On strongly convex functions

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1. Introduction

Throughout this paper $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of real, nonnegative real, natural and integer numbers respectively.

Let X be a normed space and $D \subseteq X$ be a nonempty conex subset of X. Denote by D^* the set $\{\|x-y\|, x, y \in D\}$. Let $\alpha : D^* \to \mathbb{R}_+$ be a nonnegative error function. We say that a function $f : D \to \mathbb{R}$ is strongly α-Jensen convex, if, for all $x, y \in D$,

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} - \alpha(\|x-y\|). \tag{1}$$

Observe that if $\alpha \equiv 0$, we can get the classical definition of convexity. When $\alpha(u) = cu^2$, we can get a kind of notion of strong convexity introduced by Polyak in [24] and examined by Azócar, Giménez, Nikodem and Sánchez (in [1]), Merentes and Nikodem, (in [16]) and Nikodem and Páles [22]. If $\alpha(u) = \varepsilon u^p$, then f is called *strongly* (ε, p)*-Jensen convex function*. In Section 2, we are looking connection between strong α-convexity and strong convexity type inequalities. Then, we are looking for the optimal error function. In Section 3, we will establish the connections between strong α-convexity and strong α-Jensen convexity, moreover the connections between strong convexity and Hermite–Hadamard type inequalities will be shown. In what follows we recall some Bernstein-Doetsch type theorem for approximately convex functions. A function $f : D \to \mathbb{R}$ is said to be approximately α-Jensen convex, if, for all $x, y \in D$,

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} + \alpha(\|x-y\|). \tag{2}$$

The proofs of these theorem are similar to our main theorems' proofs.

Let introduce the Takagi type functions $\mathcal{T}_\alpha : \mathbb{R} \times D^+ \to \mathbb{R}_+$ and $\mathcal{S}_\alpha : \mathbb{R} \times D^+ \to \mathbb{R}_+$ by

$$\mathcal{T}_\alpha(t,u) := \sum_{n=0}^{\infty} \frac{1}{2^n}\alpha\big(d_{\mathbb{Z}}(2^n t)u\big) \qquad \big((t,u) \in \mathbb{R} \times D^+\big) \tag{3}$$

and

$$\mathcal{S}_\alpha(t,u) := \sum_{n=0}^{\infty} \alpha\Big(\frac{u}{2^n}\Big) d_{\mathbb{Z}}(2^n t) \qquad \big((t,u) \in \mathbb{R} \times D^+\big). \tag{4}$$

Note that the first series converges uniformly if α is bounded, on the other hand, for the uniform convergence of the second series, it is sufficient if $\sum_{n=n_0}^{\infty} \alpha\big(2^{-n}\big) < \infty$ for some $n_0 \in \mathbb{N}$.

The importance of the function $\mathcal{T}_\alpha$ introduced above is enlightened by the following result which can be considered as a generalization of the celebrated Bernstein-Doetsch theorem [2].

Theorem 1. (Makó-Páles [15], Tabor–Tabor [26])
Let $f : D \to \mathbb{R}$ be locally upper bounded on D and let $\alpha : D^+ \to \mathbb{R}_+$. Then f is α-Jensen convex on D if and only if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \mathcal{T}_\alpha(t, \|x-y\|) \tag{5}$$

for all $x, y \in D$ and $t \in [0,1]$.

The other Takagi type function $\mathcal{S}_\alpha$ was introduced by Jacek Tabor and Józef Tabor. Its role and importance in the theory of approximate convexity is shown by the next theorem.

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Theorem 2. (Tabor-Tabor [26])

Let $f : D \to \mathbb{R}$ be upper semicontinuous on D and let $\alpha : D^+ \to \mathbb{R}_+$ be nondecreasing such that $\sum_{n=n_0}^{\infty} \alpha(2^{-n}) < \infty$ for some $n_0 \in \mathbb{N}$. Then f is α-Jensen convex on D if and only if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \mathcal{S}_\alpha(t, \|x-y\|) \tag{6}$$

for all $x, y \in D$ and $t \in [0,1]$.

Let $\varepsilon, q \ge 0$ be arbitrary constants. When $\alpha(u) := \varepsilon u^q, (u \in D^+)$, the two corollaries below (see [8] and [26]) are immediately consequences of the previous theorems.

For $q \ge 0$, define the Takagi type functions S_q and T_q by

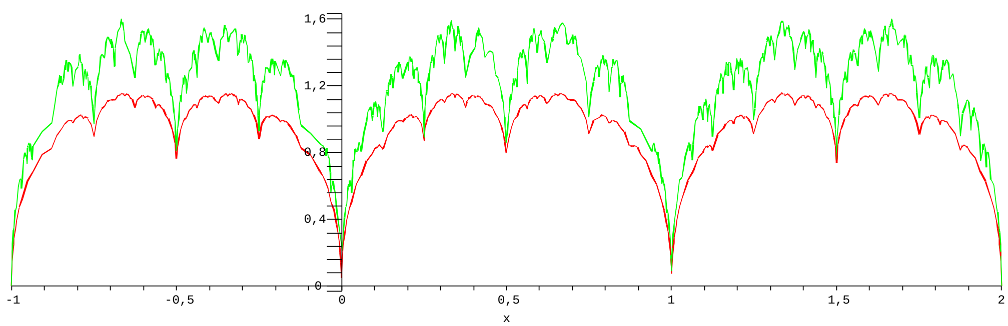
$$T_q(t) := \sum_{n=0}^{\infty} \frac{\big(d_{\mathbb{Z}}(2^n t)\big)^q}{2^n}, \qquad S_q(t) := \sum_{n=0}^{\infty} \frac{d_{\mathbb{Z}}(2^n t)}{2^{nq}} \qquad (t \in \mathbb{R}). \tag{7}$$

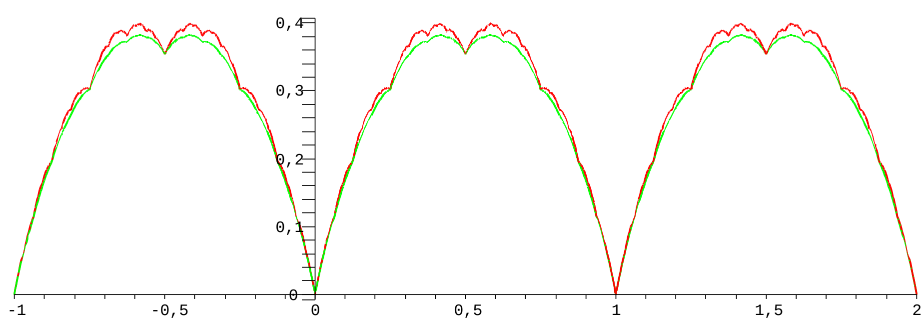
They generalize the classical Takagi function

$$T(t) := \sum_{n=0}^{\infty} \frac{\operatorname{dist}(2^n t, \mathbb{Z})}{2^n} \qquad (t \in \mathbb{R})$$

in two ways, because $T_1 = S_1 = 2T$ holds obviously. This function was introduced by Takagi in [29] and it is a well-known example of a continuous but nowhere differentiable real function.

It is less trivial, but it can be proved that $T_2(t) = S_2(t) = 4t(1-t)$ for $t \in [0,1]$. The following pictures demonstrate the comparison between T_q and S_q for $q = 0.5$ and $q = 1.5$, respectively.





Corollary 3. (Házy [4])

Let $f : D \to \mathbb{R}$ be locally upper bounded on D and $\varepsilon, q \ge 0$. Then f is (ε, q)-Jensen convex on D, if and only if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon T_q(t)\|x-y\|^q \tag{8}$$

for all $x, y \in D$ and $t \in [0,1]$.

Corollary 4. (Tabor-Tabor [26])

Let $f : D \to \mathbb{R}$ be upper semicontinuous on D and $\varepsilon, q \ge 0$. Then f is (ε, q)-Jensen convex on D if and only if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon S_q(t)\|x-y\|^q \tag{9}$$

for all $x, y \in D$ and $t \in [0,1]$.

In [3], Boros proved that if $q = 1$ and $t \in [0, 1]$ is fixed, then $S_1(t) = T_1(t) = 2T(t)$ is the smallest possible. In [25] Tabor and Tabor showed that if $1 \le q \le 2$ and $t \in [0, 1]$ is fixed, then $S_q(t)$ is the smallest possible value so that (9) be valid for all (ε, q)-Jensen convex functions f on D.

For $x, y \in D$ denote by $[x, y] = \{tx + (1-t)y \mid t \in [0, 1]\}$. It is an important question whether the error terms $\mathcal{T}_\alpha(t, \|x-y\|)$, $\mathcal{S}_\alpha(t, \|x-y\|)$ in (5) in (6) and $T_q(t)$ in (8) are the smallest possible ones. In other words, for all fixed $x, y \in D$, we want to obtain the exact upper bound of the convexity-difference of α-Jensen convex functions defined by

$$C_\alpha(x, y, t) := \sup_{f \in \mathcal{JC}_\alpha(D)} \{f(tx + (1-t)y) - tf(x) - (1-t)f(y)\}, \tag{10}$$

where

$$\mathcal{JC}_\alpha(D) := \{f : D \to \mathbb{R} \,|\, f \text{ is } \alpha\text{-Jensen convex on } D\}.$$

The statement of Theorem 1, Theorem 2, Corollary 3, and Corollary 4 can be stated as

$$C_\alpha(x, y, t) \le \tau(t, \|x-y\|), \tag{11}$$

where $\tau : \mathbb{R} \times D^+ \to \mathbb{R}_+$ is given by

$$\tau := \mathcal{T}_\alpha, \quad \tau := \mathcal{S}_\alpha, \quad \tau(t, u) := \varepsilon T_q(t) u^q, \quad \text{and} \quad \tau(t, u) := \varepsilon S_q(t) u^q,$$

respectively. To obtain also a lower bound for $C_\alpha(x, y, t)$, (and thus to prove the sharpness of the inequality (11)), the following important observation was done by Páles in [23].

Theorem 5. (Páles [23])
Let $\alpha : D^+ \to \mathbb{R}$ be continuous. Let $\tau : \mathbb{R} \times D^+ \to \mathbb{R}$ be continuous in its first variable, with $\tau(0, u) = \tau(1, u) = 0$ for all $u \in D^+$, which is Jensen convex in the following sense, for all $u \in D^+$ and $s, t \in [0, 1]$,

$$\tau\Big(\frac{t+s}{2}, u\Big) \le \frac{\tau(t, u) + \tau(s, u)}{2} + \alpha(|t-s|u).$$

Then,

$$C_\alpha(x, y, t) \ge \tau(t, \|x-y\|)$$

2. From strong α-Jensen convexity to strong convexity

Similarly as in Theorem 1 and Theorem 2, it can be proved two Bernstein–Doetsch type results for locally upper bounded strongly Jensen convex functions. Thus, these theorems give us connections between strong α-Jensen convexity and convexity type inequalities.

Theorem 6. *Let $f : D \to \mathbb{R}$ be locally upper bounded on D and let $\alpha : D^+ \to \mathbb{R}_+$. Then f is strongly α-Jensen convex on D if and only if*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - \mathcal{T}_\alpha(t, \|x-y\|) \tag{12}$$

for all $x, y \in D$ and $t \in [0, 1]$.

Theorem 7. *Let $f : D \to \mathbb{R}$ be upper semicontinuous on D and let $\alpha : D^+ \to \mathbb{R}_+$ be $\sum_{n=n_0}^{\infty} \alpha(2^{-n}) < \infty$ for some $n_0 \in \mathbb{N}$. Then f is α-Jensen convex on D if and only if*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - \mathcal{S}_\alpha(t, \|x-y\|) \tag{13}$$

for all $x, y \in D$ and $t \in [0, 1]$.

We can also look for the optimal Takagi type function. In other words, for all fixed $x, y \in D$, we want to obtain the exact upper bound of the convexity-difference of strongly α-Jensen convex functions defined by

$$SC_\alpha(x, y, t) := \sup_{f \in \mathcal{SJC}_\alpha(D)} \{f(tx + (1-t)y) - tf(x) - (1-t)f(y)\}, \tag{14}$$

where

$$\mathcal{SJC}_\alpha(D) := \{f : D \to \mathbb{R} \,|\, f \text{ is locally upper bounded and strongly } \alpha\text{-Jensen convex on } D\}.$$

By Theorem 5, it is enough to prove the Jensen-convexity of $S_\alpha(\cdot, u)$ or $\mathcal{T}_\alpha(\cdot, u)$. We shall prove that the Takagi type function $S_\alpha(\cdot, u)$ will be the optimal choice. To show this suspicion let introduce the following Takagi type function $S_\varphi : [0, 1] \to \mathbb{R}$ defined by

$$S_\varphi(x) = \sum_{n=0}^{\infty} \varphi\big(\tfrac{1}{2^n}\big) d_{\mathbb{Z}}(2^n x), \tag{15}$$

where $P := \{1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}, \dots, \}$ and $\varphi : P \to \mathbb{R}_+$ is a nonnegative function. The main results of this section state that, under certain assumptions on the function $\varphi : P \to \mathbb{R}$, $(-S_\varphi)$ is well-defined and strongly Jensen convex in the following sense: For all $x, y \in [0,1]$,

$$-S_\varphi\Big(\frac{x+y}{2}\Big) \le \frac{-S_\varphi(x) - S_\varphi(y)}{2} - \varphi \circ d_{\mathbb{Z}}\Big(\frac{x-y}{2}\Big). \tag{16}$$

First we describe the situation when the definition of S_φ is correct.

Lemma 8. *Let $\varphi : P \to \mathbb{R}_+$ be a nonnegative function. Then S_φ is well-defined, i.e., the series on the right hand side of* (15) *is convergent everywhere if and only if*

$$\sum_{n=0}^{\infty} \varphi\big(\tfrac{1}{2^n}\big) < \infty. \tag{17}$$

In the sequel, the class of nonnegative functions $\varphi : P \to \mathbb{R}_+$ satisfying the condition (17) will be denoted by $\mathcal{H}$:

$$\mathcal{H} := \Big\{\varphi : P \to \mathbb{R}_+ \mid \sum_{n=0}^{\infty} \varphi\big(\tfrac{1}{2^n}\big) < \infty\Big\}.$$

The next theorem, which was discovered by Jacek Tabor and Józef Tabor, has an important role in the proof of the main theorem of this section.

Theorem 9. *For every $x, y \in \mathbb{R}$*

$$S_2\Big(\frac{x+y}{2}\Big) \le \frac{S_2(x) + S_2(y)}{2} + d_{\mathbb{Z}}^2\Big(\frac{x-y}{2}\Big).$$

The following simple observation is a direct consequence of the previous theorem.

Corollary 10. *For every $x, y \in [0,1]$*

$$-S_2\Big(\frac{x+y}{2}\Big) = \frac{-S_2(x) - S_2(y)}{2} - d_{\mathbb{Z}}^2\Big(\frac{x-y}{2}\Big).$$

In the next result we give a representation of $S_\varphi(x)$ as an infinite linear combination of the values $S_2(2^n x)$, $n = 1, 2, \dots$.

Theorem 11. *Let $\varphi \in \mathcal{H}$. Then, for every $x \in \mathbb{R}$,*

$$S_\varphi(x) = \varphi(1) S_2(x) + \sum_{n=1}^{\infty} \Big(\varphi\big(\tfrac{1}{2^n}\big) - \tfrac{1}{4}\varphi\big(\tfrac{1}{2^{n-1}}\big)\Big) S_2(2^n x). \tag{18}$$

An immediate consequence of the previous two theorems is the next result which states the strong convexity of $(-S_\varphi)$.

Theorem 12. *Let $\varphi \in \mathcal{H}$ such that, for all $u \in \frac{1}{2}P$, $\varphi(2u) \ge 4\varphi(u)$. Then, for all $x, y \in \mathbb{R}$,*

$$-S_\varphi\Big(\frac{x+y}{2}\Big) \le \frac{-S_\varphi(x) - S_\varphi(y)}{2} - \Phi_2\Big(\frac{x-y}{2}\Big), \tag{19}$$

where $\Phi_2 : \mathbb{R} \to \mathbb{R}$ is defined by

$$\Phi_2(u) := \sum_{n=0}^{\infty} \varphi\big(\tfrac{1}{2^n}\big)\Big(d_{\mathbb{Z}}^2(2^n u) - \frac{1}{4} d_{\mathbb{Z}}^2(2^{n+1} u)\Big). \tag{20}$$

In the next proposition we describe a decomposition property of the function Φ_2.

Proposition 13. *For $\varphi \in \mathcal{H}$, for all $u \in]0, \frac{1}{2}]$,*

$$\Phi_2(u) = \Phi_2\Big(\frac{1}{2^{[\log_2 \frac{1}{u}]}} - u\Big) + \varphi\Big(\frac{1}{2^{[\log_2 \frac{1}{u}]-1}}\Big)\Big(1 - 2 \cdot 2^{[\log_2 \frac{1}{u}]} u\Big). \tag{21}$$

In the next proposition an important class of functions φ from $\mathcal{H}$ will be described.

Proposition 14. *Let $\varphi : [0,1] \to \mathbb{R}_+$. Assume that $\varphi(0) = 0$ and the mapping $x \mapsto \frac{\varphi(x)}{x}$ is convex on $]0,1]$. Then $\varphi|_P \in \mathcal{H}$, the function $x \mapsto \frac{\varphi(x)}{x^2}$ is nondecreasing on $]0,1]$ and φ is continuous on $[0,1[$.*

The next theorem has an important role in the proof of our subsequent main results.

Theorem 15. *Let $\varphi : [0,1] \to \mathbb{R}_+$. Assume that $\varphi(0) = 0$ and the mapping $x \mapsto \frac{\varphi(x)}{x}$ is convex on $]0,1]$, then, for all $u \in \mathbb{R}$,*

$$- \Phi_2(u) \leq -\varphi \circ d_{\mathbb{Z}}(u). \tag{22}$$

The main result of this section is stated in the following theorem.

Theorem 16. *Let $\varphi : [0,1] \to \mathbb{R}_+$. Assume that $\varphi(0) = 0$ and the mapping $x \mapsto \frac{\varphi(x)}{x}$ is convex on $]0,1]$. Then S_φ is approximately Jensen convex in the sense of (16).*

We shall prove that the error terms $-\mathcal{S}_\alpha(t, \|x-y\|)$ in (6) under certain assumptions on the error function α is the smallest possible one. In other words, the next theorem will provide exact upper bound for the convexity-difference of strongly α-Jensen convex functions defined by (14).

Theorem 17. *Let $\alpha : D^+ \to \mathbb{R}$ be an error function such that $\alpha(0) = 0$ and the map $u \mapsto \frac{\varphi(u)}{u}$ is convex on $D^+ \setminus \{0\}$. Then, for all $x, y \in D$ and $t \in [0,1]$,*

$$SC_\alpha(x,y,t) = -\mathcal{S}_\alpha(t, \|x-y\|). \tag{23}$$

Taking an error function α which is a combination of power functions of exponents from $[2,\infty[$, we obtain the following result.

Theorem 18. *Let ν be a nonnegative bounded Borel measure on $[2,\infty[$. Define the error function $\alpha_\nu : D^+ \to \mathbb{R}_+$ by*

$$\alpha_\nu(u) := \int\limits_{[2,\infty[} u^q d\nu(q) \qquad (u \in D^+).$$

Then, for all $x, y \in D$ and $t \in [0,1]$,

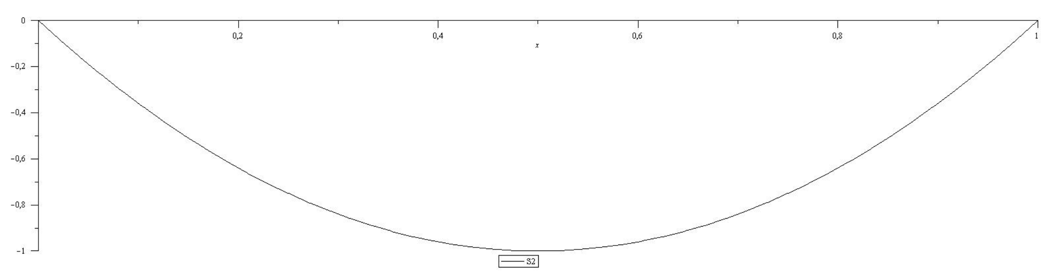
$$SC_{\alpha_\nu}(x,y,t) = -\int\limits_{[2,\infty[} S_q(t)\|x-y\|^q d\nu(q),$$

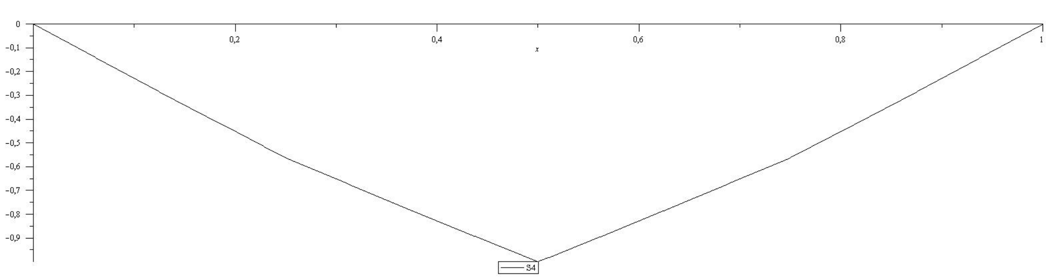
where $S_q : \mathbb{R} \to \mathbb{R}$ is given by (7).

Corollary 19. *Let $q \in [2,\infty[$ and $\varepsilon \geq 0$. Define the error function $\alpha : D^+ \to \mathbb{R}_+$ by $\alpha(u) := \varepsilon u^q$. Then, for all $x, y \in D$ and $t \in [0,1]$,*

$$SC_\alpha(x,y,t) = -\varepsilon S_q(t)\|x-y\|^q.$$

The next figures demonstrate the strong convexity of $-S_q$, when $q = 2$ and $q = 4$.





3. On a strong convexity type inequality

Given a nonnegative *even* function $\alpha : D^* \to \mathbb{R}_+$, we say that a map $f : D \to \mathbb{R}$ is *strongly α-convex*, if for all $x, y \in D$ and $t \in [0,1]$,

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - t\alpha\big((1-t)\|x-y\|\big) - (1-t)\alpha\big(t\|y-x\|\big) \tag{24}$$

holds. If (24) holds with $T = \{1/2\}$, i.e., for all $x, y \in D$,

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} - \alpha\big(\tfrac{x-y}{2}\big),$$

we can get the strong $\alpha(\frac{\cdot}{2})$-Jensen convexity of the function f By the nonnegativity of α, we have that strongly α-Jensen convex and strongly α-convex functions are always convex in the same sense, respectively.

In [12], we examined the strong α-Jensen convexity and we got the following result.

Theorem 20. (Makó-Nikodem-Páles, [12])
For any function $f : D \to \mathbb{R}$, the following conditions are equivalent:

(i) f is strongly α-convex.

(ii) f is directionally differentiable at every point of D, and for all $x_0 \in D$, the map $h \mapsto f'(x_0, h)$ is sublinear on X, furthermore for all $x_0, x \in D$,

$$f(x) \ge f(x_0) + f'(x_0, x - x_0) + \alpha(\|x - x_0\|). \tag{25}$$

(iii) For all $x_0 \in D$, there exits an element $A \in X'$ such that

$$f(x) \ge f(x_0) + A(x - x_0) + \alpha(\|x - x_0\|) \quad \textit{for all} \quad x \in D. \tag{26}$$

Thus, it can be important to look for connections between the strong α-Jensen convexity and strong α-convexity.

Theorem 21. *If $f : D \to \mathbb{R}$ is locally upper bounded and strongly α-Jensen convex, then f is strongly 2α-convex on D.*

In the following theorems, we have established relations between Hermite–Hadamard type inequalities and strong (Jensen) convexity.

Theorem 22. *Let $\rho : [0,1] \to \mathbb{R}$ be integrable function and assume that $\alpha : D^* \to \mathbb{R}_+$ be a given error function. Denote $\lambda := \int_0^1 \rho$. If $f : D \to \mathbb{R}$ is continuous and satisfies the following upper Hermite–Hadamard type inequality*

$$\int_0^1 f(tx + (1-t)y)\rho(t)dt \le \lambda f(x) + (1-\lambda)f(y) - \alpha(\|x-y\|), \qquad (x, y \in D)$$

then f is strongly $\frac{1}{\lambda}\alpha$-convex on D.

Theorem 23. *Let $\rho : [0,1] \to \mathbb{R}$ be integrable function and assume that $\alpha : D^* \to \mathbb{R}_+$ be a given error function. Denote $\lambda := \int_0^1 \rho$. If $f : D \to \mathbb{R}$ is continuous and satisfies the following lower Hermite–Hadamard type inequality*

$$f(\lambda x + (1-\lambda)y) \le \int_0^1 f(tx + (1-t)y)\rho(t)dt - \alpha(\|x-y\|) \qquad (x, y \in D)$$

then f satisfies the following Jensen-type inequality

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) - \alpha(\|x-y\|) \qquad (x, y \in D).$$

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