ON (α, β, a, b)-CONVEX FUNCTIONS

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ABSTRACT. In this paper we investigate the (α, β, a, b)-convex functions which is a common generalization of the usual convexity, the s-convexity in first and second sense, the h-convexity, the Godunova-Levin functions and the P-functions. This notion of convexity was introduced by Maksa and Páles in [16] in the following way: an (α, β, a, b)-convex function is defined as a function $f : D \to \mathbb{R}$ (where D is an open, (α, β)-convex, nonempty subset of a real or complex topological vector space) which satisfies the inequality

$$f(\alpha(t)x + \beta(t)y) \leq a(t)f(x) + b(t)f(y) \qquad (x; y \in D; t \in [0, 1]).$$

The main goal of the paper is to prove some regularity and Bernstein-Doetsch type results for (α, β, a, b)-convex functions.

1. Introduction

Maksa and Páles in [16] dealt with the following problem:

Let X be a real or complex topological vector space, $D \subset X$ be a nonempty open set, T be a nonempty set, and $\alpha, \beta, a, b : T \to \mathbb{R}$ be given functions. The problem is to find all the solutions $f : D \to \mathbb{R}$ of the functional equation

$$f(\alpha(t)x + \beta(t)y) = a(t)f(x) + b(t)f(y) \qquad (x; y \in D; t \in T) \tag{1}$$

provided that D is $(\alpha; \beta)$-convex, that is, $\alpha(t)x + \beta(t)y \in D$ whenever $x; y \in D$ and $t \in T$. To avoid the trivialities and the unimportant cases, we suppose that there exists an element $t_0 \in T$ such that

$$\alpha(t_0)\beta(t_0)a(t_0)b(t_0) \neq 0.$$

The solutions of (1) as $(\alpha; \beta; a; b)$-affine functions and the solutions f of the corresponding inequality

$$f(\alpha(t)x + \beta(t)y) \leq a(t)f(x) + b(t)f(y) \qquad (x; y \in D; t \in T) \tag{2}$$

will be called $(\alpha; \beta; a; b)$-convex functions.

In our paper we investigate the $(\alpha; \beta; a; b)$-convex functions. This notion of convexity is a common generalization of the usual convexity, the s-convexity in first and second sense, the h-convexity, the Godunova-Levin functions and the P-functions.

In the special cases when $T = \{1/2\}$, $T = \{t_0\}$ or $T = \mathbb{Q} \cap [0, 1]$, the corresponding convex functions are said to be *Jensen-*$(\alpha; \beta; a; b)$*-convex,* $t_0 - (\alpha; \beta; a; b)$*-convex* and *rationally-*$(\alpha; \beta; a; b)$*-convex.*

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Let $h : [0,1] \to \mathbb{R}$ be a given function. In the case, when $\alpha(t) = t, \beta(t) = 1 - t, a(t) = h(t), b(t) = h(1-t)$ we get the so called h-convex functions, which was introduced by Varošanec [29] and was generalized by Házy [11]. We say that $f : D \to \mathbb{R}$ is an h-convex function if, for all $x, y \in D$ and $t \in [0,1]$, we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y). \tag{3}$$

The Godunova-Levin functions was investigated by Godunova-Levin [7]. We say that $f : I \to \mathbb{R}$ (where I is a real interval) is a Godunova-Levin function, if f is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ we have

$$f\left(tx + (1-t)y\right) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

Some properties of this type of functions are given in Dragomir, Pečarič and Persson [6] Mitrinovič and Pečarič [17], Mitrinovič, Pečarič and Fink [18]. Among others, it is proved that nonnegative monotone and nonnegative convex functions belong to this class of functions. The Godunova-Levin functions are $(\alpha;\beta;a;b)$-convex functions, with $\alpha(t) = t, \beta(t) = 1-t, a(t) = 1/t, b(t) = 1/(1-t)$.

The concept of s-convexity in the first sense was introduced by Orlicz [21]. A real valued function $f : D \to \mathbb{R}$ is called *Orlicz s-convex* or *s-convex in the first sense*, if

$$f\left(t^s x + (1-t)^s y\right) \leq tf(x) + (1-t)f(y)$$

for every $x, y \in D,\ t \in]0,1]$, where $s \in [1,\infty[$ is fixed number. The Orlicz s-convex functions are $(\alpha;\beta;a;b)$-convex functions, with $\alpha(t) = t^s, \beta(t) = (1-t)^s, a(t) = t, b(t) = 1-t$.

The concept of s-convexity in the second sense was introduced by Breckner [4]. A real valued function $f : D \to \mathbb{R}$ is called *Breckner s-convex* or *s-convex in the second sense*, if

$$f\left(tx + (1-t)y\right) \leq t^s f(x) + (1-t)^s f(y)$$

for every $x, y \in D$ and $t \in [0,1]$, where $s \in]0,1]$ is a fixed number. The Breckner s-convex functions are $(\alpha;\beta;a;b)$-convex functions, with $\alpha(t) = t, \beta(t) = 1-t, a(t) = t^s, b(t) = (1-t)^s$.

The case $s = 1$ means the usual convexity of f.

In Breckner [4] and Breckner and Orban [5] Berstein-Doetsch type results were proved on rationally s-convex functions, moreover, for the s-Hölder property of s-convex functions. Pycia [26] gives a new proof of the latter statement, when f is defined on a nonempty, convex subset of a finite dimensional vector space. In the paper Hudzik and Maligranda [14] the authors collect some properties of s-convex functions defined on the nonnegative reals. In the paper Burai, Házy and Juhász [2] there are some Berstein-Doetsch type result on (H,s)-convex functions.

The P-functions was investigated in Dragomir, Pečarič and Persson [6]. A real valued function $f : D \to \mathbb{R}$ (where D is a convex, open, nonempty subset of a real (complex) linear space X) is called P-function, if for every $x, y \in D$ and $t \in [0,1]$ we have

$$f\left(tx + (1-t)y\right) \leq f(x) + f(y).$$

Some results about the P-functions there are in Pearce and Rubinov [25], Tseng, Yang and Dragomir [28]. The P-functions are $(\alpha;\beta;a;b)$ -convex functions, with $\alpha(t) = t, \beta(t) = 1 - t, a(t) = 1, b(t) = 1$.

In Bernstein and Doetsch [1] proved that if a function $f : D \to \mathbb{R}$ (where D is a convex, open, nonempty subset of a real (complex) linear space X) is locally bounded from above at a point of D, then the Jensen-convexity of the function yields its local boundedness and continuity as well, which implies the convexity of the function f (see Kuczma [15] for further references). This result has been generalized by several authors. The first such type results are due to Nikodem and Ng [20] for the approximately Jensen-convex functions (the so-called ε-Jensen-convexity), which was extended by Páles (Páles [22] and [23]) to approximately t-convex functions. Further generalizations can be found in papers Mrowiec [19], Házy [9] and [10], Házy and Páles [12] and [13]. In the paper Gilányi, Nikodem and Páles [8] there are some Bernstein-Doetsch type results for quasiconvex functions.

2. Main results

In this section we assume that $(X, \|\cdot\|)$ is a real (complex) normed space. We recall that a function $f : D \to \mathbb{R}$ is called locally bounded from above on D if, for each point of $p \in D$, there exist $\varrho > 0$ and a neighborhood $U(p, \varrho) := \{x \in X \ : \ \|x - p\| < \varrho\}$ such that f is bounded from above on $U(p, \varrho)$. We assume that $a, b : [0, 1] \to \mathbb{R}$ are nonnegative.

Proposition 1. *Let $t_0 \in [0, 1]$ be fixed such that $\alpha(t_0) + \beta(t_0) = 1$ and $f : D \to \mathbb{R}$ be an $(\alpha; \beta; a; b)$-convex function. Then*

(i) if $a(t_0) + b(t_0) > 1$ then f is nonnegative.
(ii) if $a(t_0) + b(t_0) < 1$ then f is nonpositive.

Proof. Let x be an arbitrary element of D. Using $(\alpha; \beta; a; b)$-convexity of f

$$f(x) = f(\alpha(t_0)x + \beta(t_0)x) \leq a(t_0)f(x) + b(t_0)f(x) = (a(t_0) + b(t_0))f(x),$$

which implies

$$0 \leq (a(t_0) + b(t_0) - 1)f(x).$$

If $a(t_0) + b(t_0) - 1 > 0$, then we have $f(x) \geq 0$ and if $a(t_0) + b(t_0) - 1 < 0$, then we have $f(x) \leq 0$. □

Theorem 1. *Let $]0, 1[\subset T$, $\alpha, \beta, a, b : T \to \mathbb{R}$ be given nonnegative functions and let $t_0 \in]0, 1[$ be fixed such that $\alpha(t_0)\beta(t_0)a(t_0)b(t_0) \neq 0$ and $\alpha(t_0) + \beta(t_0) = 1$. Furthermore let $D \subset X$ be open, nonempty, $(\alpha; \beta)$-convex set, let $f : D \to \mathbb{R}$ be a $t_0 - (\alpha; \beta; a; b)$-convex function, which is locally bounded from above at a p point of D. Then then f is locally bounded at every point of D.*

Proof. Since $\alpha(t_0)\beta(t_0) \neq 0$ therefore we get $\alpha(t_0), \beta(t_0) > 0$. We prove that f is locally bounded from above on D.

First we prove that f is locally bounded from above on D. Define the sequence of sets D_n by

$$D_0 := \{p\}, \qquad D_{n+1} := \alpha(t_0)D_n + \beta(t_0)D.$$

Using induction on n, we prove that f is locally upper bounded at each point of D_n. By assumption, f is locally bounded from above at $p \in D_0$. Assume that f is locally upper bounded at each point of D_n. For $x \in D_{n+1}$, there exist $x_0 \in D_n$ and $y_0 \in D$ such that $x = \alpha(t_0)x_0 + b(t_0)y_0$. By the inductive assumption, there exist $r > 0$ and a constant $M_0 \geq 0$ such that $f(x') \leq M_0$ for $\|x_0 - x'\| < r$. Then, by the $t_0 - (\alpha; \beta; a; b)$-convexity of f, for $x' \in U_0 := U(x_0, r)$ we have

$$f(\alpha(t_0)x' + \beta(t_0)y_0) \leq a(t_0)f(x') + b(t_0)f(y_0) \leq a(t_0)M_0 + b(t_0)f(y_0) =: M.$$

Therefore, for

$$y \in U := \alpha(t_0)U_0 + \beta(t_0)y_0 = U(\alpha(t_0)x_0 + \beta(t_0)y_0, t_0 r) = U(x, t_0 r),$$

we get that $f(y) \le M$. Thus f is locally bounded from above on D_{n+1}.

On the other hand, we show that

$$D = \bigcup_{n=1}^{\infty} D_n.$$

From the definition of D_n, it follows by induction that $D_n = (\alpha(t_0))^n p + (1 - (\alpha(t_0))^n)D$. For fixed $x \in D$, define the sequence x_n by

$$x_n := \frac{x - (\alpha(t_0))^n p}{1 - (\alpha(t_0))^n}.$$

Then $x_n \to x$ if $n \to \infty$. As D is open, $x_n \in D$ for some n. Therefore

$$x = \alpha(t_0)^n p + (1 - (\alpha(t_0))^n)x_n \in (\alpha(t_0))^n p + (1 - (\alpha(t_0))^n)D = D_n.$$

Thus f is locally bounded from above on D.

Now, we prove that f is locally bounded from below. Let $q \in D$ be arbitrary. Since f is locally bounded from above at the point q, there exist $\varrho > 0$ and $M > 0$ such that

$$\sup_{U(q,\varrho)} f \le M.$$

Let $x \in U(q, \beta(t_0)\varrho)$ and $y := \dfrac{q - \alpha(t_0)x}{\beta(t_0)}$. Then y is in $U(q, \varrho)$. By $t_0 - (\alpha; \beta; a; b)$-convexity,

$$f(q) \le a(t_0)f(x) + b(t_0)f(y),$$

which implies

$$f(x) \ge \frac{f(q) - b(t_0)f(y)}{a(t_0)} \ge \frac{f(q) - b(t_0)M}{a(t_0)} =: M'.$$

Therefore f is locally bounded from below at any point of D. □

Corollary 1. *Let $f : D \to \mathbb{R}$ be a Jensen-convex or t_0-convex function. If f is locally bounded from above at a point of D, then f is locally bounded at every point of D.*

Corollary 2. *Let $f : D \to \mathbb{R}$ be a Breckner (t_0, s)-convex function. If f is locally bounded from above at a point of D, then f is locally bounded at every point of D.*

Corollary 3. *Let $f : D \to \mathbb{R}$ be a (t_0, h)-convex function such that $h(t_0)$ and $h(1 - t_0)$ are not zero simultaneously. If f is locally bounded from above at a point of D, then f is locally bounded at every point of D.*

Remark 1. *It is a well-known fact that if a Jensen-convex function f is locally bounded above at a point of its domain (see* [1], [15]*), then it is continuous on its domain. This is not true for (Jensen,h)-convex functions, which implies is not true for Jensen $-(\alpha; \beta; a; b)$. Indeed, in the case $h(\lambda) = \lambda^s$ (where $0 < s < 1$ is a fixed number), in* [2] *we give an example which is (Jensen,h)-convex, bounded and nowhere continuous.*

Next theorem gives a sufficient condition when local boundedness implies continuity.

Theorem 2. *Let α, β, a, b be given nonnegative, continuous functions satisfying the limit conditions*

$$\lim_{t\to 0} a(t) = 0 \qquad \textit{and} \qquad \lim_{t\to 0} b(t) = 1.$$

and $\alpha(t) + \beta(t) = 1$.

Let the sequence $\{t_n\}_{n\in\mathbb{N}}$ be such that $t_n \in]0,1]$ and t_n tends to 0 (when $n \to \infty$) and assume that $a(t_n)$ and $b(t_n)$ are not simultaneously zero. Let $T = \{t_n\}_{n\in\mathbb{N}}$.

If $f : D \to \mathbb{R}$ is $T - (\alpha;\beta;a;b)$-convex function and f is locally bounded from above at a point of D. Then f is continuous on D.

Proof. Since $a(t_0)$ and $b(t_0)$ are not zero simultaneously, therefore, without loss generality, we may assume that $b(t_0) > 0$.

Since f is locally bounded from above at a point $x_0 \in D$, there exists a neighborhood U at x_0 and a constant $K \geq 0$ such that $f(x) \leq K$ for every $x \in U$. Let ε be an arbitrary nonnegative constant. Then there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then

$$a(t_n)K + [b(t_n) - 1]\, f(x_0) < \varepsilon,$$

whence

$$\frac{a(t_n)}{b(t_n)}K + \left[1 - \frac{1}{b(t_n)}\right] f(x_0) < \varepsilon.$$

Let V be a neighborhood of 0 such that $x_0 + V \subseteq U$, and let $U' = x_0 + \alpha(t_n)V$. We prove that

$$|f(x) - f(x_0)| < \varepsilon \qquad (x \in U').$$

For $x \in U'$ there exist $y, z \in x_0 + V$ such that

$$\begin{aligned} x &= \alpha(t_n)y + \beta(t_n)x_0, \\ x_0 &= \alpha(t_n)z + \beta(t_n)x. \end{aligned}$$

Indeed,

$$y - x_0 = \frac{1}{\alpha(t_n)}(x - x_0) \in \frac{1}{\beta(t_n)}\alpha(t_n)V = V,$$

and

$$z - x_0 = \frac{1 - \alpha(t_n)}{\alpha(t_n)}(x_0 - x) \in \frac{1 - \alpha(t_n)}{\alpha(t_n)}\alpha(t_n)V = (1 - \alpha(t_n))V \subseteq V.$$

According to $T - (\alpha;\beta;a;b)$-convexity of f,

$$\begin{aligned} f(x) &\leq a(t_n)f(y) + b(t_n)f(x_0) \leq a(t_n)K + b(t_n)f(x_0), \\ f(x_0) &\leq a(t_n)f(z) + b(t_n)f(x) \leq a(t_n)K + b(t_n)f(x). \end{aligned}$$

We get

$$f(x) - f(x_0) \leq a(t_n)K + [b(t_n) - 1]\, f(x_0) < \varepsilon \tag{4}$$

and

$$f(x) \geq \frac{f(x_0) - a(t_n)K}{b(t_n)},$$

which implies

$$f(x) - f(x_0) \geq \left[\frac{1}{b(t_n)} - 1\right] f(x_0) - \frac{a(t_n)}{b(t_n)}K > -\varepsilon. \tag{5}$$

The inequalities (4) and (5) show that $|f(x) - f(x_0)| < \varepsilon$, that is f is continuous at x_0, which was to be proved. □

Remark 2. *The previous limit conditions are not necessary, since in the case of Jensen-convexity are not fulfilled. However, the result of Bernstein and Doetsch is valid for Jensen-convex functions. In contrary, the nonnegative monotone functions - which are not necessary continuous - belongs to a special class of the $(\alpha;\beta;a;b)$-convex functions, to the class of Godunova-Levin functions. Therefore, in this setting, the limit conditions in question cannot be ignored.*

3. Convexity property of rationally-$(\alpha;\beta;a;b)$-convex

The following result offers a generalization of the theorem of Bernstein-Doetsch [1], Breckner [4], Burai-Házy-Juhász [2] and Házy [11] for rationally-$(\alpha;\beta;a;b)$-convex functions

Theorem 3. *Let α,β,a,b be given nonnegative, continuous functions satisfying the limit conditions*

$$\lim_{t\to 0} a(t)=0 \qquad \text{and} \qquad \lim_{t\to 0} b(t)=1.$$

and $\alpha(t)+\beta(t)=1$.

Assume that $a(t_0)$ and $b(t_0)$ are not zero simultaneously for all $t_0\in\mathbb{Q}\cap[0,1]$. If $f:D\to\mathbb{R}$ is rationally$-(\alpha,\beta,a,b)$-convex and locally bounded from above at a point of D, then f is continuous and (α,β,a,b)-convex.

Proof. We prove that the function f is $t_0-(\alpha;\beta;a;b)$-convex for all $t_0\in[0,1]$. Let $t_0\in[0,1]$ arbitrary. Then there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n\in\mathbb{Q}$ and $t_n\to t_0$ (when n tends to ∞). Applying rationally$-(\alpha,\beta,a,b)$-convexity of f, we get

$$f\left(\alpha(t_n)x+\beta(t_n)y\right)\le a(t_n)f(x)+b(t_n)f(y). \tag{6}$$

The local upper boundedness of f implies the continuity of f (according to Theorem 2). Therefore, taking the limit $n\to\infty$ in (6), we get

$$f\left(\alpha(t_0)x+\beta(t_0)y\right)\le a(t_0)f(x)+b(t_0)f(y),$$

which proves the (α,β,a,b)-convexity of f. □

Corollary 4. *Let $D\subset X$ be a nonempty, convex, open set and let $h:[0,1]\to\mathbb{R}$ be a given nonnegative, continuous function satisfying the limit conditions*

$$\lim_{t\to 0} h(t)=0 \qquad \text{and} \qquad \lim_{t\to 1} h(t)=1.$$

and assume that $h(t_0)$ and $h(1-t_0)$ are not simultaneously zero for all $t_0\in\mathbb{Q}\cap[0,1]$.

If $f:D\to\mathbb{R}$ is rationally-h-convex and f is locally bounded from above at a point D, then f is continuous on D and h-convex.

Corollary 5. *Let $D\subset X$ be a nonempty, convex, open set. If $f:D\to\mathbb{R}$ is rationally-Breckner s-convex and locally bounded from above at a point D, then f is continuous on D and Breckner s-convex.*

Theorem 4. *Let $T=[0,1]$, $\alpha,\beta,a,b:T\to\mathbb{R}$ be given nonnegative functions such that α,β continuous on T and $a(t)+b(t)=1$. Let $f:\mathbb{R}_+\to\mathbb{R}$ an (α,β,a,b)-convex function. Then*

(*i*) *if $(\alpha+\beta)(T)=[r,1]$ (where $r<1$), then f is nondecreasing.*
(*ii*) *if $(\alpha+\beta)(T)=[1,r]$ (where $r>1$), then f is nonincreasing.*
(*iii*) *if $(\alpha+\beta)(T)=[r_1,r_2]$ (where $r_1<1<r_2$), then f is constant.*

Proof. We have, for $x > 0$ and $t \in [0, 1]$

$$f(\alpha(t)x + \beta(t)x) \leq a(t)f(x) + b(t)f(x) = f(x).$$

Let $\gamma = \alpha + \beta$. Then γ is continuous on $[0, 1]$.

In the case (i) we get $\gamma(T) = [r, 1]$, where $r > 1$. Let $u \in [r, 1]$ be arbitrary. Then there exists a $t \in [0, 1]$ such that $\gamma(t) = u$. This yields that

$$f(ux) \leq f(x) \qquad (x \in \mathbb{R}_+, u \in [r, 1]). \tag{7}$$

If now $u \in [r^2, 1]$ then $u^{1/2} \in [r, 1]$. Therefore, by the fact that (7) holds for all $x \in \mathbb{R}_+$, we get

$$f(ux) = f(u^{1/2}(u^{1/2}x)) \leq f(u^{1/2}x) \leq f(x)$$

for all $x \in \mathbb{R}_+$. By induction we then obtain that

$$f(ux) \leq f(x) \qquad (x \in \mathbb{R}_+,\ u \in]0, 1]). \tag{8}$$

Therefore, taking $0 < u < v$ and applying (8), we get

$$f(u) = f((u/v)v) \leq f(v),$$

which means that f is nondecreasing on $\mathbb{R}_+$.

The proof of the cases (ii) and (iii) are similar. □

The above results do not hold, in general, in the case of convex functions, because a convex function $f : \mathbb{R}_+ \to \mathbb{R}$, need not be non-decreasing. But in the case of Orlicz s-convex function this is true.

Corollary 6. *Let $0 < s < 1$. Let $f : \mathbb{R}_+ \to \mathbb{R}$ an Orlicz s-convex function. Then f is nondecreasing.*

Remark 3. *In the paper of Hudzik and Maligranda* [14] *is gave an example which shows that the Orlicz s-convex function is nondecreasing on $\mathbb{R}_+$, but not necessarily on $[0, \infty)$. For the readers convenience we recall the example: let $a, b, c \in \mathbb{R}$ and let*

$$f(x) = \begin{cases} a & \textit{if } x = 0 \\ bx^s + c & \textit{if } x \neq 0. \end{cases}$$

Then if $b > 0$ and $c < a$ then f is non-decreasing on $(0, \infty)$ but not on $[0, \infty)$.

4. Optimization

It is a very well known fact that every local minimizer of a convex function is a global one. The same is true for (α, β, a, b)-convex functions under some assumptions.

Theorem 5. *Let X be a real or complex topological vector space, $D \subset X$ be a nonempty open $(\alpha; \beta)$-convex set, where $\alpha, \beta, a, b : [0, 1] \to \mathbb{R}$ be given nonnegative, continuous functions satisfying the limit conditions*

$$\lim_{t \to 0} \alpha(t) = 0 \qquad \textit{and} \qquad \lim_{t \to 0} \beta(t) = 1.$$

and assume that $a(t) + b(t) = 1$.

Then every local minimizer $x_0 \in D$ of an (α, β, a, b)-convex function $f : D \to \mathbb{R}$ is a global one.

Proof. Let $x_0 \in D$ be a local minimizer of f. Then there exists a positive real number r, such that

$$f(x_0) \leq f(y), \qquad y \in U(x_0, r)\,.$$

Assume that x_0 is not a global minimizer. Then there exists $z \in D$, such that $f(x_0) > f(z)$. Using this and the (α, β, a, b)-convexity of f, we have

$$f(\alpha(t)z + \beta(t)x_0) \leq a(t)f(x_0) + b(t)f(z) = f(x_0) + b(t)\left(f(z) - f(x_0)\right) < f(x_0)\,.$$

On the other hand, using the limit conditions, $\alpha(t)z + \beta(t)x_0 \in U(x_0, r)$, if t is sufficiently small, which contradicts to the fact that x_0 is a local minimizer.

If f is a strictly (α, β, a, b)-convex function, and $x \neq y$ are global minimizers, then

$$f(\alpha(t)x + \beta(t)y) < a(t)f(x) + b(t)f(y) = f(x),$$

which is a contradiction. □

Corollary 7. *Every local minimizer of an Orlicz-convex function $f : D \to \mathbb{R}$ is a global one. If the function f is strictly Orlicz-convex, then there is at most one global minimum.*

Corollary 8. *Every local minimizer of a convex function $f : D \to \mathbb{R}$ is a global one. If the function f is strictly convex, then there is at most one global minimum.*

References

[1] F. Bernstein and G. Doetsch, *Zur Theorie der konvexen Funktionen*, Math. Annalen **76** (1915), 514–526.

[2] P. Burai, A. Házy and T. Juhász, *Bernstein-Doetsch type results for s-convex functions* Publ. Math. Debrecen **75** (2009), vol 1-2., 23–31

[3] P. Burai, A. Házy and T. Juhász, *On approximately s-convex functions* submitted

[4] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, Publ. Inst. Math. (Beograd) **23** (1978), 13–20.

[5] W. W. Breckner and G. Orbán, *Continuity properties of rationally s-convex mappings with values in ordered topological liner space*, "Babes-Bolyai" University, Kolozsvár, 1978.

[6] S. S. Dragomir, J. Pečarič and L. E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math. **21** (1995), 335-241.

[7] E. K. Godunova and V. I. Levin, *Neravenstva dlja funkcii sirokogo klassa, soderzascego vypuklye, monotonnye i nekotorye drugie vidy funkii*, Vycislitel. Mat. i. Fiz. Mezvuzov. Sb. Nauc. Trudov, MGPI, Moskva, 1985, pp. 138142.

[8] A. Gilányi, K. Nikodem and Zs. Páles *BernsteinDoetsch type results for quasiconvex functions* Math. Ineq. and Appl. **7** (2004), no. 2, 169-175.

[9] A. Házy, *On approximately t-convexity*, Math. Ineq. and Appl. **8** (2005), no. 3, 389–402.

[10] A. Házy, *On the stability of t-convex functions*, Aequationes Math. **74** (2007) 210–218.

[11] A. Házy, *Bernstein-Doetsch type results for h-convex functions*, accepted for publication, Math. Ineq. Appl. (2011).

[12] A. Házy and Zs. Páles, *Approximately midconvex functions*, Bulletin London Math. Soc. **36** (2004) 339–350.

[13] A. Házy and Zs. Páles, *On approximately t-convex functions*, Publ. Math. Debrecen **66** (2005), no. 3-4, 489–501, Dedicated to the 75th birthday of Professor Heinz König

[14] H. Hudzik and L. Maligranda, *Some remarks on s_i-convex functions*, Aequationes Math. **48** (1994) 100–111.

[15] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe — Uniwersytet Śląski, Warszawa–Kraków–Katowice, 1985.

[16] Gy. Maksa and Zs. Páles: *The equality case in some recent convexity inequalities*, Opuscula Math. **31**/2 (2011), 269–277.

[17] D. S. Mitrinovic and J. Pecaric, *Note on a class of functions of Godunova and Levin*, C. R. Math. Rep. Acad. Sci. Can. **12** (1990),33-36.

[18] D. S. Mitrinovic, J. Pecaric and A. M. Fink, *Classical and new inequalities in analysis*, Kluwer Academic, Dordrecht, 1993.

[19] J. Mrowiec, *Remark on approximately Jensen-convex functions*, C. R. Math. Acad. Sci. Soc. R. Canada **23** (2001), 16–21.

[20] C. T. Ng and K. Nikodem, *On approximately convex functions*, Proc. Amer. Math. Soc. **118** (1993), no. 1, 103–108.

[21] W. Orlicz, *A note on modular spaces I.*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **9** (1961),157–162.

[22] Zs. Páles, *Bernstein–Doetsch-type results for general functional inequalities*, Rocznik Nauk.-Dydakt. Prace Mat. **17** (2000), 197–206, Dedicated to Professor Zenon Moszner on his 70th birthday.

[23] Zs. Páles, *On approximately convex functions*, Proc. Amer. Math. Soc. **131** (2003), 243–252 (electronic).

[24] S. Piccard, *Sur des ensembles parfaits*, Mém. Univ. Neuchâtel, vol. 16., Secrétariat de l' Université, Neuchâtel, 1942.

[25] C.E.M. Pearce and A.M. Rubinov, *P -functions, quasi-convex functions and Hadamard-type inequalities*, J. Math. Anal. Appl. **240** (1999), 92-104.

[26] M. Pycia, *A direct proof of the s-Hölder continuity of Breckner s-convex functions*, Aequationes Math., **61** (2001), 128–130.

[27] H. Steinhaus, *Sur les distances des points des ensembles de mesure positive*, Fund. Math. **1** (1920), 93-104.

[28] K.L. Tseng, G.S. Yang and S.S. Dragomir, *On quasi-convex functions and Hadamards inequality*, RGMIA Res. Rep. Coll. **6** (3) (2003), Article 1.

[29] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl. **326** (2007), 303-311.

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