Solving functional equations with computer

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Abstract for demonstration

***Abstract*—In this paper we deal with the linear two variable functional equation**

$$h_0(x,y)f_0(g_0(x,y)) + \cdots + h_n(x,y)f_n(g_n(x,y)) = F(x,y)$$

where n is a positive integer, $g_0, g_1, \ldots, g_n, h_0, h_1, \ldots, h_n$, and F are given real valued analytic functions on an open set $\Omega \subset \mathbb{R}^2$, furthermore $f_0, f_1 \ldots, f_n$ are unknown functions.

Applying the results of Páles [1] we get recursively an inhomogeneous linear differential-functional equation in one of unknown function for $f_1, f_2, \ldots, f_n$, respectively. One of our main result states that the solutions of the differential-functional equation obtained are the same as that of an ordinary differential equation (under some assumptions), whose order is usually much smaller than the order of the differential-functional equation. Our aim is also to describe a computer-program which solves functional equations of this type. This algorithm is implemented in Maple symbolic language.

I. Introduction

In the theory of functional equations there exist few methods is solving a wider class of functional equations. In several cases, with the help of the computer algebra system Maple V, which enables us to perform the tedious computations, we completely describe the general solutions of these equations. See for example the papers of Sz. Baják and Zs. Páles ([2], [3], [4], [5]), S. Czirbusz ([6], [7]), A. Házy ([8], [9]) and O. Merino ([10]).

Such a method has been found linear functional equations with constant coefficients

$$\sum_{i=1}^{n+1} c_i f_i(p_i x + q_i y) = 0 \qquad (1)$$

by L. Székelyhidi [11], [12]. The computerized algorithm of the solution of such equations has been developed by A. Gilányi [13].

An algorithm of the reduction of linear two variable functional equations

$$h_0(x,y)f_0(g_0(x,y)) + \cdots + h_n(x,y)f_n(g_n(x,y)) = F(x,y), \qquad (2)$$

to differential functional equations has been found by Zs. Páles [1]. Here $g_0, g_1, \ldots, g_n, h_0, h_1, \ldots, h_n$ and F are given real valued functions on an open set $\Omega \subset \mathbb{R}^2$ and $f_0, f_1, \ldots, f_n$ are unknown functions. Such equations are, for example the Pexider equation:

$$f(x+y) - g(x) - h(y) = 0,$$

the Jensen equation:

$$f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} = 0,$$

the monomial equation:

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+ky) = n! f(y),$$

the equation from the information theory:

$$f(xy) + g((1-x)y) + h(x(1-y)) + l((1-x)(1-y)) = 0,$$

and the equation:

$$xf(x+y) - yg(xy) - h(y) = 0.$$

According to the results of Páles, it can be shown that, assuming sufficiently high-order differentiability, there exists a linear partial differential operator

$$\mathbf{D} = \sum_{\substack{i,\, j \,\geq\, 0, \\ i \,+\, j \,\leq\, k}} \alpha_{ij}(x,y)\partial_x^i \partial_y^j$$

(with variable coefficients and order k less than $2n-1$) which "kills" all of the members in the left hand side of the equation, except the first. That is, applying $\mathbf{D}$ to the equation (2), it yields

$$\mathbf{D}[h_0(x,y)f_0(g_0(x,y))] = \mathbf{D}F(x,y), \qquad (x,y) \in \Omega,$$

which is a linear differential-functional equation of order at most k.

Based on the results of Páles we can obtain the following algorithm to solve (2):

(A) Take data $n, h_0, \ldots, h_n, g_0, \ldots, g_n, F$.
(B) Let $k = 0$.
(C) Increase k by 1.
(D) Let $m = (k+1)(k+2)/2$, and $\{\mathbf{D}_1, \ldots, \mathbf{D}_m\}$ be equal to $\{id, \partial_x, \partial_y, \ldots, \partial_x^k, \partial_x^{k-1}\partial_y, \ldots, \partial_y^k\}$.
(E) Determine $H_0, \ldots, H_n$.
(F) Determine rank(H) and rank(H^*).
(G) If $m - 1 <$ rank(H), then go to (C), else go to (H).
(H) If rank$(H) =$ rank(H^*) then go to (C), else go to (I).
(I) Determine $\mathbf{D}$.
(J) Derive the differential-functional equation for f_0.

At the end of this algorithm we get a differential-functional equation for f_0. In the next part of this paper, we deal with the equation obtained. We will show that there exists an ordinary differential equation (under some assumptions) whose $(k+1)$-times differentiable solutions are the same as the that of obtained differential-functional equation.

In the second part of this paper, we deal with the obtained differential-functional equations. It is shown, that if we consider the equation

$$l_k(x,y)f^{(k)}(g(x,y)) + \ldots + l_0(x,y)f(g(x,y)) = L(x,y), \tag{3}$$

for all $(x,y) \in \Omega$, where $l_0, l_1, \ldots, l_k, g$ and L are given real valued analytic functions on Ω, furthermore f is an unknown real function on $g(\Omega)$, then there exists a differential equation (under some assumptions) of the form

$$f^{(m)}(g(x,y)) + \sum_{i=0}^{m-1} K_i(g(x,y))f^{(i)}(g(x,y)) = K(g(x,y)), \tag{4}$$

whose $(k+1)$-times differentiable solutions coincide with that of (3). Hence after simplification and with the substitution $t = g(x,y)$, (4) reduces to an ordinary differential equation with respect to f whose order is usually much smaller than the order of (3).

The solving method is the following. Consider the n-th order inhomogeneous linear differential-functional equation. In the first step we construct another differential-functional equation, whose order is not greater than the order of the original equation and the solutions of this system of equations are the same as that of the original equation. In the second step, we create a smaller order differential-functional equation whose solutions are the same as that of the system of equations. We repeat this two steps until getting such differential-functional equation whose coefficients satisfy a certain system of partial differential equations. Finally, with substitution $t = g(x,y)$, we get an ordinary differential equation.

More precisely, the next lemma enables us to produce a new differential-functional equation from (3) without increasing the order of equation.

Lemma 1: Let $l_0, l_1, \ldots, l_k, g$ and L be given real valued, differentiable functions on Ω (such that $g(\Omega)$ is an open set), furthermore f be a $(k+1)$-times differentiable real function on $g(\Omega)$. Then, applying the differential operator

$$\mathbf{D_0} = \partial_y g \cdot \partial_x - \partial_x g \cdot \partial_y$$

to the equation (3), the order of the resulting equation does not increase.

Clearly, the solutions of system of equations (3), (**??**) are identical to that of equation (3).

Applying the differential operator $\mathbf{D_0}$ to a k-th order differential-functional equation we get an m-th $(m \le k)$ order equation. In the next lemma we prove that there exists a smaller order differential-functional equation, whose solutions are the same as that of the system of these equations.

Lemma 2: Consider the following two equations

$$\sum_{i=0}^{k} h_i(x,y) \cdot f^{(i)}(g(x,y)) = H(x,y) \qquad (x,y) \in \Omega, \tag{5}$$

$$\sum_{i=0}^{m} l_i(x,y) \cdot f^{(i)}(g(x,y)) = L(x,y) \qquad (x,y) \in \Omega, \tag{6}$$

where the functions $h_0, h_1, \ldots, h_k, l_0, l_1, \ldots, l_m, g, H, L : \Omega \to \mathbb{R}$ are analytic and $h_k, l_m \neq 0$ on Ω. Assume that $k \ge m$. Then either there is no common solution of these equations, or there exist analytic functions $w_0, w_1, \ldots, w_s, W : \Omega \to \mathbb{R}$ (where $s \le m$) such that the k-times differentiable solutions of the equation

$$\sum_{i=0}^{s} w_i(x,y) \cdot f^{(i)}(g(x,y)) = W(x,y) \qquad (x,y) \in \Omega \tag{7}$$

are the same as that of the system of equations (5),(6).

Our main results are the following two teheomrems.

Theorem 1: Consider the equation

$$l_n(x,y)f^{(n)}(g(x,y)) + \ldots + l_0(x,y)f(g(x,y)) = F(x,y) \tag{8}$$

for all $(x,y) \in \Omega$, where $\Omega \subset \mathbb{R}^2$ is an open, connected set and $l_0, l_1, \ldots, l_n, g$ and F are given real valued, analytic functions on Ω (such that $g(\Omega)$ is an open set), furthermore f is an unknown real function on $g(\Omega)$. Then there exists a functional-differential equation

$$h_m(x,y)f^{(m)}(g(x,y)) + \ldots + h_0(x,y)f(g(x,y)) = H(x,y) \tag{9}$$

for all $(x,y) \in \Omega$, (where $m \le n$) whose $(n+1)$-times differentiable solutions coincide with that of (8) such that $h_0, h_1, \ldots, h_m, H$ satisfy the following system of equations

$$\partial_x g \cdot (h_m \cdot \partial_y h_i - h_i \cdot \partial_y h_m) = \partial_y g \cdot (h_m \cdot \partial_x h_i - h_i \cdot \partial_x h_m) \tag{10}$$

for all $i = 1, \ldots, m-1$ and

$$\partial_x g \cdot (h_m \cdot \partial_y H - H \cdot \partial_y h_m) = \partial_y g \cdot (h_m \cdot \partial_x H - H \cdot \partial_x h_m). \tag{11}$$

Theorem 2: Consider the equation

$$l_n(x,y)f^{(n)}(g(x,y)) + \ldots + l_0(x,y)f(g(x,y)) = F(x,y), \tag{12}$$

for all $(x,y) \in \Omega$, where $\Omega \subset \mathbb{R}^2$ is an open, smoothly g-connected set and $l_0, l_1, \ldots, l_n, g$ and F are given real valued, analytic functions on Ω (such that $g(\Omega)$ is an open set, $(\partial_x g, \partial_y g) \neq (0,0)$ on Ω), furthermore f is an unknown real function on $g(\Omega)$. Then there exists an ordinary differential equation

$$f^{(m)}(t) + K_{m-1}(t)f^{(m-1)}(t) + \ldots + K_0(t)f(t) = K(t). \tag{13}$$

(where $m \le n$ and $K_0, K_1, \ldots, K_{m-1}$ and K are differentiable real valued functions) whose $(n+1)$-times differentiable solutions locally coincide with that of (12).

To illustrate this idea, we present the following example:

Consider the functional equation

$$(x+y)f(x+y) - yg(x) - h(y) = x+y. \quad (14)$$

Applying the differential operator $\mathbf{D} = \partial_x - y\partial_x\partial_y$ to this equation the unknown functions g and h will be eliminated from (14) and we get the following second order differential-functional equation for f

$$f(x+y) + (x-y)f'(x+y) - (xy+y^2)f''(x+y) = 1. \quad (15)$$

If we apply the differential operator $\mathbf{D_0} = \partial_x - \partial_y$ to this equation then the order will not increase and we get

$$(x+y)f''(x+y) + 2f'(x+y) = 0. \quad (16)$$

Multiplying equation (16) by $(xy+y^2)$, equation (15) by $(x+y)$ and adding up these equations, we obtain

$$(x+y)^2 f'(x+y) + (x+y)f(x+y) = x+y. \quad (17)$$

It is easy to see that the solutions of (17) are the same as that of (15). However (17) has smaller order and with substitution $t = x+y$ it transforms to

$$t^2 f'(t) + tf(t) = t,$$

which implies that f is of the form:

$$f(t) = 1 + \frac{c}{t}.$$

Substituting f back to (14), we get

$$yg(x) + h(y) = c.$$

It is now easy to see that there exists constant d such that

$$g(x) = d, \qquad h(y) = c - dy.$$

Conversely, these functions are indeed solutions of (14).

II. The computer-program *feqsolve*

The computer-program *feqsolve* is written in the Computeralgebra-System MAPLE [1]. It can be used for solving functional-equations of type (2). The program can be started with the command:

$$feqsolve(h_0(x,y) * f_0(g_0(x,y)) + \cdots + h_n(x,y)$$
$$* f_n(g_n(x,y)) = F(x,y), [f_0, f_1, \ldots, f_n]);$$

where the first parameter of *feqsolve* is the functional-equation to be solved and the second parameter is the list of unknown functions in the functional-equation. The program solves the functional-equation for the first element of the list.

The source code of the program can be downloaded from the webpage

www.uni-miskolc.hu/ ∼ matha

or please send your request to the e-mail address:

matha@uni-miskolc.hu

[1]MAPLE is a registered trademark of Waterloo Maple software

III. A testing result of the program

Finally we give two testing result of the program *feqsolve*.

Determine the solutions of the equation $f(x+y) - g(x-y) - 4h(xy) = 0$.

```
>  feqsolve(f(x+y)-g(x-y)-4*h(x*y)=0,[h,f,g]);
```

The functional equation is

$$f(x+y) - g(x-y) - 4\,h(x\,y) = 0$$

Search for the differential operator

Search for 1st order operator

Search for 2nd order operator

Thedifferentialequationobtainedfromfunctionalequation :

$$-16\,(D^{(2)})(h)(x\,y)\,(x-y)\,(x+y) = 0$$

After dividing $-16(x^2-y^2)$ *the following differential equation is obtained*

$$(D^{(2)})(h)(x\,y) = 0$$

The differential equation obtained with the substitution $t = xy$

$$(D^{(2)})(h)(t) = 0$$

The solution of the differential equation

$$h(t) = C_1 + C_2\,t$$

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>  feqsolve(f(x+y)-g(x-y)=4*(C_1+C_2*x*y),[g,f]);
```

The functional equation is

$$f(x+y) - g(x-y) = 4\,C_1 + 4\,C_2\,x\,y$$

Search for the differential operator

Search for 1st order operator

The differential equation obtained from functional equation :

$$-2\,\mathrm{D}(g)(x-y) = -4\,C_2\,(x-y)$$

After dividing -2 *the following differential equationisobtained*

$$\mathrm{D}(g)(x-y) = 2\,C_2\,(x-y)$$

The differential equation obtained with thesubstitution $t = x-y$

$$\mathrm{D}(g)(t) = 2\,C_2\,t$$

The solution of the differential equation

$$g(t) = C_3 + C_2\,t^2$$

Finally we get $f(t) = 4C_1 + C_3 + C_2\,t^2$.

The second example: determine the solutions of the equation $(x+y)f(x+y) - yg(x) - h(y) = x+y$.

```
>   feqsolve((x+y)*f(x+y)-y*g(x)-h(y)
    =x+y,[f,g,h]);
```

The functional equation is

$$(x+y)\,\mathrm{f}(x+y) - y\,\mathrm{g}(x) - \mathrm{h}(y) = x+y$$

Search for the differential operator

Search for 1st order operator

Search for 2nd order operator

The differential functional equation obtained from functional equation :

$$\mathrm{f}(x+y) + \mathrm{D}(f)(x+y)\,x - \mathrm{D}(f)(x+y)\,y - y\,(\mathrm{D}^{(2)})(f)(x+y)\,x - (\mathrm{D}^{(2)})(f)(x+y)\,y^2 = 1$$

After transformation the following differentialequation is obtained :

$$x\,\mathrm{f}(x+y) + \mathrm{D}(f)(x+y)\,x^2 + y\,\mathrm{f}(x+y) + y^2\,\mathrm{D}(f)(x+y) + 2\,y\,x\,\mathrm{D}(f)(x+y) = x+y$$

Dividing by$(x^2 + y^2 + 2xy)$ *we get the following differential equation*

$$\mathrm{D}(f)(x+y) + \frac{\mathrm{f}(x+y)}{x+y} = \frac{1}{x+y}$$

The differential equation obtained with the substitution$t = x+y$

$$\mathrm{D}(f)(t) + \frac{\mathrm{f}(t)}{t} = \frac{1}{t}$$

The solution of the differential equation :

$$\mathrm{f}(t) = 1 + \frac{C_1}{t}$$

It is now easy to see that $g(t) = C_2$ and $h(t) = C_1 - C_2 t$.

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