# Coloring signed graphs using DFS

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**Abstract**

We show that depth first search can be used to give a proper coloring of connected signed graphs $G$ using at most $\Delta(G)$ colors, provided $G$ is different from a balanced complete graph, a balanced cycle of odd length, and an unbalanced cycle of even length, thus giving a new, short proof to the generalization of Brooks' theorem to signed graphs, first proved by Máčajová, Raspaud, and Škoviera.

## 1 Introduction

All graphs considered in this paper are connected, finite, and simple. A signed graph is a graph where each edge is labelled with a sign that is either $+1$ or $-1$. The concept of signed graphs is due to Harary [2]. Vertex-colorings of signed graphs were introduced by Zaslavsky [5] in the following way. Let

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$G$ be a signed graph and $r \geq 0$ an integer, now a function $c : V(G) \to \{-r, -r+1, \ldots -1, 0, 1, \ldots, r\}$ is a (proper) vertex-coloring of $G$ if for each edge $e = (a, b)$ we have $c(a) \neq s(e) \cdot c(b)$, where $s(e)$ is the sign of $e$. This natural definition is an extension of the coloring of ordinary graphs, as a coloring of a signed graph with all positive signs is obviously a coloring of the corresponding ordinary graph. However, the Zaslavsky's definition of the chromatic number of a signed graph was not an extension of the chromatic number of ordinary graphs: it was defined as the smallest $r$, such that a coloring $c : V(G) \to \{-r, -r+1, \ldots, -1, 0, 1, \ldots, r\}$ exists. Actually, he defined two different chromatic numbers, depending on whether the color 0 is allowed or not. A much more natural definition of the chromatic number of signed graphs is due to Máčajová, Raspaud, and Škoviera [3, 4]: if $r = 2k$ for some $k$, then let $M_r = \{-k, -k+1, \ldots, -1, 1, 2, \ldots, k\}$, while if $r = 2k+1$ for some $k$, then let $M_r = \{-k, -k+1, \ldots, -1, 0, 1, 2, \ldots, k\}$. Now an $r$-coloring is a coloring using colors from $M_r$, and the smallest $r$, such that an $r$-coloring of the signed graph $G$ exists is called the signed chromatic number of $G$, denoted by $\chi_{\pm}(G)$. It is easy to see that this definition is an extension of the ordinary chromatic number of (ordinary) graphs.

The operation *switching* is defined as reversing the signs of the edges incident to a certain vertex. It is easy to see that switching does not change the chromatic number of a signed graph. A signed graph $G$ can be switched to a signed graph $H$ if there is a sequence of switchings applied to $G$ that results in $H$. A signed graph is called *balanced* if it can be switched to the graph with all positive signs and *unbalanced* otherwise. It is well-known [6] that a signed graph is balanced if and only if all cycles of the graph contain an even number of negative edges.

The fact that the abovementioned coloring of signed graphs is a generalization of the ordinary graph coloring makes it possible to naturally extend known coloring theorems to signed graphs. One such extension is due to Máčajová, Raspaud, and Škoviera [3, 4], they generalize the well-known theorem of Brooks [1] to signed graphs. Here we show that depth first search and greedy coloring can be used to find a proper coloring of connnected signed graphs $G$ using at most $\Delta(G)$ colors, provided $G$ is different from a balanced complete graph, a balanced cycle of odd length, and an unbalanced cycle of even length, thus giving a new, short proof to this generalized version of Brooks' theorem.

## 2 Brooks' theorem

In order to use depth first search for coloring signed graphs, first we prove that graphs that are not isomorphic to $K_n$, $K_{n,n}$, or $C_n$ for some $n$, have a DFS tree that contains a branch.

**Lemma 1.** *Assume that all DFS trees of a graph $G$ are paths starting at the vertex of DFS number 1. Then $G$ is either a cycle, a complete graph, or $G \cong K_{n,n}$ for some $n$.*

We prove Lemma 1 with the help of the following claim.

**Claim 2.** *Assume that $G$ is traceable and the terminals of any hamiltonian path of $G$ are adjacent. Then $G \cong C_n$, $G \cong K_n$, or $G \cong K_{n,n}$ for some $n$.*

*Proof.* Let $C = (v_1, v_2, \dots, v_n)$ be a hamiltonian cycle of $G$. If $G$ is not a cycle, then $G$ has some edge $(v_i, v_j)$ where $j \neq i \pm 1$ (addition is modulo $n$). As $(v_{j+1}, v_{j+2}, \dots, v_{i-1}, v_i, v_j, v_{j-1}, \dots, v_{i+1})$ is a hamiltonian path of $G$, its terminals are adjacent, hence $(v_{i+1}, v_{j+1}) \in E(G)$. Therefore $(v_{i+k}, v_{j+k}) \in E(G)$ holds for any $k$, that is, each "rotation" along $C$ of any edge of $G$ also belongs to $G$. As $(v_{i-1}, v_{j-1}, v_j, v_i, v_{i+1}, \dots, v_{j-2}, v_{i-2}, v_{i-3}, \dots, v_{j+1})$ is a hamiltonian path of $G$, its terminals are adjacent, hence $(v_{i-1}, v_{j+1}) \in E(G)$. As $(v_{i-2}, v_{i-3}, \dots, v_{j+1}, v_{i-1}, v_i, v_j, v_{j-1}, \dots, v_{i+1})$ is a hamiltonian path of $G$, its terminals are adjacent, hence $(v_{i-2}, v_{i+1}) \in E(G)$. These observations imply that two vertices must be adjacent if their indices have the same parity. If $n$ is odd, then the first observation yields that vertices with the opposite parity indices are also adjacent, so $G$ is complete. For $n$ even, we obtain that a complete bipartite graph with color classes $\{v_1, v_3, v_5, \dots\}$ and $\{v_2, v_4, v_6, \dots\}$ is a subgraph of $G$. If $G$ has any further edges, then $G$ is complete by the first two observations. □

*Proof of Lemma 1.* As any DFS tree of $G$ is a path, $G$ is traceable. If $P = (v_1, v_2, \dots, v_n)$ is a hamiltonian path of $G$, then there is a DFS tree $T$ of $G$ rooted at $v_2$ that contains the path $(v_2, v_3, \dots, v_n)$. As $T$ is a path starting at $v_2$, $(v_n, v_1) \in E(T)$, therefore the terminals of any hamiltonian path $P$ are adjacent. Now Lemma 1 immediately follows from Claim 2. □

**Definition 3.** Let $G$ be a graph and $v \in V(G)$. A *connectivity order with last vertex $v$* is an order $v_1, v_2, \dots, v_{k-1}, v_k = v$ of all vertices of $G$, such that $G[\{v_i, v_{i+1}, \dots, v_k\}]$ is connected for all $i = 1, 2, \dots, k-1$.

It is easy to see that for an arbitrary $v \in V(G)$ there exists a connectivity order with last vertex $v$: let $v_i$ be a leaf of a spanning tree of $G[V - \{v_1, \ldots, v_{i-1}\}]$ different from $v$. Now we prove two easy lemmas concerning the signed chromatic number of graphs that are not regular using greedy coloring and the connectivity order.

**Lemma 4.** *If a graph $G$ has a vertex $v$, such that $d_G(v) \leq \Delta - 1$, then $\chi_{\pm}(G) \leq \Delta$. If furthermore $d_G(v) \leq \Delta - 2$ or $G$ has another vertex $w \neq v$, such that $d_G(w) \leq \Delta - 1$, then there exists a $\Delta$-coloring of $G$, such that the color of $v$ is different from 0.*

*Proof.* A greedy coloring in a connectivity order with last vertex $v$ uses at most $\Delta$ colors: when coloring a vertex different from $v$ there may be at most $\Delta - 1$ forbidden colors, since such a vertex always has a not yet colored neighbour, and when coloring $v$ obviously there may be at most $\Delta - 1$ forbidden colors. If $d_G(v) \leq \Delta - 2$, then we have at least two options for the color of $v$, therefore the color 0 can be avoided. If there is a vertex $w \neq v$, such that $d_G(w) \leq \Delta - 1$, then let us consider a greedy coloring in a connectivity order with last vertex $w$. This also uses at most $\Delta$ colors, moreover when coloring $v$ we have at least two options, so the color 0 can be avoided in this case, as well. □

**Lemma 5.** *If a graph $G$ has two adjacent vertices $v$ and $w$, such that $d_G(v) \leq \Delta - 1$ and $G - w$ is connected, then there exists a $\Delta$-coloring of $G$, such that the color of $v$ is different from 0.*

*Proof.* Let us start a greedy coloring of $G$ by coloring the vertex $w$ with the color 0. Since $G - w$ is connected, there is a connectivity order of $G - w$ with last vertex $v$. It is straightforward that proceeding with the coloring of the vertices of $G - w$ greedily in this order we obtain a $\Delta$-coloring of $G$. In this coloring $v$ cannot have color 0, since one of its neighbours, namely $w$ has color 0. □

Now we are in a position to use depth first search to prove the following generalization of Brooks' theorem.

**Theorem 6** (Máčajová, Raspaud, Škoviera)**.** *Let $G$ be a connected signed graph, different from a balanced complete graph, a balanced cycle of odd length, and an unbalanced cycle of even length. Then $\chi_{\pm}(G) \leq \Delta(G)$.*

*Proof.* If $\Delta(G) = 0$ or $\Delta(G) = 1$, then the proposition is straightforward. If $\Delta(G) = 2$, then $G$ is either a path or a cycle. Paths have signed chromatic number at most 2, by Lemma 4. If $G$ is an even cycle, then $G$ is balanced, therefore obviously has signed chromatic number 2. If $G$ is an odd cycle, then it is unbalanced, thus can be switched to an odd cycle $(v_1, v_2, \ldots, v_{2k+1}, v_1)$ with exactly one negative edge, say the edge $(v_{2k+1}, v_1)$. Now let $c(v_{2i+1}) = 1$ and $c(v_{2i+2}) = -1$ for $i = 0, 1, \ldots, k$. It is obvious that $c$ is a proper 2-coloring of $G$.

This means that we may assume that $\Delta := \Delta(G) \geq 3$. We also may assume that $G$ is not isomorphic to $G \cong K_{n,n}$ for some $n \geq 3$, because all bipartite graphs $G = (A, B, E)$ have signed chromatic number at most 3: we assign color 0 to the vertices in $A$ and color 1 to the vertices in $B$ (notice that this is not a 2-coloring, since we used color 0).

Now if $G$ is a complete graph, then we use a simple induction, while if $G$ is not complete, then we use depth first search. Suppose first that $G$ is complete. Then $G$ is not balanced, thus $G$ has at least 3 vertices. If $G$ has exactly 3 vertices, then the proposition is easy to check. Now we use induction on the number of vertices of $G$. Let $G$ be an unbalanced complete graph on $n + 1 \geq 4$ vertices. Then $G$ has a cycle $C$ that contains an odd number of negative edges. We may suppose that $C$ is not a hamiltonian cycle. Indeed, if $C$ is a hamiltonian cycle $C = (v_1, v_2, \ldots, v_{n+1}, v_1)$ and $e = (v_1, v_i)$ is an arbitrary edge not in $C$, then one of the cycles $C' = (v_1, v_2, \ldots, v_i, v_1)$ and $C'' = (v_i, v_{i+1}, \ldots, v_{n+1}, v_1, v_i)$ contains an odd number of negative edges, and we may use this cycle instead of $C$. Let now $v$ be a vertex of $G$ not in $C$ and let us consider the graph $G'$ obtained from $G$ by deleting $v$. $G'$ is unbalanced, since it contains the cycle $C$, which has an odd number of negative edges, thus by the induction hypothesis it can be properly colored using $n-1$ colors. If $n-1$ is even, then a proper coloring of $G$ using $n$ colors is easy to give: use the coloring of $G'$ and assign color 0 to the vertex $v$ (notice that since $n - 1$ is even, the color 0 is not used in the coloring of $G'$). If $n-1$ is odd, then let $k := \frac{n}{2}$, $k$ is obviously an integer. The signed graph $G'$ is colored using the colors $-(k-1), -(k-2), \ldots, -1, 0, 1, \ldots, k-2, k-1$. If the color 0 does not appear in the coloring, then once again we simply use the coloring of $G'$ and assign color 0 to the vertex $v$. If 0 appears, then obviously it appears exactly once, say it is assigned to the vertex $w$. Now we use the coloring of $G'$, except that we assign color $k$ to $w$, instead of 0. Now if the edge $(v, w)$ is positive, we assign the color $-k$ to $v$ and if the edge $(v, w)$ is negative, we assign the color $k$ to $v$. It is easy to see that in both cases we obtain a proper coloring of $G$ with the colors $-k, -(k-1), \ldots, -1, 1, \ldots, k-1, k$, finishing the proof by induction.

Now let us turn our attention to the case when $G$ is not complete. We also know that $G$ is not a cycle or a $K_{n,n}$ and that $\Delta \geq 3$ and we may assume that all vertices have degree $\Delta$, otherwise the theorem follows directly from Lemma 4. By Lemma 1, there is a DFS tree $T$ of $G$ with a branching vertex. Let $u$ be the branching vertex of $T$ with the greatest DFS number, $u_1$ and $u_2$ children of $u$, and let $T_1$ and $T_2$ be the subtrees rooted at $u_1$ and $u_2$, respectively. Let furthermore $G_1$ and $G_2$ be the subgraphs of $G$ spanned by $V(T_1)$ and $V(T_2)$, respectively.

We show that $G_i$ can be colored with $\Delta$ colors, such that $u_i$ has an arbitrary color $c_i \neq 0$ for $i = 1, 2$. We distuingish two cases.
Case 1. There is a backward edge from $G_i$. If this edge is incident to $u_i$, then $d_{G_i}(u_i) \leq \Delta - 2$, thus by Lemma 4, $G_i$ can be colored with $\Delta$ colors, such that $u_i$ has color $c' \neq 0$. Now it is possible to change the colors $c'$ and $c_i$, and $-c'$ and $-c_i$ and obtain the desired coloring. If the backward edge is not incident to $u_i$, then there is a backward edge from some $u_i \neq w \in V(G_i)$, thus $d_{G_i}(w) \leq \Delta - 1$. Again by Lemma 4, $G_i$ can be colored with $\Delta$ colors, such that $u_i$ has color $c' \neq 0$, and the desired coloring can be obtained.
Case 2. There is no backward edge from $G_i$. Notice that in this case $d_{G_i}(u_i) = \Delta - 1 \geq 2$, since there are no cross edges either. Since $u$ is a branching vertex with a maximum DFS number, there are no branches in $T_i$ and therefore $T_i$ is a path. Now let $w$ be the child of $u_i$ in $T$ (since $d_{G_i}(u_i) \geq 2$, $w$ exists). $G_i - w$ is easily seen to be connected then and therefore by Lemma 5, the desired coloring can be obtained again.

Now we give a proper $\Delta$-coloring of $G$. Let $c$ be an arbitrary color different from 0. First let us color the vertices of $G_1$, such that $u_1$ has color $c$. Then let us color the vertices of $G_2$ using the same $\Delta$ colors (this is possible, since there are no edges between $G_1$ and $G_2$), such that $u_2$ has color $s((u, u_1))s((u, u_2))c$. It is easy to see that $G' = G[V(G) - V(G_1) - V(G_2)]$ is connected, therefore a connectivity order of $G'$ with last vertex $u$ exists. Let us continue the coloring of the vertices greedily in this order. Now $\Delta$ colors suffice indeed: the connectivity order ensures that all vertices except $u$ can be colored with one of the $\Delta$ colors and when coloring $u$ two neighbours of $u$ (namely $u_1$ and $u_2$) forbid the same color, therefore there must be at least one color left for $u$, which finishes the proof. □

## References


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