(A)}(B)) \leq \operatorname{rk}\phi(A) = 1.$$

Hence $R = AB$ is a rank-one matrix and $0 \neq \phi(R) = \phi(AB) = \phi(A)M = yy^tM$ where $M = \phi^{\kappa(A)}(B)$. Defining $g^t = y^tM$ finishes the proof of the claim for $k = 1$. □

Let us define sets $\mathcal{L}_x$, $x \neq 0$, of rank-one matrices as

$$\mathcal{L}_x = \{xf^t : f \in \mathbb{F}^n \backslash \{0\}\}.$$

Lemma 12. *Let $k \geq 0$ be an integer. If $y \in \mathbb{F}^n$ is a nonzero vector, then there exists a nonzero vector $x \in \mathbb{F}^n$, such that*

$$\phi^k(\mathcal{L}_x) = \mathcal{L}_y.$$

Proof. The case $k = 0$ is trivial so let $k = 1$. By Lemma 11 there exists a rank-one matrix xf^t, where $x, f \in \mathbb{F}^n$, such that $\phi(xf^t) = yg^t$ for some nonzero vector $g \in \mathbb{F}^n$. By Lemma 3 the set of invertible matrices is mapped surjectively onto itself, therefore

$$\phi(\mathcal{L}_x) = \phi(xf^t\,\mathrm{GL}_n) = \phi(xf^t)\phi^{\kappa(xf^t)}(\mathrm{GL}_n) = yg^t\,\mathrm{GL}_n = \mathcal{L}_y.$$

The case $k \geq 2$ now follows trivially. □

Lemma 13. *The set of rank-one matrices is mapped surjectively onto itself by ϕ.*

Proof. Let $A = xf^t \in M_n$ be a rank-one matrix. By Lemma 10, $\phi(A) \neq 0$. So there exists a rank-one matrix $B \in M_n$ such that $\operatorname{rk}(\phi(A)B) = 1$. By Lemma 12 there exists a rank-one matrix $R \in M_n$ with $\phi^{\kappa(A)}(R) = B$. So

$$\operatorname{rk}\phi(AR) = \operatorname{rk}(\phi(A)\phi^{\kappa(A)}(R)) = \operatorname{rk}(\phi(A)B) = 1$$

and therefore by Lemma 10 we have $AR \neq 0$. Since $AR = x(R^tf)^t$ it is easy to see that there exists an invertible matrix $S \in M_n$ such that $ARS = x(S^tR^tf)^t = xf^t = A$. Hence by Lemma 3 we obtain

$$\operatorname{rk}\phi(A) = \operatorname{rk}\phi(ARS) = \operatorname{rk}(\phi(AR)\phi^{\kappa(AR)}(S)) = \operatorname{rk}\phi(AR) = 1.$$

We proved that ϕ maps the set of rank-one matrices into the set of rank-one matrices. The surjectivity of the restriction of ϕ to rank-one matrices follows by Lemma 12. □

Lemma 14. *If $\operatorname{rk} A \in \{0, 1, n-1, n\}$, then $\operatorname{rk}\phi(A) = \operatorname{rk} A$.*

Proof. We already know this if $\operatorname{rk} A \in \{0, 1, n\}$. So let $\operatorname{rk} A = n - 1$. By Lemma 3, $\operatorname{rk}\phi(A) \leq n - 1$. Assume that $\operatorname{rk}\phi(A) < \operatorname{rk} A$. Then there exist at least 2 linearly independent vectors y_1, y_2 in the kernel of $\phi(A)$. By

Lemma 13 there exist rank-one matrices R_1, R_2 such that $\phi^{\kappa(A)}(R_i) = y_i y_i^t$. Observe that

$$\phi(AR_i) = \phi(A)\phi^{\kappa(A)}(R_i) = \phi(A)y_i y_i^t = 0,$$

so by Lemma 10 also $AR_i = 0$. If $R_1, R_2 \in L_x$ for some x, then $R_2 = R_1 S$ for some invertible matrix S, hence $\phi^{\kappa(A)}(R_2) = \phi^{\kappa(A)}(R_1)\phi^{\kappa(A)+\kappa(\phi(A))+\cdots+\kappa(\phi^{\kappa(A)-1}(A))}(S)$, and so $\phi^{\kappa(A)}(R_2) \in L_{y_1}$ which contradicts the linear independence of y_1, y_2. Thus, $R_i = x_i f_i^t$ and x_1, x_2 are linearly independent vectors in the kernel of A. This implies $\operatorname{rk} A \leq n-2$, a contradiction. □

Proof of Theorem 8. Recall that by the assumption (5), $\kappa(\mathrm{GL}_n) \geq 1$. Next, Sylvester's rank inequality states that for $A, B \in M_n$, $\operatorname{rk}(AB) \geq \operatorname{rk} A + \operatorname{rk} B - n$, so by induction, for $A_1, \ldots, A_k \in M_n$,

$$\operatorname{rk}(A_k \cdots A_1) \geq \operatorname{rk} A_k + \cdots + \operatorname{rk} A_1 - (k-1)n. \tag{8}$$

Now, let $B \in M_n$ and let $r = \operatorname{rk} B$, i.e., $B = S(0_{n-r} \oplus \mathrm{Id}_r)T$ for some $S, T \in \mathrm{GL}_n$. Then $B = \prod_{i=1}^{n-r} B_i$ is the product of $n-r$ matrices with rank equal to $n-1$, where $B_1 = S(\mathrm{Id} - E_{11})$, $B_{n-r} = (\mathrm{Id} - E_{(n-r)\,(n-r)})T$, and $B_i = (\mathrm{Id} - E_{ii})$, $i = 2, \ldots, n-r-1$. By (1) it follows that

$$\phi(B) = \phi^{s_1}(B_1) \cdots \phi^{s_{n-r}}(B_{n-r})$$

for some nonnegative integers $s_1, \ldots, s_{n-r}$ with $s_1 = 1$. Hence by Lemma 14 and inequality (8),

$$\operatorname{rk} \phi(B) \geq (n-r)(n-1) - (n-r-1)n = r = \operatorname{rk} B. \tag{9}$$

Suppose there exists $B \in M_n$ with $\operatorname{rk} \phi(B) > \operatorname{rk} B$. Among all such matrices we choose B_0 with the smallest possible rank and set $r_0 = \operatorname{rk} B_0$. Then, (9) implies that

$$\operatorname{rk} \phi(B) = \operatorname{rk} B \quad \text{for every matrix } B \text{ with} \quad \operatorname{rk} B < r_0. \tag{10}$$

Moreover, for arbitrary matrices $S, T \in \mathrm{GL}_n$ we obtain (see (7) for the definition of $\kappa_{\kappa(S)}$)

$$\phi(SB_0T) = \phi(S)\phi^{\kappa(S)}(B_0)\phi^{\kappa_{\kappa(S)}(B_0)}(T). \tag{11}$$

By (5), $\kappa(S) \geq 1$ for every invertible matrix S, so inequality (9) implies $\operatorname{rk} \phi^{\kappa(S)}(B_0) > \operatorname{rk} B_0$. In addition, by Lemma 3, $\operatorname{rk} \phi(SB_0T) = \operatorname{rk} \phi^{\kappa(S)}(B_0) > \operatorname{rk} B_0$. Since every matrix with rank equal to $r_0 = \operatorname{rk} B_0$ can be written as SB_0T, $S, T \in \mathrm{GL}_n$, we see that

$$\operatorname{rk} \phi(B) > r_0 \quad \text{whenever} \quad \operatorname{rk} B = r_0.$$

By (9) the same inequality holds also if $\operatorname{rk} B > r_0$. However, we already showed that $\operatorname{rk} \phi(B) = \operatorname{rk} B$ if $\operatorname{rk} B < r_0$. This contradicts the surjectivity of ϕ. □

In the rest of the paper we describe surjective skew-morphisms ϕ with respect to the values that the power function κ takes on GL_n. The case when $\kappa(G_0)=0$ for some $G_0\in\mathrm{GL}_n$ was treated in Lemma 1. We continue with the case when $\kappa(G)=1$ for every $G\in\mathrm{GL}_n$, i.e., $\phi|_{\mathrm{GL}_n}$ is multiplicative. Note that in this case ϕ maps GL_n surjectively onto itself, see Lemma 3.

2.3. **Surjective skew-morphisms that are multiplicative on GL_n.** Let us start with the following remark.

Remark 15. *Observe that each surjective group homomorphism $\phi\colon \mathrm{GL}_n\to\mathrm{GL}_n$ is nontrivial on SL_n, since otherwise ϕ would induce a surjective group homomorphism from the abelian quotient group $\mathrm{GL}_n/\mathrm{SL}_n$ onto GL_n, a contradiction. Therefore by Guralnick, Li, and Rodman* [5, Theorem 2.7] *surjective group homomorphisms are of the following two forms*

$$(i)\ \ A\mapsto\rho(\det A)S^{-1}A_\sigma S \quad or \quad (ii)\ \ A\mapsto \rho(\det A)S^{-1}(A_\sigma^{-1})^tS, \tag{12}$$

where ρ is a multiplicative function of the underlying field, A_σ denotes the matrix obtained from A by applying the field automorphism σ entry-wise, and $S\in\mathrm{GL}_n$.
When $n=2$ only the case (i) appears because for 2-by-2 matrices, $(A_\sigma^{-1})^t=KA_\sigma K^{-1}$ for $K=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$.

Lemma 16. *Let $n\geq 2$. Assume a surjective skew-morphism $\phi\colon M_n\to M_n$ satisfies $\kappa|_{\mathrm{GL}_n}=1$. Then $\phi|_{\mathrm{GL}_n}\colon A\mapsto S^{-1}A_\sigma S$ for some $S\in\mathrm{GL}_n$ and some field automorphism σ. Moreover, for every vectors x and f we have*

$$\phi(xf^t)\in\{S^{-1}x_\sigma g^t:\ \ g\in\mathbb{F}^n\}. \tag{13}$$

Proof. By the assumptions and Lemma 3, $\phi|_{\mathrm{GL}_n}$ is a surjective group homomorphism on GL_n. By Remark 15, if $n=2$, then the restriction $\phi|_{\mathrm{GL}_n}$ takes the form (i) in equation (12). Let us show that for $n\geq 3$ the restriction $\phi|_{\mathrm{GL}_n}$ cannot take the form (ii) in (12).

Assume otherwise and consider the n-by-n invertible matrices ($n\geq 3$)

$$A=\mathrm{Id}+J,\qquad B=1\oplus C$$

where $J=\sum_{i=1}^{n-1}E_{i\,(i+1)}$ is an upper-triangular Jordan cell and where C is a companion matrix of a polynomial $\lambda^{n-1}+(-1)^n\lambda+(-1)^{n-1}$. Clearly, 1 is not an eigenvalue of C. Thus, it easily follows that $\det A=\det B=1$, and that $\mathrm{Ker}(A-\mathrm{Id})=\mathrm{Ker}(B-\mathrm{Id})=\mathbb{F}e_1$, where $e_1,e_2,\ldots,e_n$ is the standard basis of column vectors for $\mathbb{F}^n$. Clearly, $E_{11}=AE_{11}=BE_{11}$, and so

$$\begin{aligned}\phi(A)\phi(E_{11})&=\phi(A)\phi^{\kappa(A)}(E_{11})=\phi(AE_{11})\\&=\phi(E_{11})=\phi(BE_{11})=\phi(B)\phi^{\kappa(B)}(E_{11})\\&=\phi(B)\phi(E_{11}).\end{aligned}\tag{14}$$

Yet, one sees that $A_\sigma=A$, $B_\sigma=B$, and $\rho(\det A)=\rho(\det B)=\rho(1)=1$, and so $\phi(A)=\rho(\det A)S^{-1}(A_\sigma^{-1})^tS=S^{-1}(A^{-1})^tS$ and likewise $\phi(B)=S^{-1}(B^{-1})^tS$. It further follows that 1 is an eigenvalue for both $\phi(A)$ and

$\phi(B)$ and that the corresponding eigenvectors for $\phi(A)$ are all spanned by $S^{-1}e_n$, while for $\phi(B)$ all the corresponding eigenvectors are spanned by $S^{-1}e_1$. However by Lemma 13, $\phi(E_{11}) = uv^t$ is of rank-one, so by (14), u is a common fixed point for both $\phi(A), \phi(B)$, a contradiction. Thus,

$$\phi|_{\mathrm{GL}_n} \colon A \mapsto \rho(\det A) S^{-1} A_\sigma S. \tag{15}$$

Let x, f be fixed nonzero vectors and let A be an invertible unipotent matrix (i.e., its spectrum equals $\{1\}$) such that $\mathrm{Ker}(A - \mathrm{Id}) = \mathbb{F}x$. It then follows from $xf^t = A(xf^t)$ that

$$\phi(xf^t) = \phi(Axf^t) = \phi(A)\phi^{\kappa(A)}(xf^t) = \rho(\det A)S^{-1}A_\sigma S \phi(xf^t),$$

and thus $(\rho(\det A)S^{-1}A_\sigma S - \mathrm{Id})\phi(xf^t) = 0$. By Lemma 13, $\phi(xf^t) = yg^t$ is also of rank-one. Note that map $X \mapsto S^{-1}X_\sigma S$ is multiplicative, and from its Jordan structure we see that $S^{-1}A_\sigma S$ is also unipotent, and the geometric multiplicity of its eigenvalue is one. Thus, y is an eigenvector of $\rho(\det A)S^{-1}A_\sigma S$, corresponding to eigenvalue 1, which is possible only if $\rho(\det A) = 1$ and $y \in \mathbb{F}S^{-1}x_\sigma$. This proves (13).

To finish the proof we only need to show that in (15), $\rho(\det X) = 1$ for every invertible X. To this end, pick a scalar $\lambda \in \mathbb{F}\backslash\{0\}$ and consider the invertible matrix $A_\lambda = \mathrm{Id}_{n-1} \oplus \lambda$. Then, $E_{11} = A_\lambda E_{11}$, and as $(e_1)_\sigma = e_1$ we have, by (13), that there is a vector g such that $S^{-1}e_1 g^t = \phi(E_{11}) = \phi(A_\lambda E_{11}) = \phi(A_\lambda)\phi^{\kappa(A_\lambda)}(E_{11}) = \phi(A_\lambda)\phi(e_1 e_1^t) = \rho(\det A_\lambda)S^{-1}(A_\lambda)_\sigma S \cdot S^{-1}e_1 g^t = \rho(\lambda)S^{-1}e_1 g^t$. Comparing both sides yields $\rho(\lambda) = 1$, wherefrom $\rho(\det X) = 1$ for invertible X. □

Theorem 17. *Let $n \geq 2$. Assume a surjective skew-morphism $\phi \colon M_n \to M_n$ satisfies $\kappa|_{\mathrm{GL}_n} = 1$. Then there exist $S \in \mathrm{GL}_n$, a field automorphism σ, and an integer $s \geq 0$ such that*

$$\phi(X) = \begin{cases} S^{-1}X_\sigma S, & X \in \mathrm{GL}_n \\ \gamma S^{-1} X_\sigma G, & X \in M_n \backslash \mathrm{GL}_n, \end{cases}$$

where $\gamma \in \mathbb{F}\backslash\{0\}$, $G = S_{\sigma^{s-1}} \cdots S_\sigma S$ for $s > 0$ and $G = \mathrm{Id}$ for $s = 0$. Moreover, $\sigma^s = \sigma$.

Proof. By Lemma 16, $\phi|_{\mathrm{GL}_n} \colon A \mapsto S^{-1}A_\sigma S$. So it remains to consider ϕ on singular matrices.

Step 1. We start by proving the Theorem for rank-one matrices. Fix an arbitrary rank-one matrix $\mathring{R} = \mathring{x}\mathring{f}^t$ and define $s = \kappa(\mathring{x}\mathring{f}^t) \geq 0$. Let $\{x_1, \ldots, x_{n-1}\}$ be a basis for $\mathring{f}^\perp$ and let $R_i = x_i f^t$ for some nonzero vector $f \in \mathbb{F}^n$. Then $0 = \mathring{R}R_i$, $i = 1, \ldots, n-1$, so by (13) and Lemma 16, $0 = \phi(0) = \phi(\mathring{R}R_i) = \phi(\mathring{R})\phi^s(R_i) = S^{-1}(\mathring{x}_\sigma)\mathring{g}^t \cdot S^{-1}(S_\sigma^{-1})(S_{\sigma^2}^{-1}) \ldots (S_{\sigma^{s-1}}^{-1})((x_i)_{\sigma^s})g_i^t$ for suitable nonzero vectors $\mathring{g}, g_i$ (if $s = 0$ we get $0 = S^{-1}(\mathring{x}_\sigma)\mathring{g}^t \cdot x_i f^t$). Setting

$$G = \begin{cases} (S_{\sigma^{s-1}}) \cdots (S_{\sigma^2})S_\sigma S; & s \geq 1 \\ \mathrm{Id}; & s = 0 \end{cases} \tag{16}$$

we see that $\mathring{g}^t$ is annihilated by $n-1$ linearly independent vectors $G^{-1}((x_i)_{\sigma^s})$, $i=1,\dots,n-1$, and hence

$$\mathring{g}^t = \gamma \cdot (\mathring{f}^t_{\sigma^s})G$$

for some nonzero scalar γ. Therefore,

$$\phi(\mathring{R}) = \gamma S^{-1}(\mathring{x}_\sigma)(\mathring{f}^t_{\sigma^s})G.$$

Choose nonzero vectors x, f and invertible matrices A, B with $x = A\mathring{x}$ and $f^t = \mathring{f}^t B$. Then $R = xf^t = A\mathring{R}B$, so

$$\begin{aligned}\phi(R) &= \phi(A\mathring{R}B) = \phi(A)\phi^{\kappa(A)}(\mathring{R}B) = \phi(A)\phi(\mathring{R}B) = S^{-1}A_\sigma S\cdot\phi(\mathring{R})\phi^s(B)\\ &= S^{-1}A_\sigma S\cdot S^{-1}(\mathring{x}_\sigma)(\gamma\mathring{f}^t_{\sigma^s}G)\cdot G^{-1}B_{\sigma^s}G = \gamma\, S^{-1}(A\mathring{x})_\sigma(\mathring{f}^t B)_{\sigma^s}G\\ &= \gamma\, S^{-1}x_\sigma f^t_{\sigma^s}G.\end{aligned}$$

It remains to show that $\sigma^s = \sigma$. Pick any nonzero scalar λ. Then

$$\begin{aligned}\gamma\sigma(\lambda)\, S^{-1}x_\sigma(f^t_{\sigma^s})G &= \gamma\, S^{-1}(\lambda x)_\sigma(f^t_{\sigma^s})G\\ &= \phi((\lambda x)f^t) = \phi(x(\lambda f)^t) = \gamma\, S^{-1}x_\sigma(\lambda f^t)_{\sigma^s}G\\ &= \gamma\,\sigma^s(\lambda)S^{-1}x_\sigma(f^t_{\sigma^s})G,\end{aligned}$$

and since $\gamma\neq 0$ we obtain $\sigma^s(\lambda)=\sigma(\lambda)$. Therefore

$$\phi(R) = \gamma S^{-1}R_\sigma G, \qquad \operatorname{rk} R = 1. \tag{17}$$

Step 2. Next, we consider the action of ϕ on idempotent matrices. Let P be an idempotent with $\operatorname{rk} P = k \in \{2,\dots,n-1\}$ and let $\{g_1,\dots,g_n\}$ be a basis of $\mathbb{F}^n$ such that $Pg_i = g_i$, $i=1,\dots,k$, and $\{g_{k+1},\dots,g_n\}\subseteq \ker P$. For each matrix $X\in \mathrm{GL}_n$ and nonzero scalar $\beta\in\mathbb{F}$ set

$$\begin{aligned}X^{(P)} &= (X_{\sigma^{\kappa(P)-1}})\cdots(X_{\sigma^2})X_\sigma X,\\ \beta^{(P)} &= (\beta_{\sigma^{\kappa(P)-1}})\cdots(\beta_{\sigma^2})\beta_\sigma\beta\end{aligned}$$

(if $\kappa(P)=0$, then $X^{(P)}=\mathrm{Id}$ and $\beta^{(P)}=1$). Then for an arbitrary nonzero vector $f\in\mathbb{F}^n$ and an arbitrary nonzero scalar $\lambda\in\mathbb{F}$ we have for $i=1,\dots,k$,

$$\begin{aligned}\sigma(\lambda)\gamma S^{-1}(g_if^t)_\sigma G &= \gamma S^{-1}(\lambda g_i f^t)_\sigma G = \phi(\lambda g_i f^t) = \phi(\lambda P g_i f^t)\\ &= \phi(P)\phi^{\kappa(P)}(\lambda g_i f^t)\\ &= \gamma^{(P)}\,\phi(P)(S^{(P)})^{-1}((\lambda g_i f^t)_{\sigma^{\kappa(P)}})G^{(P)}\\ &= \sigma^{\kappa(P)}(\lambda)\gamma^{(P)}\,\phi(P)(S^{(P)})^{-1}((g_i f^t)_{\sigma^{\kappa(P)}})G^{(P)}.\end{aligned} \tag{18}$$

Observe that $\sigma, \sigma^{\kappa(P)}$ are field isomorphisms and so $\sigma(0) = \sigma^{\kappa(P)}(0) = 0$ and $\sigma(1) = \sigma^{\kappa(P)}(1) = 1$. Inserting $\lambda = 1$ into (18), the latter gives $\gamma^{(P)}\,\phi(P)(S^{(P)})^{-1}((g_if^t)_{\sigma^{\kappa(P)}})G^{(P)} = \gamma S^{-1}(g_if^t)_\sigma G \neq 0$. Hence, it follows from (18) that

$$\sigma = \sigma^{\kappa(P)} \tag{19}$$

and therefore since $f \in \mathbb{F}^n$ was arbitrary we further obtain that $G^{(P)} = \alpha_P G$ for some nonzero scalar $\alpha_P \in \mathbb{F}$. So for $i = 1, \dots, k$ we have

$$\gamma S^{-1}(g_i f^t)_\sigma G = \alpha_P \gamma^{(P)}\, \phi(P)(S^{(P)})^{-1}(g_i f^t)_\sigma G. \tag{20}$$

In addition, for $i = k+1, \dots, n$ we have

$$\begin{aligned} 0 = \phi(0) &= \phi(P g_i f^t) = \phi(P)\phi^{\kappa(P)}(g_i f^t) \\ &= \gamma^{(P)}\, \phi(P)(S^{(P)})^{-1}((g_i f^t)_{\sigma^{\kappa(P)}})G^{(P)} \\ &= \alpha_P \gamma^{(P)}\, \phi(P)(S^{(P)})^{-1}(g_i f^t)_\sigma G. \end{aligned} \tag{21}$$

Comparing (20) and (21) we deduce that P_σ and $\frac{\alpha_P \cdot \gamma^{(P)}}{\gamma} \cdot S\phi(P)(S^{(P)})^{-1}$ coincide on the basis $\{(g_1)_\sigma, \dots, (g_n)_\sigma\}$, therefore

$$\phi(P) = \gamma S^{-1} P_\sigma \, \tfrac{S^{(P)}}{\alpha_P \cdot \gamma^{(P)}}.$$

Step 3. Lastly, we consider the action of ϕ on arbitrary singular matrices. First recall that each matrix M of rank k can be written as $M = APB$ for some $A, B \in \mathrm{GL}_n$ and thus

$$\begin{aligned} \phi(M) &= \phi(APB) = \phi(A)\phi(PB) = \phi(A)\phi(P)\phi^{\kappa(P)}(B) \\ &= S^{-1}A_\sigma S \cdot \gamma S^{-1} P_\sigma \tfrac{S^{(P)}}{\alpha_P \cdot \gamma^{(P)}} \cdot (S^{(P)})^{-1} B_{\sigma^{\kappa(P)}} S^{(P)} \\ &= \gamma S^{-1} A_\sigma P_\sigma B_\sigma \tfrac{S^{(P)}}{\alpha_P \cdot \gamma^{(P)}} \\ &= \gamma S^{-1} M_\sigma \tfrac{S^{(P)}}{\alpha_P \cdot \gamma^{(P)}}, \quad \operatorname{rk} M = k \geq 2. \end{aligned} \tag{22}$$

Hence, if $P_k = E_{11} + \cdots + E_{kk}$ is the standard idempotent of rank k and if we set $\gamma^{(k)} = \gamma^{(P_k)}$, $S^{(k)} = S^{(P_k)}$, and $\alpha_k = \alpha_{P_k}$, then $\gamma S^{-1} M_\sigma \frac{S^{(P)}}{\alpha_P \cdot \gamma^{(P)}} = \gamma S^{-1} M_\sigma \frac{S^{(k)}}{\alpha_k \cdot \gamma^{(k)}}$ for every matrix M of rank k. After simplification, this gives

$$M_\sigma \big(\tfrac{S^{(P)}}{\alpha_P \cdot \gamma^{(P)}} - \tfrac{S^{(k)}}{\alpha_k \cdot \gamma^{(k)}} \big) = 0$$

for every matrix M of rank k and hence

$$\tfrac{S^{(P)}}{\alpha_P \cdot \gamma^{(P)}} = \tfrac{S^{(k)}}{\alpha_k \cdot \gamma^{(k)}}. \tag{23}$$

We finish the proof of the assertion that $\phi(M) = \gamma S^{-1} M_\sigma G$, $\operatorname{rk} M = k \leq n-1$, by using induction on $k = 1, \dots, n-1$. Take any idempotent K of rank $(k-1)$, $2 \leq k \leq n-1$. We can find two idempotents P, P' of rank k such that $PP' = K$. Then by the inductive step and using (19), (22) and

(23) we obtain

$$\begin{aligned}\gamma S^{-1}K_\sigma G &= \phi(PP') = \phi(P)\phi^{\kappa(P)}(P') \\ &= \gamma S^{-1}P_\sigma \tfrac{S^{(P)}}{\alpha_P\cdot\gamma^{(P)}}\cdot\gamma^{(P)}(S^{(P)})^{-1}P'_{\sigma^{\kappa(P)}}\left(\tfrac{S^{(P)}}{\alpha_P\cdot\gamma^{(P)}}\right)^{(P)} \\ &= \gamma S^{-1}P_\sigma P'_\sigma \frac{1}{\alpha_P}\left(\tfrac{S^{(P)}}{\alpha_P\cdot\gamma^{(P)}}\right)^{(P)} \\ &= \gamma S^{-1}K_\sigma \frac{1}{\alpha_P}\left(\tfrac{S^{(P)}}{\alpha_P\cdot\gamma^{(P)}}\right)^{(P)} \\ &= \gamma S^{-1}K_\sigma \frac{1}{\alpha_P}\left(\tfrac{S^{(k)}}{\alpha_k\cdot\gamma^{(k)}}\right)^{(P)}.\end{aligned}$$

If r is the order of the automorphism σ, then by (19), $r|(\kappa(P)-1)$ and $r|(\kappa(P_k)-1)$. Hence, if for each $X\in \mathrm{GL}_n$ we define $X^{(k)}$ in the same way as $S^{(k)}$ we easily deduce that

$$\begin{aligned}(X^{(k)})^{(P)} &= ((X_{\sigma^{\kappa(P_k)-1}})\cdots(X_{\sigma^2})X_\sigma X)^{(P)} = (X(X_{\sigma^{r-1}}\cdots X_\sigma X)^{\frac{\kappa(P_k)-1}{r}})^{(P)} \\ &= X(X_{\sigma^{r-1}}\cdots X_\sigma X)^{\frac{\kappa(P_k)\cdot\kappa(P)-1}{r}} = (X^{(P)})^{(k)}\end{aligned}$$

which by (23) and some simplifications gives

$$K_\sigma G = \tfrac{(\alpha_P)^{(k)}}{\alpha_P\cdot(\alpha_k)^{(P)}}\cdot K_\sigma\big(\tfrac{S^{(k)}}{\alpha_k\gamma^{(k)}}\big)^{(k)}.$$

Hence for each idempotent K of rank $k-1$ we obtain that

$$K_\sigma\left(G-\tfrac{(\alpha_P)^{(k)}}{\alpha_P\cdot(\alpha_k)^{(P)}}\cdot(\tfrac{S^{(k)}}{\alpha_k\cdot\gamma^{(k)}})^{(k)}\right)=0. \tag{24}$$

Recall that σ is a bijective homomorphism. So by multiplying the last equation from the left with a rank-one matrix we obtain that

$$f^t(G-\beta_f(\tfrac{S^{(k)}}{\alpha_k\cdot\gamma^{(k)}})^{(k)})=0$$

for every vector $f\in\mathbb{F}^n$ and some $\beta_f\in\mathbb{F}$ which depends on f. It follows that the transpose of G and the transpose of $(\frac{S^{(k)}}{\alpha_k\cdot\gamma^{(k)}})^{(k)}$ are locally linearly dependent matrices and since both are invertible, they are linearly dependent. So also

$$(\tfrac{S^{(k)}}{\alpha_k\cdot\gamma^{(k)}})^{(k)}=\varepsilon_k G, \tag{25}$$

where $\varepsilon_k\in\mathbb{F}$. By (24) and since G is invertible, it follows that $\frac{(\alpha_P)^{(k)}}{\alpha_P\cdot(\alpha_k)^{(P)}}=\frac{1}{\varepsilon_k}$ for every rank k idempotent P.

Let P be any idempotent of rank k. Then simplifying $\phi(P)=\phi(P^2)=\phi(P)\phi^{\kappa(P)}(P)$ using (22), (23), and (25) we obtain

$$P_\sigma\cdot\tfrac{S^{(k)}}{\alpha_k\gamma^{(k)}}=P_\sigma\cdot\tfrac{(\alpha_P)^{(k)}}{\alpha_P\cdot(\alpha_k)^{(P)}}\big(\tfrac{S^{(k)}}{\alpha_k\gamma^{(k)}}\big)^{(k)}=P_\sigma G.$$

It follows that $\frac{S^{(k)}}{\alpha_k\gamma^{(k)}}=G$, hence $\phi(M)=\gamma S^{-1}M_\sigma G$ for every matrix M of rank k. This finishes the inductive step. □

2.4. **On general surjective skew-morphisms.**

Theorem 18. *Assume a surjective skew-morphism $\phi\colon M_n \to M_n$ satisfies $\kappa|_{\mathrm{GL}_n} \geq 1$ and $\kappa(B) \geq 2$ for some $B \in \mathrm{GL}_n$. Then there exists an integer $s \geq 1$ such that ϕ^s is the identity map on M_n and hence ϕ is bijective.*

Proof. First, let us assume that $\kappa(\mathrm{Id}) = 1$. Then $\phi(\mathrm{Id}) = \phi(\mathrm{Id}\cdot\mathrm{Id}) = \phi(\mathrm{Id})\cdot\phi^{\kappa(\mathrm{Id})}(\mathrm{Id}) = \phi(\mathrm{Id})\phi(\mathrm{Id})$ and since ϕ maps GL_n onto GL_n by Lemma 3, it follows that $\phi(\mathrm{Id}) = \mathrm{Id}$. Using the identity $X = (BB^{-1})X = B(B^{-1}X)$, $X \in M_n$, and since $\kappa(B) \geq 2$ it further follows that

$$\begin{aligned}\phi(B)\phi^{\kappa(B)}(B^{-1})\phi(X) &= \phi(BB^{-1})\phi(X) = \phi(\mathrm{Id})\phi(X) = \phi(X)\\ &= \phi(BB^{-1}X) = \phi(B(B^{-1}X)) = \phi(B)\phi^{\kappa(B)}(B^{-1}X)\\ &= \phi(B)\phi^{\kappa(B)}(B^{-1})\phi^{\kappa(B^{-1})+\kappa(\phi(B^{-1}))+\cdots+\kappa(\phi^{\kappa(B)-1}(B^{-1}))}(X).\end{aligned}$$

After canceling out the invertible matrix $\phi(B)\phi^{\kappa(B)}(B^{-1})$ we obtain

$$\phi(X) = \phi^{\kappa(B^{-1})+\kappa(\phi(B^{-1}))+\cdots+\kappa(\phi^{\kappa(B)-1}(B^{-1}))}(X).$$

By our hypothesis, $\kappa(A) \geq 1$ for each $A \in \mathrm{GL}_n$ and $\kappa(B) \geq 2$, so $r = \kappa(B^{-1}) + \kappa(\phi(B^{-1})) + \cdots + \kappa(\phi^{\kappa(B)-1}(B^{-1})) \geq \kappa(B^{-1}) + \kappa(\phi(B^{-1})) \geq 2$. Hence $\phi(X) = \phi^r(X)$ for each $X \in M_n$. Let $Y \in M_n$ be arbitrary. By surjectivity there exists $X \in M_n$ with $\phi(X) = Y$, and hence $Y = \phi(X) = \phi^r(X) = \phi^{r-1}(\phi(X)) = \phi^{r-1}(Y)$. So ϕ^s is the identity for $s = r - 1 \geq 1$.

Second, let us assume that $\kappa(\mathrm{Id}) \geq 2$. Then $\phi(\mathrm{Id}) = \phi(\mathrm{Id}\cdot\mathrm{Id}) = \phi(\mathrm{Id})\phi^{\kappa(\mathrm{Id})}(\mathrm{Id})$, so $\phi^{\kappa(\mathrm{Id})}(\mathrm{Id}) = \mathrm{Id}$. Let $p \geq 1$ be the smallest integer such that $\phi^p(\mathrm{Id}) = \mathrm{Id}$. Consider an arbitrary matrix $A \in \mathrm{GL}_n$. Then $\phi(A) = \phi(A\cdot\mathrm{Id}) = \phi(A)\phi^{\kappa(A)}(\mathrm{Id})$, so $\phi^{\kappa(A)}(\mathrm{Id}) = \mathrm{Id}$ which implies that $\kappa(A)$ is a multiple of p. In particular $\kappa(\phi(\mathrm{Id})) \geq p$. Then for each $X \in M_n$,

$$\begin{aligned}\phi^p(X) &= \phi^p(\mathrm{Id}\cdot X) = \phi^p(\mathrm{Id})\phi^{\kappa(\mathrm{Id})+\kappa(\phi(\mathrm{Id}))+\cdots+\kappa(\phi^{p-1}(\mathrm{Id}))}(X)\\ &= \phi^{\kappa(\mathrm{Id})+\kappa(\phi(\mathrm{Id}))+\cdots+\kappa(\phi^{p-1}(\mathrm{Id}))}(X).\end{aligned}$$

Since $\kappa(\mathrm{Id}) \geq \max\{p, 2\}$ and therefore $s = \kappa(\mathrm{Id}) + \kappa(\phi(\mathrm{Id})) + \cdots + \kappa(\phi^{p-1}(\mathrm{Id})) - p \geq 1$, and since ϕ^p is surjective, we obtain as before that $\phi^s(Y) = Y$ for every $Y \in M_n$. □

Remark 19. *If G is an arbitrary group, $\phi\colon G \to G$ is a restricted skew-morphism, and $\kappa(g_0) \geq 2$ for some $g_0 \in G$, then we can apply the same arguments as in the proof of Theorem 18 to show that ϕ is bijective and has finite order.*

Acknowledgment. The authors are deeply grateful to the anonymous referee for suggesting many improvements of the initial draft. In addition we acknowledge that it was suggested by the referee to use matrices K in Remark 15 and B in Lemma 16 to shorten the initial draft.

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