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Abstract

A common generalization of earlier results on arborescence packing and the covering of intersecting bi-set families was presented by the authors in [2]. The present paper investigates the algorithmic aspects of that result and gives a polynomial-time algorithm for the corresponding optimization problem.

1 Introduction

In [2], we presented a result that joined two different directions of research on extensions of Edmonds' disjoint branchings theorem [5]. One direction, initiated by Frank [6] and pursued by Szegő [14] and by Bérczi and Frank [1], considered an abstract problem involving the covering of families of sets or bi-sets (see section 1.1 for definitions). The other direction of research, represented by the work of Katoh and Tanigawa [10], Durand de Gevigney, Nguyen, and Szigeti [4], and Cs. Király [11], involved a matroid given on the possible roots, with the requirement that roots of the branchings reaching a given node v should be a maximal independent set in the set of roots from which v is reachable. The common generalization in [2], described in detail in section 1.2, considers a digraph, bi-set families $\mathcal{F}_i, \mathcal{G}_i$ ($i = 1, \dots, t$), and a matroid on ground set $[t] = \{1, \dots, t\}$. A succinct description of the problem is that the digraph should be partitioned into arc sets A_1, \dots, A_t in such a way that for every $X \subseteq V$ the index set $\{i : A_i \text{ enters } X\} \cup \{j : X \in \mathcal{G}_j\}$ should span the index set $\{i : X \in \mathcal{F}_i \cup \mathcal{G}_i\}$ in the matroid. The paper proved that for bi-set families with certain properties the natural cut condition is sufficient, but the question of algorithmic complexity was left open.

In this paper we show that if the bi-set families are given by digraph representations, then there is a polynomial-time algorithm for finding a valid decomposition of the arc

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set. Moreover, the problem of finding a minimum cost arc subset that has a valid decomposition is also in P. This result gives the first polynomial-time algorithm for the minimum-cost version of the problem in [11].

1.1 Notation

Let $D = (V, A)$ be a directed graph (digraph, for short). The **tail** and **head** nodes of an arc a are denoted by $t(a)$ and $h(a)$, respectively. A **bi-set** is a pair $X = (X_I, X_O)$ such that $X_I \subseteq X_O \subseteq V$ where X_I and X_O are called the **inner** and the **outer set** of X , respectively. The family of all bi-sets on ground-set V is denoted by $\mathcal{P}_2(V) = \mathcal{P}_2$. The **intersection** and **union** of bi-sets can be defined in a straightforward manner: for bi-sets X and Y , we define $X \cap Y = (X_I \cap Y_I, X_O \cap Y_O)$ and $X \cup Y = (X_I \cup Y_I, X_O \cup Y_O)$. An edge $a \in A$ **enters** or **covers** a bi-set X if $h(a) \in X_I$ and $t(a) \notin X_O$, and it **leaves** X if $h(a) \notin X_O$ and $t(a) \in X_I$. A subset of edges $A' \subseteq A$ **covers** a bi-set family \mathcal{F} if each member of \mathcal{F} is covered by at least one arc in A' . The set of arcs entering (leaving) a bi-set X are denoted by $\Delta^{in}(X)$ ($\Delta^{out}(X)$), while the number of arcs entering (leaving) a bi-set X are denoted by $\varrho(X)$ ($\delta(X)$). An arc is **contained** in a bi-set X if $t(a) \in X_O$ and $h(a) \in X_I$. We say that $X \subseteq Y$ if $X_I \subseteq Y_I$ and $X_O \subseteq Y_O$. Two bi-sets are **intersecting** if $X_I \cap Y_I \neq \emptyset$. A family \mathcal{F} of bi-sets is called **intersecting** if $X, Y \in \mathcal{F}, X_I \cap Y_I \neq \emptyset$ implies $X \cap Y, X \cup Y \in \mathcal{F}$.

A **bi-set function** is a function $p : \mathcal{P}_2 \rightarrow \mathbb{R}$. A bi-set function p is called **fully supermodular** (respectively, **intersecting supermodular**) if

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$$

for every $X, Y \in \mathcal{P}_2$ (respectively, for every intersecting $X, Y \in \mathcal{P}_2$). If the reverse inequality holds, we call p **fully submodular**. A basic example for a submodular bi-set function is the in-degree function ϱ . We call p **positively intersecting supermodular** or **positively intersecting submodular** if the corresponding inequality holds whenever X and Y are intersecting and $p(X), p(Y) > 0$.

1.2 Problem definition

Motivated by results of de Geigney, Nguyen and Szigeti [4] and Cs. Király [11] on packing arborescences under matroid constraints, and by results of Frank [6], Szegő [14], and Bérczi and Frank [1] on covering intersecting set families, we considered the following problem in [2].

Let $\mathcal{M} = ([t], r)$ be a matroid. The **closure** of $I \subseteq [t]$ is denoted by $\text{Span}(I)$, that is, $\text{Span}(I) = \{i : r(I + i) = r(I)\}$. A set $I \subseteq [t]$ is called **closed** if $\text{Span}(I) = I$. Let $D = (V, A)$ be a digraph and $\mathcal{F}_1, \dots, \mathcal{F}_t$ be intersecting, $\mathcal{G}_1, \dots, \mathcal{G}_t$ be arbitrary bi-set families over \mathcal{P}_2 with the property that $\mathcal{F}_i \cap \mathcal{G}_i = \emptyset$ for each $i \in [t]$. We denote the family of bi-sets appearing in at least one of the families \mathcal{F}_i by $\mathcal{F} = \bigcup \mathcal{F}_i$.

For a bi-set $X \in \mathcal{P}_2$, let

$$\begin{aligned} I_X &= \{i : X \in \mathcal{F}_i\}, \\ J_X &= \{i : X \in \mathcal{G}_i\}. \end{aligned}$$

The disjointness of \mathcal{F}_i and \mathcal{G}_i means that $I_X \cap J_X = \emptyset$ for each bi-set $X \in \mathcal{P}_2$. We introduce the following bi-set function defined on \mathcal{P}_2 :

$$p(X) = \begin{cases} r(I_X \cup J_X) - r(J_X) & \text{if } X \in \mathcal{F}, \\ 0 & \text{otherwise.} \end{cases}$$

A bi-set X is said to be **active** if $p(X) > 0$, or equivalently $I_X \setminus \text{Span}(J_X) \neq \emptyset$. We denote the set $I_X \setminus \text{Span}(J_X)$ by I_X^{act} and say that X is **active for** i if $i \in I_X^{\text{act}}$. A bi-set X is called **tight** if $\varrho(X) = p(X) > 0$ and $I_X \neq [t]$. A bi-set is **tight for** i if it is tight and $i \notin I_X^{\text{act}}$. Note that every active or tight bi-set is in \mathcal{F} .

We say that $\mathcal{F}_1, \dots, \mathcal{F}_t, \mathcal{G}_1, \dots, \mathcal{G}_t$ satisfy the **active intersection property** if

$$\text{(AIP)} \quad X \in \mathcal{F}_i, Y \in \mathcal{F}_j, X_I \cap Y_I \neq \emptyset, i \in I_X^{\text{act}} \Rightarrow i \in I_{X \cap Y}^{\text{act}}.$$

The main result of [2] is the following.

Theorem 1. *Let $\mathcal{M} = ([t], r)$ be a matroid and $D = (V, A)$ a digraph. Let $\mathcal{F}_1, \dots, \mathcal{F}_t$ be intersecting families, $\mathcal{G}_1, \dots, \mathcal{G}_t$ arbitrary bi-set families over \mathcal{P}_2 such that $\mathcal{F}_i \cap \mathcal{G}_i = \emptyset$ for every $i \in [t]$. Assume that the active intersection property holds and*

- (a) $I_{X'} \cup J_{X'} = I_X \cup J_X$ for all $X, X' \in \mathcal{F}, X' \subseteq X$,
- (b) $\varrho(X) \geq r(I_X \cup J_X) - r(J_X)$ for all $X \in \mathcal{F}$.

Then there are pairwise disjoint arc-sets $A_1, \dots, A_t \subseteq A$ such that

$$r(J_X \cup \{i \in I_X : \varrho_{A_i}(X) \geq 1\}) = r(J_X \cup I_X) \quad (1)$$

for every $X \in \mathcal{F}$.

Let us call a family of pairwise disjoint arc-sets A_1, \dots, A_t of A that satisfies (1) a **valid arc-cover**. In this note, we consider the algorithmic aspects of finding a valid arc-cover.

The rest of the paper is organized as follows. In Section 2, a polynomial-time algorithm is given for finding a valid arc-cover. In Section 3 we show that a minimum cost valid arc-cover can be found in polynomial time via the ellipsoid method; moreover, if \mathcal{F} is an intersecting family, then there is a strongly polynomial algorithm by reduction to submodular flow.

2 Finding a valid arc-cover

Let V' be a copy of V , and identify $(X_I, X_O) \in \mathcal{P}_2(V)$ with $X_I \cup X'_O \in 2^{V \cup V'}$, where $X'_O \subseteq V'$ is the counterpart of $X_O \subseteq V$. Note that the union and intersection operations are consistent with this correspondence. Therefore, $\mathcal{F}_i \subseteq \mathcal{P}_2(V)$ can be regarded as an intersecting family of $2^{V \cup V'}$. In what follows in this section, we regard each bi-set in $\mathcal{P}_2(V)$ as a subset of $V \cup V'$. The definitions of \mathcal{F} , p , and active sets are the same as before.

To discuss polynomiality of the algorithm, we need a compact representation of each intersecting family $\mathcal{F}_i \subseteq 2^{V \cup V'}$. For $v \in V$, let $\mathcal{F}_i^v := \{X \mid v \in X \in \mathcal{F}_i\}$. Since \mathcal{F}_i^v is closed under the union and intersection operations, it is a ring family (or a distributed lattice), which can be represented by a digraph whose vertex set is a subset of $V \cup V'$ by Birkhoff's Representation Theorem [3] (see also [12]). In what follows, we assume that we are given a digraph representation of $\mathcal{F}_i^v \subseteq 2^{V \cup V'}$ for $v \in V$, and \mathcal{F}_i is given as $\bigcup_{v \in V} \mathcal{F}_i^v$. Note that the size of this representation is polynomial in $|V|$. We also assume the following:

1. \mathcal{G}_i is given as an oracle. That is, for $X \subseteq V \cup V'$, we can check whether $X \in \mathcal{G}_i$ or not.
2. The rank function r of the matroid is given as an oracle. That is, for $S \subseteq [t]$, we can compute $r(S)$.

With these assumptions, we show the following.

Proposition 2. *In the statement of Theorem 1, suppose that \mathcal{F}_i , \mathcal{G}_i , and r are given as above. Assume that Condition (a) holds. Then, we can check whether Condition (b) holds or not in polynomial time.*

Proof. Let $\mathcal{F}_+ \subseteq \mathcal{F}$ be the set of all active sets, and for $v \in V$, let $\mathcal{F}_+^v := \{\bigcup X_i \mid v \in X_i \in \mathcal{F}_+ \text{ for each } i\}$.

Claim 3. *For each $v \in V$, \mathcal{F}_+^v is a ring family, i.e., it is closed under the union and intersection operations. Furthermore, the digraph representation of \mathcal{F}_+^v can be computed in polynomial time.*

Proof. It is obvious that \mathcal{F}_+^v is closed under the union operation. For $X, Y \in \mathcal{F}_+^v$, suppose that $X = \bigcup X_i$ and $Y = \bigcup Y_j$, where $v \in X_i \in \mathcal{F}_+$ and $v \in Y_j \in \mathcal{F}_+$. By (AIP) (or [2, Proposition 2.2]), $X_i \cap Y_j \in \mathcal{F}_+$ for each i, j . Therefore,

$$X \cap Y = \bigcup_{i,j} (X_i \cap Y_j) \in \mathcal{F}_+^v,$$

which shows that \mathcal{F}_+^v is closed under the intersection operation.

Since each irreducible element in \mathcal{F}_+^v is irreducible also in \mathcal{F}_i^v for some i , by checking whether each irreducible element in \mathcal{F}_i^v is in \mathcal{F}_+ or not, we can enumerate all irreducible elements in \mathcal{F}_+^v in polynomial time. Then we can obtain the digraph representation of \mathcal{F}_+ . \square

Claim 4. *For each $v \in V$, p is supermodular on the ring family \mathcal{F}_+^v .*

Proof. Let $X, Y \in \mathcal{F}_+^v$. We consider the following three cases.

1. If $p(X) > 0$ and $p(Y) > 0$, then $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ by [2, Proposition 2.3].
2. If $p(X) = p(Y) = 0$, then it is obvious that $p(X) + p(Y) = 0 \leq p(X \cap Y) + p(X \cup Y)$.

3. Suppose that $p(X) > 0$ and $p(Y) = 0$. Since $X \cap Y \in \mathcal{F}_+^v$ by Claim 3, $X \cap Y$ can be represented as $\bigcup Z_j$, where $v \in Z_j \in \mathcal{F}_+$. By (AIP), Z_j is active for any $i \in I_X^{act}$, which implies that $X \cap Y = \bigcup Z_j \in \mathcal{F}_i \subseteq \mathcal{F}$. Then, by [2, Proposition 2.2] and Condition (a), we obtain $p(X) \leq p(X \cap Y)$. By combining this with $p(Y) = 0 \leq p(X \cup Y)$, we have $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$.

This completes the proof. \square

The digraph D can also be regarded as a bipartite digraph on node set $V \cup V'$, where every arc has its head in V and its tail in V' . It is easy to check that this is consistent with the notion of covering. Condition (b) holds if and only if the condition

$$\varrho(X) - p(X) \geq 0 \text{ for every } X \in \mathcal{F}_+^v$$

holds for every $v \in V$, and this is equivalent to $\min_{X \in \mathcal{F}_+^v} \{\varrho(X) - p(X)\} \geq 0$ for every $v \in V$. Since $\varrho - p$ is a submodular function on \mathcal{F}_+^v , we can compute this minimum value in polynomial time by submodular function minimization algorithms (see [12, 13]). Therefore, Condition (b) can be checked in polynomial time. \square

Corollary 5. *In the statement of Theorem 1, suppose that \mathcal{F}_i , \mathcal{G}_i , and r are given as above. Assume that Conditions (a) and (b) hold. Then, in polynomial time, we can find arc-sets $A_1, \dots, A_t \subseteq A$ satisfying the conditions in Theorem 1.*

Proof. For any arc $a = uv \in A$ and for any index $i \in [t]$, we can check using the digraph representation of \mathcal{F}_i^v whether the unique smallest bi-set in \mathcal{F}_i that contains a is active for i . If it is not, then we try the following procedure:

- adding a to A_i and removing it from A ;
- adding bi-sets in \mathcal{F}_i covered by a to \mathcal{G}_i ;
- deleting bi-sets covered by a from \mathcal{F}_i .

The new instance obtained by this procedure still satisfies the active intersection property because a is not contained in any bi-set active for i . The new digraph representations for \mathcal{F}_i and the oracle for \mathcal{G}_i are easy to construct using the original ones.

It is possible that the new instance does not satisfy Condition (b). However, by the results of [2], there exists a pair (a, i) such that Condition (b) still holds after this procedure, and by Proposition 2 we can check it in polynomial time. Thus we can check all pairs (a, i) and choose one where Condition (b) is satisfied in the new instance. By repeating this procedure, A becomes empty and the obtained arc sets A_1, \dots, A_t satisfy the conditions in Theorem 1. \square

Remark 6. *Note that in the case of [2, Theorem 3.2], the arcs can be chosen in such an order that each A_i is an arborescence at any given point in the algorithm. This gives an algorithmic proof of the theorem of Cs. Király.*

3 Finding a minimum cost arc-cover

The main tool used in this section is the following theorem of Frank [7], see also [8, Theorem 17.1.11].

Theorem 7 (Frank). *Let $D = (V, A)$ be a digraph, q a positively intersecting supermodular bi-set function on V , and $g \in \mathbb{Z}_+^A$ a nonnegative upper bound on the arcs. Then the system*

$$\{x \in \mathbb{R}^A : 0 \leq x_a \leq g_a \ \forall a \in A, \ \varrho_x(Z) \geq q(Z) \ \forall Z \in \mathcal{P}_2(V)\}$$

is total dual integral. If q is the nonnegative part of an intersecting supermodular bi-set function, then the system defines a submodular flow polyhedron.

We first describe a polynomial-time algorithm that uses the ellipsoid method.

Theorem 8. *Let $\mathcal{M} = ([t], r)$ be a matroid, $D = (V, A)$ a digraph and $c : A \rightarrow \mathbb{R}$ a cost function. Let $\mathcal{F}_1, \dots, \mathcal{F}_t$ be intersecting families, $\mathcal{G}_1, \dots, \mathcal{G}_t$ arbitrary bi-set families over \mathcal{P}_2 such that $\mathcal{F}_i \cap \mathcal{G}_i = \emptyset$ for every $i \in [t]$. Assume that the active intersection property holds and*

$$(a) \ I_{X'} \cup J_{X'} = I_X \cup J_X \text{ for all } X, X' \in \mathcal{F}, X' \subseteq X.$$

We can find a minimum cost arc set $A' \subseteq A$ that satisfies

$$\varrho(X) \geq p(X) \text{ for every } X \in \mathcal{F} \tag{2}$$

in polynomial time.

Proof. Let

$$P = \{x \in \mathbb{R}^A : 0 \leq x_a \leq 1 \ \forall a \in A, \ \varrho_x(Z) \geq p(Z) \ \forall Z \in \mathcal{F}\}.$$

The set function p is positively intersecting supermodular by the results of [2], so by Theorem 7 the system of inequalities defining P is TDI. As a consequence, the polyhedron P is integer, so finding a minimum cost arc-cover amounts to optimization over P . By Proposition 2, there is a polynomial-time separation algorithm for the linear system defining P , thus we can optimize on P in polynomial time using the ellipsoid method by the results of Grötschel, Lovász and Schrijver [9]. \square

It is left as an open question whether there is a strongly polynomial combinatorial algorithm for finding the minimum cost arc-cover. However, in the special case when \mathcal{F} is an intersecting family, this is possible using submodular flows.

Theorem 9. *Let $\mathcal{M} = ([t], r)$ be a matroid, $D = (V, A)$ a digraph and $c : A \rightarrow \mathbb{R}$ a cost function. Let \mathcal{F} be an intersecting family over \mathcal{P}_2 , $\mathcal{F}_1, \dots, \mathcal{F}_t$ be intersecting subfamilies of \mathcal{F} , $\mathcal{G}_1, \dots, \mathcal{G}_t$ arbitrary bi-set families over \mathcal{P}_2 such that $\mathcal{F}_i \cap \mathcal{G}_i = \emptyset$ for every $i \in [t]$. Assume that the active intersection property holds and*

$$(a) \ I_{X'} \cup J_{X'} = I_X \cup J_X \text{ for } X, X' \in \mathcal{F}, X' \subseteq X.$$

We can find a minimum cost arc set $A' \subseteq A$ that satisfies

$$\varrho(X) \geq p(X) \text{ for every } X \in \mathcal{F} \quad (3)$$

in strongly polynomial time.

Proof. The following proposition was proved in [2] (Proposition 2.2).

Proposition 10. *If $X, X' \in \mathcal{F}$ and $X' \subseteq X$, then $J_{X'} \subseteq \text{Span}(J_X)$.*

Let

$$p'(X) = \begin{cases} r(I_X \cup J_X) - r(J_X) & \text{if } X \in \mathcal{F}, \\ -\infty & \text{otherwise.} \end{cases}$$

The main advantage of \mathcal{F} being intersecting is that it implies the intersecting supermodularity of p' .

Proposition 11. *The bi-set function p' is intersecting supermodular on $\mathcal{P}_2(V)$.*

Proof. Let X and Y be intersecting bi-sets. If $p'(X)$ or $p'(Y)$ is $-\infty$ then the supermodular inequality trivially holds.

If $p'(X), p'(Y) \geq 0$ then $X, Y \in \mathcal{F}$. As \mathcal{F} is intersecting, $X \cap Y$ and $X \cup Y$ are also in \mathcal{F} . Note that Proposition 10 implies $J_{X \cap Y} \subseteq \text{Span}(J_X) \cap \text{Span}(J_Y)$. As the families \mathcal{F}_i are also intersecting, we have $I_{X \cup Y} \supseteq I_X \cap I_Y$. This, together with (a), implies $J_{X \cup Y} \subseteq J_X \cup J_Y$. Thus

$$\begin{aligned} r(J_X) + r(J_Y) &= r(\text{Span}(J_X)) + r(\text{Span}(J_Y)) \\ &\geq r(\text{Span}(J_X) \cap \text{Span}(J_Y)) + r(\text{Span}(J_X) \cup \text{Span}(J_Y)) \\ &\geq r(J_{X \cap Y}) + r(J_{X \cup Y}). \end{aligned}$$

Using this and (a) we get

$$\begin{aligned} p'(X) + p'(Y) &= r(I_X \cup J_X) - r(J_X) + r(I_Y \cup J_Y) - r(J_Y) \\ &\leq r(I_{X \cap Y} \cup J_{X \cap Y}) - r(J_{X \cap Y}) + r(I_{X \cup Y} \cup J_{X \cup Y}) - r(J_{X \cup Y}) \\ &= p'(X \cap Y) + p'(X \cup Y). \end{aligned}$$

□

As p is the nonnegative part of p' , Theorem 7 of Frank implies that the system

$$\{x \in \mathbb{R}^A : 0 \leq x_a \leq 1 \forall a \in A, \varrho_x(Z) \geq p(Z) \forall Z \in \mathcal{P}_2(V)\}$$

describes a submodular flow polyhedron. Moreover, the proof of Frank gives a construction of a corresponding submodular flow problem that in our case can be solved in strongly polynomial time, since the support of p' is given by the digraph representations. □

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