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Regenerative delay, parametric forcing and machine tool chatter: A review

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Abstract: Two intrinsic component of machine tool chatter modeling is regenerative time delay and parametric forcing. The corresponding governing equations are therefore given in the form of delay-differential equations with time-periodic coefficients. In this paper, a brief review is given on the mechanism of these two effects, and recent numerical techniques from the literature are categorized and discussed.

Keywords: regenerative delay, parametric forcing, stability, numerical techniques.

1. INTRODUCTION

One of the most important fields of engineering where large time delays appear in the model equations is machine tool vibration, where the delay time could be several times larger than the characteristic time periods in the system, while the damping effects in the machine tool system are very small. After the pioneering work of Tobias (1965) and Tlusty et al. (1962), the so-called regenerative effect became the most commonly accepted explanation for machine tool chatter. This effect is related to the cutting-force variation due to the wavy workpiece surface cut one revolution ago. The phenomenon can be described by involving time delay in the model equations. Stability properties of the machining process are depicted by so-called stability lobe diagrams, which plot the maximum stable axial depths of cut versus the spindle speed. These diagrams provide a guide to the machinist to select optimal technological parameters in order to achieve maximum material removal rate without chatter. Although there exist many sophisticated methods to optimize manufacturing processes, machine tool chatter is still an existing problem in manufacturing centers (Altintas and Weck, 2004; Schmitz and Smith, 2009; Quintana and Ciurana, 2011; Altintas, 2012).

In case of turning operations, regenerative chatter can be described by time-invariant delay-differential equations (DDEs). Stability of these systems can be analyzed by the classical D-subdivision method (Stepan, 1989). In the case of milling, surface regeneration is coupled with parametric excitation of the cutting teeth, resulting in a DDE with time-periodic coefficients. Stability analysis of these systems requires the application of the Floquet theory of DDEs. In the recent years, several numerical techniques have been developed in order to determine stability lobe diagrams for milling operations, such as the semi-discretization method (Insperger and Stepan, 2002a, 2011), the temporal finite element method (Bayly et al., 2003); the multi-frequency solution (Altintas and Budak, 1995; Budak and Altintas, 1998; Bachrathy and Stepan, 2013; Otto et al., 2014), just to mention a few.

This paper aims to give a brief overview on the main issues of machine tool chatter. First the regenerative delay is explained in details for an orthogonal turning operation. Then parametric forcing is described for a one-degree-of-freedom model of milling operations. Finally, a series of numerical techniques, which were developed for the stability prediction of machining operations in the last 15-20 years, are categorized and discussed

2. REGENERATIVE DELAY

Time delay in machine tool chatter shows up as result of the so-called surface regeneration. In this section, the phenomenon is explained briefly. Figure 1 shows the chip removal process in an orthogonal turning operation for an ideally rigid tool and for a compliant tool. In the latter case, the tool experiences bending vibrations in directions x and y and leaves a wavy surface behind. The system can be modeled as a two-degrees-of-freedom oscillator excited by the cutting force, as shown in Fig. 2. If there is no dynamic coupling between x and y directions, then the governing equation can be given as

$$m\ddot{x}(t) + c_x\dot{x}(t) + k_x x(t) = F_x(t)\ , \tag{1}$$
$$m\ddot{y}(t) + c_y\dot{y}(t) + k_y y(t) = F_y(t)\ , \tag{2}$$

where m, c_x, c_y, k_x, and k_y are the modal mass and the damping and stiffness parameters in the x and y directions, respectively. The cutting force is given in the form

$$F_x(t) = K_x\, w\, h^q(t)\ , \tag{3}$$
$$F_y(t) = K_y\, w\, h^q(t)\ , \tag{4}$$

where K_x and K_y are the cutting-force coefficients in the tangential (x) and the normal (y) directions, w is the depth of cut (also known as the width of cut or the chip width in the case of orthogonal cutting), $h(t)$ is the instantaneous chip thickness, and q is the cutting-force exponent. Note that other formulas for the cutting force are also used in the literature; see, e.g., Kienzle (1957); Shi and Tobias (1984); Dombovari et al. (2008).

If the tool was rigid, then the chip thickness would be constant $h(t) \equiv h_0$, which is equal to the feed per

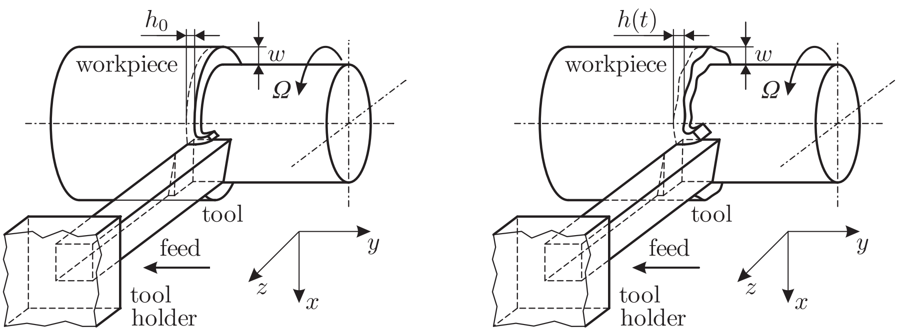

Fig. 1. Chip removal in orthogonal turning processes in the case of an ideally rigid tool and real compliant tool.

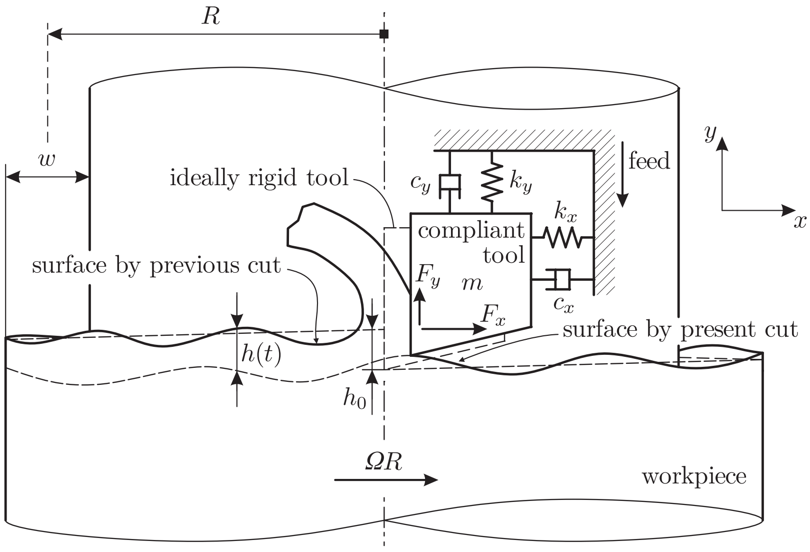

Fig. 2. Surface regeneration in an orthogonal turning process: the instantaneous chip thickness $h(t)$ is varying due to the vibrations of the tool.

revolution (in case of orthogonal cutting). However, in reality, the tool experiences vibrations, which are recorded on the workpiece, and after one revolution, the tool cuts this wavy surface. The chip thickness $h(t)$ is determined by the feed motion, by the current and by an earlier position of the tool. If the displacement $x(t)$ is negligible compared to the radius R of the workpiece, then the time delay τ between the present and the previous cuts is

$$\tau = \frac{60}{\Omega}, \tag{5}$$

where Ω is the spindle speed given in [rpm].

The chip thickness can be given as the linear combination of the feed and the present and the delayed positions of the tool in the form

$$h(t) = h_0 + y(t-\tau) - y(t)\ . \tag{6}$$

Thus, the governing equations can be written as

$$m\ddot{x}(t) + c_x\dot{x}(t) + k_x x(t) = K_x w\left(h_0 + y(t-\tau) - y(t)\right)^q , \tag{7}$$

$$m\ddot{y}(t) + c_y\dot{y}(t) + k_y y(t) = K_y w\left(h_0 + y(t-\tau) - y(t)\right)^q . \tag{8}$$

Let $x_{\rm st}$ and $y_{\rm st}$ denote the constant solution that satisfies (7)–(8). The general solution can be written as $x(t) = x_{\rm st} + \xi(t)$ and $y(t) = y_{\rm st} + \eta(t)$ with $\xi(t)$ and $\eta(t)$ being perturbations around $x_{\rm st}$ and $y_{\rm st}$, respectively. Substitution into (7)–(8), expansion into power series with respect to $\xi(t)$ and $\eta(t)$, and elimination of higher-order terms give the variational system in the form

$$m\ddot{\xi}(t) + c_x\dot{\xi}(t) + k_x\xi(t) = K_x wqh_0^{q-1}\left(\eta(t-\tau) - \eta(t)\right), \tag{9}$$

$$m\ddot{\eta}(t) + c_y\dot{\eta}(t) + k_y\eta(t) = K_y wqh_0^{q-1}\left(\eta(t-\tau) - \eta(t)\right). \tag{10}$$

Note that (9) is an ODE with state variable ξ forced by η, while (10) is a linear time-invariant DDE with state variable η. Since the homogeneous part of (9) is a simple damped oscillator, the stability of the system is determined by (10) only.

By parameter transformation, (10) can be written in the form

$$\ddot{\eta}(t) + 2\zeta\omega_{\rm n}\dot{\eta}(t) + \omega_{\rm n}^2\eta(t) = H\left(\eta(t-\tau) - \eta(t)\right) , \tag{11}$$

where $\omega_{\rm n} = \sqrt{k_y/m}$ is the natural angular frequency, $\zeta = c_y/(2m\omega_{\rm n})$ is the damping ratio of the tool in the y direction, and $H = K_y wqh_0^{q-1}/m$ is the specific cutting-force coefficient. Note that H is linearly proportional to the depth of cut w, which is an important technological parameter for the machinist. Equation (11) is the simplest mathematical model that describes regenerative machine tool chatter.

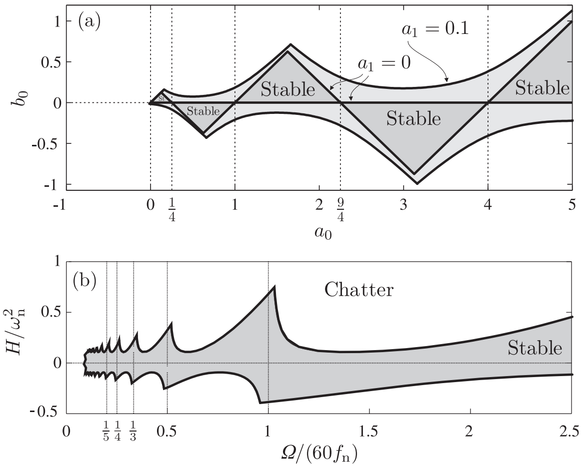

Fig. 3. (a) HsuBhatt stability chart for $\tau = 2\pi$. (b) Stability diagram for (10).

Another form of (11) is the so-called delayed oscillator

$$\ddot{\eta}(t) + a_1\dot{\eta}(t) + a_0\eta(t) = b_0\eta(t-\tau), \tag{12}$$

where the new parameters are $a_1 = 2\zeta\omega_n$, $a_0 = \omega_n^2 + H$ and $b_0 = H$. The stability chart of the delayed oscillator was first published in 1966 by Hsu and Bhatt (1966). Since then, this equation has become a basic example for delayed Newtonian problems; see, for instance, (Stepan, 1989; Butcher et al., 2004; Insperger and Stepan, 2011). The stability diagram of (12) is shown in Fig. 3a. For the undamped case ($a_1 = 0$), stable regions are bounded by straight lines crossing the $a_0 = 0$ line at $1/4, 1, 9/4$ etc. Apart from the lines $b_0 = 0$ and $b_0 = a_0$, the stability boundaries represent a complex conjugate pair of characteristic exponents crossing the imaginary axis, which corresponds to a Hopf bifurcation for the underlying nonlinear system.

The stability diagram for (11) in the plane of dimensionless spindle speed $\Omega/(60 f_n)$ and dimensionless specific cutting-force coefficient H/ω_n^2 is shown in Fig. 3b. Here, $f_n = \omega_n/2\pi$ is the natural frequency of the tool in [Hz]. This diagram can be obtained by a parameter transformation of the Hsu-Bhatt diagram. From practical point of view, only the region associated with the positive depth of cut ($H > 0$) makes sense.

3. PARAMETRIC EXCITATION

In case of milling operations, surface regeneration is coupled with parametric excitation caused by the rotating cutting tool. The mechanical model of a thin-wall milling operation is shown in Fig. 4. The workpiece is assumed to be flexible in direction x (perpendicular to the feed) with modal mass m, damping coefficient c, and spring stiffness k, while the tool is assumed to be rigid. In this model, the tool has N equally distributed cutting teeth with zero helix angle. The equation of motion reads

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = -F_x(t)\ , \tag{13}$$

where $F_x(t)$ is the x component of the cutting force vector acting from the tool on the workpiece. Let the teeth of the tool be indexed by $j = 1, 2, \ldots, N$.

The tangential and radial components of the cutting force acting on tooth j reads

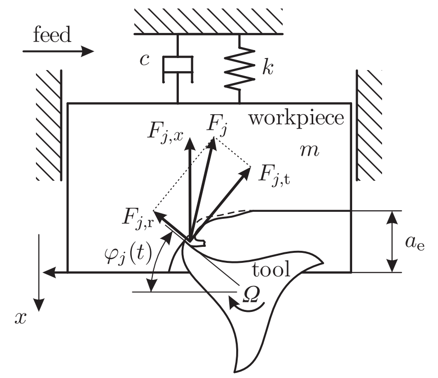

Fig. 4. Mechanical model of thin-wall milling operation.

$$F_{j,\mathrm{t}}(t) = g_j(t) K_\mathrm{t} a_\mathrm{p} h_j^q(t)\ , \tag{14}$$

$$F_{j,\mathrm{r}}(t) = g_j(t) K_\mathrm{r} a_\mathrm{p} h_j^q(t)\ , \tag{15}$$

where K_t and K_r are the tangential and radial cutting-force parameters, respectively, a_p is the axial depth of cut, $h_j(t)$ is the instantaneous chip thickness cut by tooth j, and q is the cutting-force exponent. Function $g_j(t)$ is a screen function; it is equal to 1 if tooth j is in the cut, and 0 if it is not. If φ_en and φ_ex denote the angular locations where the cutting teeth enter and exit the cut, then the screen function reads

$$g_j(t) = \begin{cases} 1 & \text{if } \varphi_\mathrm{en} < (\varphi_j(t) \bmod 2\pi) < \varphi_\mathrm{ex}\ , \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

where

$$\varphi_j(t) = \frac{2\pi\,\Omega}{60} t + j\,\frac{2\pi}{N} \tag{17}$$

is the angular position of tooth j and mod is the modulo function. The instantaneous chip thickness $h_j(t)$ is determined by the actual feed per tooth and the angular position of the cutting teeth. A circular approximation of the tooth path gives

$$h_j(t) = (f_\mathrm{z} + x(t) - x(t-\tau)) \sin\varphi_j(t), \tag{18}$$

where $\tau = 60/(N\Omega)$ is the tooth-passing period (the regenerative delay) and f_z is the feed per tooth. The x component of the cutting force acting on tooth j is obtained as the projection of $F_{j,\mathrm{t}}$ and $F_{j,\mathrm{r}}$ in the x direction, i.e.,

$$F_{j,x}(t) = F_{j,\mathrm{t}}(t)\cos\varphi_j(t) + F_{j,\mathrm{r}}(t)\sin\varphi_j(t)\ . \tag{19}$$

The x component of the resultant cutting force acting on the tool reads

$$F_x(t) = Q(t)\left(f_\mathrm{z} + x(t) - x(t-\tau)\right)^q\ , \tag{20}$$

where

$$Q(t) = \sum_{j=1}^{N} a_\mathrm{p} g_j(t) \sin^q\varphi_j(t)\left(K_\mathrm{t}\cos\varphi_j(t) + K_\mathrm{r}\sin\varphi_j(t)\right)\ . \tag{21}$$

Thus, the equation of motion is the following nonlinear DDE:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = -Q(t)\left(f_\mathrm{z} + x(t) - x(t-\tau)\right)^q\ . \tag{22}$$

The general solution of (22) can be written as

$$x(t) = x_\mathrm{p}(t) + \xi(t)\ , \tag{23}$$

where $x_\mathrm{p}(t) = x_\mathrm{p}(t+\tau)$ is a periodic function (the particular solution of (22)) and $\xi(t)$ is the perturbation around $x_\mathrm{p}(t)$. Substitution of (23) into (22), expansion into

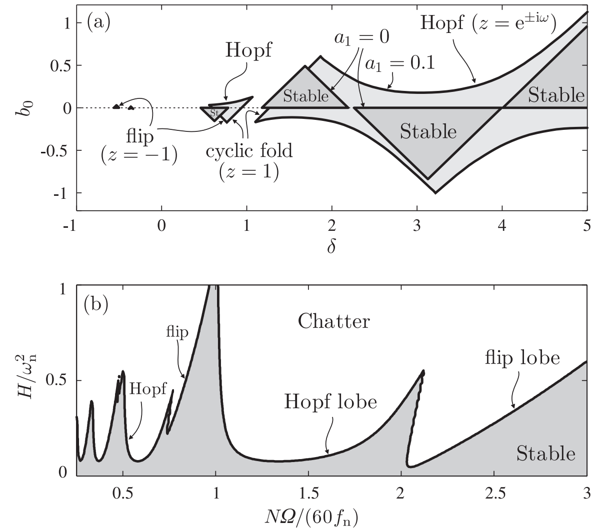

Fig. 5. (a) Stability chart for the delayed Mathieu equation (25) with $\varepsilon = 1$. (b) Stability diagram for a one-degree-of-freedom milling process.

power series with respect to $\xi(t)$ and elimination of higher-order terms give the variational system

$$m\ddot{\xi}(t) + c\dot{\xi}(t) + k\xi(t) = -qf_z^{q-1}Q(t)\left(\xi(t) - \xi(t-\tau)\right) . \tag{24}$$

This is now a linear DDE with time-periodic coefficients.

A paradigm for delayed periodic systems is the delayed Mathieu equation

$$\ddot{x}(t) + a_1\dot{x}(t) + (\delta + \varepsilon \cos t)\, x(t) = b_0 x(t - 2\pi), \tag{25}$$

which is a generalization of the delayed oscillator (12) by introducing a time-periodic stiffness $a_0 = \delta + \varepsilon \cos t$. The stability diagram for the delayed Mathieu equation was presented by Insperger and Stepan (2002b) and Insperger and Stepan (2003) for the undamped ($a_1 = 0$) and the damped ($a_1 \neq 0$) cases, respectively. A sample stability diagram is shown in Fig. 5a. If $a_1 = 0$ then the stability boundaries are straight lines of slopes -1, 0 and 1, where the characteristic multiplier are $z = 1$, $z = e^{\pm i\omega}$ ($\omega \in \mathbb{R}$) and $z = -1$ representing cyclic fold, secondary Hopf and flip (period doubling) bifurcations of the underlying nonlinear problem, respectively. If $a_1 \neq 0$ then nonlinear stability boundaries associated with secondary Hopf bifurcation also show up.

A sample stability chart for the milling problem described by (24) is shown in Fig. 5b. This diagram can be considered as a special transformation of the stability diagram of the delayed Mathieu equation. In case of milling, stability boundaries are typically associated with secondary Hopf and flip bifurcations. The arising vibrations are often called quasi-periodic chatter and period doubling chatter, respectively. Note that in this one-degree-of-freedom model of the milling operation, no cyclic fold bifurcation is possible (Insperger and Stepan, 2011).

4. NUMERICAL METHODS

In this section, we list several numerical techniques used for the stability prediction of machining processes. All these methods are based on a finite dimensional approximation of the infinite dimensional system. Different techniques are categorized with respect to

- the form of the equation under study, which is either some form of a retarded functional differential equation (RFDE) or an operator differential equation (OpDE);
- the discretized operator, which is either the monodromy operator or the infinitesimal generator; and
- the method of discretization.

Semi-discretization (Insperger and Stepan, 2002a, 2011)
Equation under study: RFDE, strong form
Discretized operator: monodromy operator
Method of discretization: direct discretization of the functional

Full discretization (Ding et al., 2010)
Equation under study: RFDE, integral equation
Discretized operator: monodromy operator
Method of discretization: direct discretization of the functional and the integral term

Complete discretization (Li et al., 2013)
Equation under study: RFDE, strong form
Discretized operator: monodromy operator
Method of discretization: direct discretization of both the functional and the differential operator

Continuous-time approximation (Sun, 2009)
Equation under study: OpDE, strong form
Discretized operator: infinitesimal generator
Method of discretization: direct discretization of both the functional and the differential operator

Pseudospectral collocation (Breda et al., 2005, 2015), **Chebyshev spectral continuous-time approximation** (Butcher and Bobrenkov, 2011)
Equation under study: OpDE, weak form
Discretized operator: infinitesimal generator
Method of discretization: via trial solution using Lagrange polynomial interpolation of the solution on Lobatto-Chebyshev node set and collocation

Spectral least-square (Vyasarayani et al., 2014)
Equation under study: OpDE, weak form
Discretized operator: infinitesimal generator
Method of discretization: via trial solution using Legendre base functions for expansion and least-square method for the test functions

Spectral Legendre tau (Vyasarayani et al., 2014)
Equation under study: OpDE, weak form
Discretized operator: infinitesimal generator
Method of discretization: via trial solution using Legendre base functions for expansion and Bubnov-Galerkin-type test functions

Method proposed by Wahi and Chatterjee (2005)
Equation under study: OpDE, weak form
Discretized operator: infinitesimal generator
Method of discretization: via trial solution using trigonometric base functions for expansion and Bubnov-Galerkin-type test functions

Temporal finite element analysis (Bayly et al., 2003; Mann and Patel, 2010)
Equation under study: RFDE, weak form
Discretized operator: monodromy operator
Method of discretization: via trial solution using Her-

mite interpolation on a Lobatto-equidistant node set and Petrov-Galerkin-type test functions

Spectral element method (Khasawneh and Mann, 2011, 2013)
Equation under study: RFDE, weak form
Discretized operator: monodromy operator
Method of discretization: via trial solution using Lagrange polynomial interpolation on Lobatto-Legendre node set and Petrov-Galerkin-type test functions.

Pseudospectral collocation (Breda et al., 2014, 2015)
Equation under study: RFDE, weak form
Discretized operator: monodromy operator
Method of discretization: via trial solution using Lagrange polynomial interpolation on Lobatto-Chebyshev node set and collocation

Multi-interval Chebyshev collocation (Khasawneh et al., 2011; Totis, 2009; Totis et al., 2014)
Equation under study: RFDE, weak form
Discretized operator: monodromy operator
Method of discretization: via trial solution using Chebyshev base functions for expansion and collocation

Method proposed by Butcher et al. (2004)
Equation under study: RFDE, integral equation
Discretized operator: monodromy operator
Method of discretization: via trial solution using Chebyshev base functions for expansion

Spectral method (Ding et al., 2011)
Equation under study: RFDE, integral equation
Discretized operator: monodromy operator
Method of discretization: via trial solution using Lagrange polynomial interpolation on Lobatto-Chebyshev node set

Multi-frequency solution (Altintas and Budak, 1995; Merdol and Altintas, 2004; Bachrathy and Stepan, 2013)
Equation under study: RFDE, strong form
Discretized operator: monodromy operator
Method of discretization: via trial solution using Floquet exponential form for the expansion

Method using characteristic matrices (Szalai et al., 2006; Sieber and Szalai, 2011)
Equation under study: RFDE, strong form
Discretized operator: monodromy operator
Method of discretization: via trial solution using Floquet exponential form for the expansion

5. CONCLUSION

In this brief review, simple models were presented to show how time-delay and periodic coefficients appear in the governing equations of machine tool chatter. The models presented here were simplified one-degree-of-freedom models, however, real-life machining operations are described by more sophisticated models. For instance, there exist more complex models for the chip thickness calculation, such as the trochoidal tooth path model, which results in time-dependent delays (Faassen et al., 2007) and models including the vibrations of the tool–workpiece system that results in state-dependent delays in the model equations (Insperger et al., 2007; Bachrathy et al., 2011). In addition, general tool geometry can also be modeled such as ball end milling (Ozturk and Budak, 2010) or tools with varying pitch and with varying helix angles, which result in multiple and distributed delays (Sellmeier and Denkena, 2011; Stepan et al., 2014). An intriguing combination of point delay and time varying delay shows up in case of digital control of machining processes (Lehotzky and Insperger, 2012). There are several fundamental concepts to suppress machine tool chatter. Spindle speed selection (van Dijk et al., 2010), continuous spindle speed variation (Otto and Radons, 2013) and active chatter control techniques (Pakdemirli and Ulsoy, 1997; van Dijk et al., 2012) can be mentioned as examples.

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