Rigid representations of multiplicative coalescents with and without deletion

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Contents

1 Introduction

??? Balazs says: the interval coalescent representation should not be stated as a main result, because we want to keep the introduction fairly short. In this section let's just refer briefly to Section 3.2

Let

$$\ell_\infty^\downarrow = \{\ \underline{m} = (m_1, m_2, \dots)\ :\ m_1 \geq m_2 \geq \dots \geq 0\ \}$$
$$\ell_1^\downarrow = \{\ \underline{m} \in \ell_\infty^\downarrow\ :\ \sum_{i=1}^\infty m_i < \infty\}, \quad \ell_2^\downarrow = \{\ \underline{m} \in \ell_\infty^\downarrow\ :\ \sum_{i=1}^\infty m_i^2 < \infty\}$$
$$\ell_0^\downarrow = \{\ \underline{m} \in \ell_\infty^\downarrow\ :\ \exists\, i_0 \in \mathbb{N}\ :\ m_i = 0 \ \text{ for any } \ i \geq i_0\ \}$$

We will use the topology of coordinate-wise convergence on $\ell_\infty^\downarrow$.

For $\underline{m}, \underline{m}' \in \ell_2^\downarrow$ we define the distance

$$\mathrm{d}(\underline{m}, \underline{m}') = \|\underline{m} - \underline{m}'\|_2 = \left(\sum_{i\geq 1}(m_i - m_i')^2\right)^{1/2}. \tag{1.1}$$ eq_def_d_metric

The metric space $\left(\ell_2^\downarrow, \mathrm{d}(\cdot,\cdot)\right)$ is complete and separable.

Definition of the multiplicative coalescent: process $\mathbf{m}(t), t \geq 0$. Markov, Aldous, natural state space is $\ell_2^\downarrow$, Feller.

Let $\lambda \in \mathbb{R}_+$. For any $\underline{m} \in \ell_2^\downarrow$ we want to define a continuous time Markov process $\mathbf{m}_t$ with state space $\ell_2^\downarrow$ where $\mathbf{m}_0 = \underline{m}$ and $\mathbf{m}_t$ represents the ordered sequence of sizes of components of a coagulation-deletion process at time t. We want the dynamics of the process $(\mathbf{m}_t)$ to satisfy

(i) two components of size m_i and m_j merge with rate $m_i \cdot m_j$,
(ii) a component of size m_i is deleted with rate $\lambda \cdot m_i$. (1.2) mcld_informal_def

We are going to call such a process a *multiplicative coalescent with linear deletion* with rate λ, and briefly denote it by MCLD(λ).

If $\underline{m} \in \ell_0^{\downarrow}$ (i.e., if the initial state has finitely many components) then the MCLD(λ) process obviously exists and $\mathbf{m}_t \in \ell_0^{\downarrow}$ for any $t \geq 0$.

For initial conditions with infinitely many blocks, the construction of MCLD(λ) is non-trivial: in Section 2.2 we will give a *graphical construction* of the process $\mathbf{m}_t$ with initial state $\underline{m} \in \ell_2^{\downarrow}$ and deletion rate λ. Our construction indeed gives rise to a well-behaved continuous-time Markov process on $\ell_2^{\downarrow}$:

⟨thm:feller_basic⟩ **Theorem 1.1** (Feller property). *Let $\underline{m}^{(n)}, n \in \mathbb{N}$ be a convergent sequence of elements of $\ell_2^{\downarrow}$ with limit $\underline{m}^{(\infty)}$, i.e., $\lim_{n\to\infty} \mathrm{d}(\underline{m}^{(n)}, \underline{m}^{(\infty)}) = 0$. For any $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+ \cup \{\infty\}$, denote by $\mathbf{m}_t^{(n)}$ the* MCLD(λ) *process with initial condition $\underline{m}^{(n)}$ at time t. For any $t \geq 0$ we have*

$$\mathbf{m}_t^{(n)} \xrightarrow{d} \mathbf{m}_t^{(\infty)}, \quad n \to \infty, \tag{1.3}$$?feller_convergence_

where $\xrightarrow{d}$ denotes convergence in distribution of random variables on the Polish space $(\ell_2^{\downarrow}, \mathrm{d}(\cdot,\cdot))$.

We will prove Theorem 1.1 in Section 2.3.

1.1 Excursions

⟨def:excursion⟩ **Definition 1.2.** Consider a c.à.d.l.à.g. function $g : [0,\infty) \to \mathbb{R} \cup \{-\infty\}$. For $0 \leq l < r < \infty$, the interval (l, r) is an *excursion above the minimum* of g if:

(i) $g(x) > g(l)$ for all $x < l$.

(ii) $r = \inf\{\, x : g(x) < g(l)\,\}$.

We say that $r - l$ is the *length* of the excursion and $g(l)$ is the *height* of the excursion. We say that the excursion in *strict* if $g(x) > g(l)$ for any $x \in (l, r)$.

From now on, we say simply “excursion” to mean excursion above the minimum, and we say “minimum” for a (strict) running minimum; i.e. g has a minimum at l if l is the left endpoint of an excursion.

⟨def:ORDX⟩ **Definition 1.3.** Suppose that for any $\varepsilon > 0$, g has only finitely many excursions with length greater than ε. Then let $\mathrm{ORDX}(g) \in \ell_\infty^{\downarrow}$ be the sequence of the lengths of the excursions of g, arranged in non-increasing order.

We write $\bar{g}$ for the function defined by

$$\bar{g}(x) = \inf_{0 \leq u \leq x} g(u). \tag{1.4}$$ eq:def_barg

Note that a c.à.d.l.à.g. function g and $\bar{g}$ have the same excursions, thus

$$\mathrm{ORDX}(g) = \mathrm{ORDX}(\bar{g}). \tag{1.5}$$ ordx_of_bar_is_ordx

1.2 Tilt operator representation of the multiplicative coalescent

Definition 1.4. Given a locally finite measure μ on $(-\infty, 0]$, we define the *inverse cumulative distribution function* $f_\mu : [0, +\infty) \to [-\infty, 0]$ of μ by

$$f_\mu(x) = \sup\{\, y \le 0 \;:\; \mu[y, 0) > x \,\}, \qquad x \ge 0. \tag{1.6}$$ def_eq_inverse_sdf_

In particular, $f_\mu(x) = -\infty$ for any $x \ge \mu(-\infty, 0)$.

Note that f_μ is non-increasing and c.à.d.l.à.g.

Given some $\underline{m} = (m_1, m_2, \dots) \in \ell_2^\downarrow$, we define the independent exponential random variables

$$E_i \sim \mathrm{Exp}(m_i),\; i = 1, 2, \dots. \tag{1.7}$$ exponentials_E_i

If $m_i = 0$, we formally define $E_i = +\infty$.

Let us define the random point measure μ_0 with point masses of weight m_i at locations $-E_i$, $i \in \mathbb{N}$:

$$\mu_0 = \sum_{i=1}^{\infty} m_i \cdot \delta_{Y_i}, \quad Y_i = -E_i. \tag{1.8}$$ mu_0_def

The total mass $\mu_0(-\infty, 0]$ is infinite if $\underline{m} \notin \ell_1^\downarrow$. However, as long as $\underline{m} \in \ell_2^\downarrow$, the mass distribution is locally finite:

⟨lemma:exponential⟩ **Lemma 1.5.** *Let $\underline{m} \in \ell_2^\downarrow$. Define μ_0 by (1.7) and (1.8). With probability 1,*

onential_i_bounded⟩? *(i) $\mu_0(A) < \infty$ for every bounded set $A \subseteq (-\infty, 0]$.*

onential_ii_sparser⟩ *(ii) $\mu_0[y, y+1] \to 0$, as $y \to -\infty$.*

The proof of Lemma 1.5 is postponed until Section 5.

_from_exponentials⟩ **Definition 1.6.** Let $\underline{m} \in \ell_2^\downarrow$. Define E_i, $i \in \mathbb{N}$ by (1.7) and μ_0 by (1.8). Let $f_0 : [0, +\infty) \to [-\infty, 0]$ be the inverse cdf of μ_0, i.e.,

$$f_0(x) \stackrel{(1.6)}{=} f_{\mu_0}(x). \tag{1.9}$$ f0def_b

⟨remark:f_0⟩ **Remark 1.7.** Let f_0 be defined by Definition 1.6.

emark_f0_i_lebesgue⟩

(i) An alternative characterization of the function f_0 is as follows: f_0 is the non-increasing c.à.d.l.à.g. function such that the interval I_j on which it takes the value $-E_j$ has length m_j, moreover the Lebesgue measure of the complement of $\cup_{j=1}^{\infty} I_j$ is zero.

emark_f0_ii_finite⟩ (ii) If $\underline{m} = (m_1, \dots, m_n) \in \ell_0^{\downarrow}$ and $E_{\sigma_1} < \cdots < E_{\sigma_n}$ is the increasing rearrangement of E_i, $1 \le i \le n$, then an equivalent way to write the function f_0 is

$$f_0(x) = \begin{cases} -E_{\sigma_k} & \text{if } \sum_{l=1}^{k-1} m_{\sigma_l} \le x < \sum_{l=1}^{k} m_{\sigma_l}, \quad 1 \le k \le n, \\ -\infty & \text{if } x \ge \sum_{i=1}^{n} m_{\sigma_l}. \end{cases} \tag{1.10}$$

f0def_for_lzeroord

??? see Figure ???

(iii) Recalling Definitions 1.2 and 1.3 we see that the excursion lengths of f_0 are given by the entries of $\underline{m}$; that is, $\mathrm{ORDX}(f_0) = \underline{m}$.

_tilt_from_f_zero⟩? **Definition 1.8.** Let f_0 be defined by Definition 1.6. Let us define

$$f_t(x) = f_0(x) + tx, \qquad x \ge 0. \tag{1.11}$$

def_eq_f_t_from_f_0

We say that the function f_t is a "tilt" of f_0.

In Lemma 6.2 we will show that, with probability 1, for all t the function f_t has only finitely many excursions with length greater than any given $\varepsilon > 0$. Then we can consider its list of excursions $\mathrm{ORDX}(f_t)$.

⟨thm:tilt⟩ **Theorem 1.9.** *Let $\underline{m} \in \ell_2^{\downarrow}$. The process $\mathrm{ORDX}(f_t), t \ge 0$ has the law of the multiplicative coalescent $\mathbf{m}_t, t \ge 0$ started from $\mathbf{m}_0 = \underline{m}$.*

We will prove Theorem 1.9 for $\underline{m} \in \ell_0^{\downarrow}$ in Section 3, and extend this result to $\underline{m} \in \ell_2^{\downarrow}$ in Section 7.

We say that Theorem 1.9 gives a "rigid" representation of the multiplicative coalescent process, because all of the randomness is contained in the initial state of the representation and the rest of the dynamics is rigid, i.e., deterministic.

1.3 Tilt & shift representation of $\mathrm{MCLD}(\lambda)$

Similarly to the rigid representation of the multiplicative coalescent in terms of the excursion lengths of $(f_t(\cdot))$ in Theorem 1.9, we will give a rigid representation of the $\mathrm{MCLD}(\lambda)$ in terms of the excursion lengths of another function $(g_t(\cdot))$ in Theorem 1.12 below. We begin with the case of finitely many blocks.

shift_step_function⟩ **Definition 1.10.** Given $\underline{m} \in \ell_0^{\downarrow}$, we define $g_0(x) \equiv f_0(x)$, were $f_0(x)$ is defined by (1.9) (or, equivalently, (1.10)). The time evolution of $g_t(\cdot)$ consists of two parts:

1. *Tilt:* If $g_t(0) < 0$ then we let $\frac{\mathrm{d}}{\mathrm{d}t} g_t(x) = \lambda + x$.
2. *Shift:* If $g_{t-}(0) = 0$, then we let $g_t(x) = g_{t-}(x + x^*(t))$, where

$$x^*(t) = \inf\{\, x > 0 \;:\; g_{t-}(x) < 0 \} \tag{1.12}$$ eq_def_finite_shift

   is the length of the *first excursion* of $g_{t-}(\cdot)$ (see Definition 1.2).

Let us define ν to be the measure on $[0, \infty)$ given by

$$\nu = \sum_{0 \le t < \infty} x^*(t) \cdot \delta_t \tag{1.13}$$ def_eq_nu_sum_dirac

where $x^*(t) > 0$ is the size of the shift to the left at time t (see (1.12)); and if no shift occurred at time t, then we let $x^*(t) = 0$. Let us also define

$$\Phi(t) = \nu[0, t], \tag{1.14}$$ def_eq_Phi_from_nu

the total amount of left shifts up to time t.

Recall the definition of the MCLD(λ) from (1.2) and the notion of ORDX from Definition 1.3.

o:tilt_and_shift_l0⟩ **Proposition 1.11.** *Let $\underline{m} \in \ell_0^{\downarrow}$, $\lambda > 0$ and let f_0 be defined by Definition 1.6. The process* $\mathrm{ORDX}(g_t), t \ge 0$ *has the law of the* MCLD(λ) *process* $\mathbf{m}_t, t \ge 0$ *started from* $\mathbf{m}_0 = \underline{m}$.

We will prove Proposition 1.11 in Section 3.

Note that in the MCLD(λ) interpretation, $\Phi(t)$ corresponds to the total amount of mass deleted up to time t.

From Definition 1.10 it follows that we have

$$g_t(x) = g_0(x + \Phi(t)) + \lambda t + \int_0^t (x + \Phi(t) - \Phi(s)) \, \mathrm{d}s. \tag{1.15}$$ g_t_from_g_0

Extending the dynamics of $g_t(x)$ for initial states in $\underline{m} \in \ell_2^{\downarrow}$ will amount to finding $\Phi(\cdot)$ corresponding to $g_0(\cdot)$. We will then define $g_t(x)$ using the formula (1.15): the question is how to define $\Phi(\cdot)$ in way that will appropriately extend Definition 1.10 from $\underline{m} \in \ell_0^{\downarrow}$ to $\underline{m} \in \ell_2^{\downarrow}$. The technical issue that we have to overcome is that for a typical $t \ge 0$ our functions $g_{t-}(\cdot)$ will not have a "first excursion" (c.f. (1.12)) if $\underline{m} \in \ell_2^{\downarrow} \setminus \ell_1^{\downarrow}$.

ension_introduction⟩ **Theorem 1.12.** *For any $\underline{m} \in \ell_2^{\downarrow}$ let us define $g_0(x) \equiv f_0(x)$, where f_0 is constructed using Definition 1.6. There exists a unique random measure ν such that if we define $\Phi(t) = \nu[0, t]$ and $g_t(x)$ by* (1.15) *then*

⟨t_shift_statement_i⟩ *(i) the process* ORDX$(g_t), t \geq 0$ *has the law of the* MCLD(λ) *process* $\mathbf{m}_t, t \geq 0$ *started from* $\mathbf{m}_0 = \underline{m}$*,*

⟨hi_stopping_time_ii⟩ *(ii) ??? prove it ??? for any* $t, x \geq 0$

$$\begin{gathered}\textit{the event } \{\, \Phi(t) \leq x \,\} \textit{ is measurable} \\ \textit{with respect to the sigma-algebra } \sigma\left(g_0(x'), 0 \leq x' \leq x\right), \end{gathered} \tag{1.16}$$

kind_of_stopping_ti

⟨g_t_zero_is_zero⟩ *(iii) if* $\underline{m} \in \ell_2^{\downarrow} \setminus \ell_1^{\downarrow}$*, then* $g_t(0) = 0$ *for any* $t \geq 0$*.*

We will prove Theorem 1.12 in Section 7.

_extend_mcld_to_l2⟩? **Remark 1.13.** (i) In words, (1.16) means that for any fixed t, the random variable $\Phi(t) := \nu[0,t]$ is a stopping time with respect to the filtration $\mathcal{F}_x := \sigma\left(g_0(x), 0 \leq x' \leq x\right), x \geq 0$.

(ii) The control function $\Phi(\cdot)$ of Theorem 1.12 is measurable with respect to the sigma-algebra $\sigma\left(g_0(x), x \geq 0\right)$ (see (1.16)), therefore we have obtained a "rigid" representation of MCLD(λ), because the function $g_t(\cdot)$ defined by (1.15) is determined by the initial state $g_0(\cdot)$.

(iii) We construct the measure ν that appears in Theorem 1.12 by extending our earlier construction given in Definition 1.10 from $\ell_0^{\downarrow}$ to $\ell_2^{\downarrow}$ in the sense that we obtain ν as the weak limit as $n \to \infty$ of the measures $\nu^{(n)}$ corresponding to initial conditions $\underline{m}^{(n)}$ truncated at index n, see Lemma 7.5 and Corollary 7.6.

(iv) If $\underline{m} \in \ell_2^{\downarrow} \setminus \ell_1^{\downarrow}$, then $g_0(\cdot)$ is a continuous function satisfying $g_0(x)/x \to -\infty$ as $x \to \infty$ (see Lemmas 6.2, 7.4). We conjecture that for such a $g_0(\cdot)$, there is a unique measure ν such that the function $g_t(x)$ defined by $\Phi(t) = \nu[0,t]$ and (1.15) satisfies $g_t(0) \equiv 0, t \geq 0$.

1.4 Context, recalling related results by others

Aldous's result and Armenariz's extension. N.B. Broutin and Marckert.

⟨proposition:Aldous⟩? **Proposition 1.14** (Aldous)**.**

⟨theorem:Armendariz⟩? **Theorem 1.15** (Armendariz)**.**

2 MCLD

We define ℓ_2^+ to be the space of square-summable sequences with non-negative entries:

$$\ell_2^+ = \left\{ x = (x_1, x_2, \dots) \, : \, \forall i \;\; x_i \geq 0, \quad \sum_{i \geq 1} x_i^2 < +\infty \right\}.$$

We have $\ell_2^\downarrow \subseteq \ell_2^+$. Define the mapping

$$\text{ord} : \ell_2^+ \to \ell_2^\downarrow \tag{2.1}$$ eq:def_ord

by letting $\text{ord}(\underline{x})$ be the decreasing rearrangement of $\underline{x} \in \ell_2^+$.

_weights_from_graph⟩ **Definition 2.1.** If $\underline{m} \in \ell_2^\downarrow$ and G is a graph with vertex set $V \subseteq \mathbb{N}_+$, denote by $\text{ord}(\underline{m}, G)$ the ordered sequence of the weights of the connected components of G. More precisely, if $\mathcal{C}_1, \mathcal{C}_2, \dots$ is the sequence of the vertex sets of the connected components of G, we define

$$\underline{x}_G = \left(\sum_{i \in \mathcal{C}_1} m_i, \sum_{i \in \mathcal{C}_2} m_i, \dots \right) \quad \text{and} \quad \text{ord}(\underline{m}, G) \overset{(2.1)}{=} \text{ord}(\underline{x}_G), \tag{2.2}$$ eq_def_ord_um_G

assuming that $\underline{x}_G \in \ell_2^+$.

Let us now state an elementary yet useful result which involves the metric $\text{d}(\cdot, \cdot)$ defined in (1.1).

raph_compare_aldous⟩ **Lemma 2.2.** *If $\underline{m} \in \ell_2^\downarrow$ and G, G' are graphs with vertex sets $V, V' \subseteq \mathbb{N}_+$ such that $V \subseteq V'$, $G \subseteq G'$ and $\text{ord}(\underline{m}, G) \in \ell_2^\downarrow$ then we have*

$$\text{d}\left(\text{ord}(\underline{m}, G), \text{ord}(\underline{m}, G')\right) \leq \sqrt{\|\text{ord}(\underline{m}, G')\|_2^2 - \|\text{ord}(\underline{m}, G)\|_2^2}.$$

Proof. This is a special case of [1, Lemma 17].

□

2.1 Basic results on the multiplicative coalescent

?⟨subsection:mc⟩? The aim of this section is to collect some basic results about the multiplicative coalescent.

??? Known, Aldous, Limic, but we want to be self-contained.

⟨def_mc_graphical⟩ **Definition 2.3.** Let $(\xi_{i,j})_{i,j=1}^{\infty}$ be independent random variables with EXP(1) distribution. Given $\underline{x} \in \ell_2^+$ let us define the simple graph G_t with vertex set $\mathbb{N}_+$ and an edge between i and j if and only if $\xi_{i,j} \leq t x_i x_j$. For $i, j \in \mathbb{N}_+$ we denote by $i \overset{G_t}{\longleftrightarrow} j$ the fact that i and j are connected by a simple path in the graph G_t.

Given G_t we recursively define the connected components $(\mathcal{C}_k(t))_{k=1}^{\infty}$ of G_t by

$$i_k = \min\{\, \mathbb{N}_+ \setminus \cup_{l=1}^{k-1} \mathcal{C}_l(t)\,\}, \quad k \geq 1 \tag{2.3}$$

eq_def_componnets_o

$$\mathcal{C}_k(t) = \{\, i \in \mathbb{N}_+ \;:\; i \overset{G_t}{\longleftrightarrow} i_k \,\}, \quad k \geq 1. \tag{2.4}$$

eq_def_componnets_o

We define $S_2^{G_t}$ to be the sum of the squares of the weights of the components of G_t:

$$S_2^{G_t} = \sum_{k=1}^{\infty} \left(\sum_{j \in \mathcal{C}_k(t)} x_j \right)^2 = S_2^{G_0} + \sum_{i \neq j} x_i x_j \mathbb{1}[i \overset{G_t}{\longleftrightarrow} j] \tag{2.5}$$

S2_def

Note that we have $S_2^{G_0} = \sum_{i=1}^{\infty} x_i^2 < +\infty$ if $\underline{x} \in \ell_2^+$.

Lemma 2.4. *For any $\underline{x} \in \ell_2^+$ and $i, j \in \mathbb{N}_+$ and $t < \frac{1}{S_2^{G_0}}$ we have*

$$\mathbf{P}\left(i \overset{G_t}{\longleftrightarrow} j \right) \leq \frac{x_i \cdot x_j \cdot t}{1 - t \cdot S_2^{G_0}}. \tag{2.6}$$

eq_i_j_conn_in_G_t

Proof.

$$\mathbf{P}\left(i \overset{G_t}{\longleftrightarrow} j \right) \leq \sum_{k=1}^{\infty} \mathbf{P}\left(\begin{array}{c} \exists\, i_0, \ldots, i_k \in \mathbb{N}_+ \;:\; i_0 = i,\, i_k = j \text{ and} \\ (i_0, i_1, \ldots, i_{k-1}, i_k) \text{ is a simple path in } G_t \end{array} \right) \leq$$
$$\sum_{k=1}^{\infty} \sum_{(i_1,\ldots,i_{k-1}) \in \mathbb{N}_+^{k-1}} \prod_{l=1}^{k} (1 - \exp(-x_{i_{l-1}} x_{i_l} t)) \leq$$
$$\sum_{k=1}^{\infty} \sum_{(i_1,\ldots,i_{k-1}) \in \mathbb{N}_+^{k-1}} \prod_{l=1}^{k} x_{i_{l-1}} x_{i_l} t = x_i x_j t \cdot \sum_{k=1}^{\infty} \sum_{(i_1,\ldots,i_{k-1}) \in \mathbb{N}_+^{k-1}} \prod_{l=1}^{k-1} x_{i_l}^2 t =$$
$$x_i x_j t \cdot \sum_{k=1}^{\infty} (t \cdot S_2^{G_0})^{k-1} = \frac{x_i \cdot x_j \cdot t}{1 - t \cdot S_2^{G_0}}. \tag{2.7}$$

eq_proof_connect_i_

□

s_not_increase_much⟩ **Lemma 2.5.** *For any* $\underline{x} \in \ell_2^+$, $t \geq 0$ *and* $i, j \in \mathbb{N}_+$, *if*

$$S_2^{G_0} \leq \frac{1}{2t} \tag{2.8}$$ eq_bound_on_initial

holds then we have

$$\mathbf{E}\left(S_2^{G_t} \right) \leq 2 S_2^{G_0} \tag{2.9}$$ small_S2_if_small_i

Proof.

$$\mathbf{E}\left(S_2^{G_t} \right) \overset{(2.5)}{=} S_2^{G_0} + \sum_{i \neq j} x_i x_j \mathbf{P}\left(i \overset{G_t}{\longleftrightarrow} j \right) \overset{(2.6),(2.8)}{\leq}$$
$$S_2^{G_0} + 2t \sum_{i \neq j} x_i^2 x_j^2 \leq S_2^{G_0} + 2t \cdot (S_2^{G_0})^2 \overset{(2.8)}{\leq} 2 S_2^{G_0}. \tag{2.10}$$ eq_proof_expect_con

□

⟨lemma_bipartite⟩ **Lemma 2.6.** *Let* $\underline{x}, \underline{y} \in \ell_2^+$ *and* $t \geq 0$. *Denote the index set of* $\underline{x}$ *by* I *and the index set of* $\underline{y}$ *by* J. *Denote by* $\alpha = \|\underline{x}\|_2^2 < +\infty$ *and* $\beta = \|\underline{y}\|_2^2 < +\infty$. *Consider the bipartite random graph* B_t *with vertex set* $I \cup J$, *where* $i \in I$ *and* $j \in J$ *are connected with probability* $1 - \exp(-t x_i y_j)$. *Then we have*

$$|I| < +\infty \quad \Longrightarrow \quad \mathbf{E}\left(S_2^{B_t} \right) < +\infty. \tag{2.11}$$ eq_bipartite_S_2_B_

Moreover, if

$$t^2 \alpha \beta \leq \frac{1}{2}, \tag{2.12}$$ eq_assumption_bipar

holds then we have

$$\mathbf{E}\left(S_2^{B_t} \right) - \alpha \leq 2\beta \cdot (1 + t\alpha)^2. \tag{2.13}$$ eq_bipartite_S_2_B_

Proof. First note that, similarly to the first step of (2.10), we have

$$\mathbf{E}\left(S_2^{B_t} \right) = \alpha + \beta + \sum_{i_1 \neq i_2 \in I} x_{i_1} x_{i_2} \mathbf{P}\left(i_1 \overset{B_t}{\longleftrightarrow} i_2 \right) +$$
$$\sum_{j_1 \neq j_2 \in J} y_{j_1} y_{j_2} \mathbf{P}\left(j_1 \overset{B_t}{\longleftrightarrow} j_2 \right) + 2 \sum_{i \in I,\, j \in J} x_i y_j \mathbf{P}\left(i \overset{B_t}{\longleftrightarrow} i \right) \tag{2.14}$$ eq_bipartite_S_2_ex

Similarly to (2.7), we obtain the inequalities

$$\mathbf{P}\left(i_1 \overset{B_t}{\longleftrightarrow} i_2\right) \leq (x_{i_1}x_{i_2}\cdot\beta\cdot t^2)\cdot\sum_{k=1}^{|I|}(t^2\alpha\beta)^{k-1}, \quad i_1\neq i_2,\ i_1,i_2\in I$$

$$\mathbf{P}\left(j_1 \overset{B_t}{\longleftrightarrow} j_2\right) \leq (y_{j_1}y_{j_2}\cdot\alpha\cdot t^2)\cdot\sum_{k=1}^{|I|}(t^2\alpha\beta)^{k-1}, \quad j_1\neq j_2,\ j_1,j_2\in J$$

$$\mathbf{P}\left(i \overset{B_t}{\longleftrightarrow} j\right) \leq (x_i y_j t)\cdot\sum_{k=1}^{|I|}(t^2\alpha\beta)^{k-1}, \quad i\in I,\ j\in J$$

Combining these inequalities with (2.14) we obtain (2.11) as well as

$$\mathbf{E}\left(S_2^{B_t}\right)-\alpha \overset{(2.12)}{\leq} \beta+2\left(\alpha^2\cdot\beta\cdot t^2+\beta^2\cdot\alpha\cdot t^2+2\alpha\cdot\beta\cdot t\right) \overset{(2.12)}{\leq} \beta\cdot\left(1+2\alpha^2t^2+1+4\alpha t\right)=2\beta\left(1+\alpha t\right)^2.$$

This completes the proof of (2.13). □

Definition 2.7. For any $\underline{x}\in\ell_2^+$ and $t\geq 0$, $m\in\mathbb{N}_+$ define the graph $G_t^{m\downarrow}$ to be the subgraph of G_t spanned by the vertex set $\{1,\dots,m\}$ and $G_t^{m\uparrow}$ to be the subgraph of G_t spanned by the vertex set $\{m+1,m+2,\dots\}$.

a:S_2_G_t_as_finite⟩ **Lemma 2.8.** *For any $t\geq 0$ and $\underline{x}\in\ell_2^+$ we have*

$$\mathbf{P}\left(S_2^{G_t}<+\infty\right)=1. \tag{2.15}$$

S_2_G_t_as_finite

In particular, for any $t\in\mathbb{R}_+$ the weights of the connected components of G_t are almost surely finite:

$$\mathbf{P}\left(\forall\, k\in\mathbb{N}_+\ :\ \sum_{i\in\mathcal{C}_k(t)}x_i<+\infty\right)=1. \tag{2.16}$$

finite_components

Proof. Given $\underline{x}\in\ell_2^+$ and $t\geq 0$ we first choose m big enough so that $S_2^{G_0^{m\uparrow}}=\sum_{i=m+1}^{\infty}x_i^2\leq\frac{1}{2t}$ holds. Then we apply Lemma 2.5 to deduce $\mathbf{E}\left(S_2^{G_t^{m\uparrow}}\right)<+\infty$. Now if we condition on the component sizes of $G_t^{m\downarrow}$ and $G_t^{m\uparrow}$, we can apply Lemma 2.6 to construct the graph G_t as B_t and use (2.11) to deduce that $S_2^{G_t}$ is almost surely finite. □

We have obtained a graphical representation of the $\ell_2^\downarrow$-valued multiplicative coalescent process with initial state $\underline{m}\in\ell_2^\downarrow$ in the form $\mathrm{ord}(\underline{m},G_t), t\geq 0$, see Definitions 2.1 and 2.3.

raphical_rep_cadlag⟩ **Lemma 2.9.** *With probability* 1*, the function* $t \mapsto \mathrm{ord}(\underline{m}, G_t)$ *is c.à.d.l.à.g. with respect to the* $\mathrm{d}(\cdot,\cdot)$*-metric (defined in* (1.1)*).*

Proof. Let us fix some $T \geq 0$. Denote by A the event

$$A = \{S_2^{G_T} < +\infty\} \cap \left\{ \begin{array}{c} \text{for any } i, j \in \mathbb{N} \text{ the number of simple} \\ \text{paths connecting } i \text{ and } j \text{ in } G_T \text{ is finite} \end{array} \right\} \quad (2.17)$$ event_A_cadlag_on_t

By Lemma 2.8 the event A almost surely holds. Assuming that A holds, we will show that $t \mapsto \mathrm{ord}(\underline{m}, G_t)$ is c.à.d.l.à.g. on $[0, T)$.

Since $G_s \subseteq G_t$ if $s \leq t$, we can apply Lemma 2.2 in order to reduce our task to showing that the function $t \mapsto S_2^{G_t}$ is c.à.d.l.à.g. on $[0, T)$. If A holds, then for any $i, j \in \mathbb{N}$ the function $t \mapsto \mathbb{1}[i \xleftrightarrow{G_t} j]$ is c.à.d.l.à.g. on $[0, T)$. Using this fact, (2.5) and the dominated convergence theorem, we obtain that indeed $t \mapsto S_2^{G_t}$ is also c.à.d.l.à.g. on $[0, T)$.

□

2.2 Graphical construction of MCLD(λ)

ubsection:deletions⟩ Recall the informal definition of the MCLD(λ) process $\mathbf{m}_t$ from (1.2). We now give a graphical construction of the process $\mathbf{m}_t$ with initial state $\underline{m} \in \ell_2^{\downarrow}$ and deletion rate λ. Let

$$\begin{array}{l} (\xi_{i,j})_{i,j=1}^{\infty} \text{ be random variables with EXP(1) distribution,} \\ (\lambda_i)_{i=1}^{\infty} \text{ be random variables with EXP}(\lambda) \text{ distribution,} \end{array} \quad (2.18)$$ exponetials_xi_lamb

and let us also assume that all of these random variables are independent.

The heuristic description of our graphical construction is as follows: we increase t and if the event $\xi_{i,j} = t m_i m_j$ occurs for some $i, j \in \mathbb{N}_+$, we merge the components of the vertices i and j, moreover if $\lambda_i = t m_i$ for some $i \in \mathbb{N}_+$, then we say that a *lightning* strikes vertex i and delete the connected component of vertex i. Since the total rate of merger and deletion events is infinite if $\underline{m} \in \ell_2^{\downarrow} \setminus \ell_1^{\downarrow}$, we need to be careful with the above heuristic definition if we want to make it precise: we will now provide the graphical construction.

In Definition 2.3 we defined the simple graph G_t with vertex set $\mathbb{N}_+$. We enumerated the connected components $\mathcal{C}_k(t), k \in \mathbb{N}_+$ of G_t in (2.3), (2.4).

We will define for any $t \in \mathbb{R}_+$

$$\begin{array}{l} \text{the set of intact vertices } \mathcal{V}_t \subseteq \mathbb{N}_+ \text{ and} \\ \text{the set of burnt vertices } \mathbb{N}_+ \setminus \mathcal{V}_t. \end{array} \quad (2.19)$$ {?}

The graph H_t will be the subgraph of G_t spanned by $\mathcal{V}_t$ and $\mathbf{m}_t$ will be the ordered sequence of component weights of H_t.

By the properties of exponential random variables, (2.16) and the independence of $(\xi_{i,j})_{i,j=1}^{\infty}$ and $(\lambda_i)_{i=1}^{\infty}$, we see that for every $t \geq 0$

$$\mathbf{P}\left(\forall k \in \mathbb{N}_+ : \sum_{i \in \mathcal{C}_k(t)} \mathbb{1}[\lambda_i \leq t m_i] < +\infty \right) = 1. \tag{2.20}$$ finitely_many_light

In words: every connected component of G_t is exposed to only finitely many lightning strikes on $[0,t]$.

Equation (2.20) implies that for every $t \geq 0$ and $k \in \mathbb{N}_+$, there exists an almost surely finite $\mathbb{N}$-valued random variable N (the number of lightnings that hit the component $\mathcal{C}_k(t)$ by time t), indices $i_1, \dots, i_N \subseteq \mathcal{C}_k(t)$ (the vertices that are hit by lightning) and times $0 < t_1 < \cdots < t_N \leq t$ (the ordered sequence of the times of the lightnings) such that

$$\{\, i \in \mathcal{C}_k(t) \;:\; \lambda_i \leq t m_i \,\} = \{\, i_1, \dots, i_N \,\} \quad \text{and} \quad \forall\, 1 \leq l \leq N \;:\; t_l = \frac{\lambda_{i_l}}{m_{i_l}}.$$

We now define the set of intact vertices $\mathcal{V}_t \subseteq \mathbb{N}_+$ by constructing $\mathcal{V}_t \cap \mathcal{C}_k(t)$ for every $k \in \mathbb{N}_+$.

Let us fix $k \in \mathbb{N}_+$. We recursively define $\mathcal{V}_{t_l} \cap \mathcal{C}_k(t)$ for each $1 \leq l \leq N$ in the following way.

(i) At $t_0 = 0$ we have $\mathcal{V}_{t_0} \cap \mathcal{C}_k(t) = \mathcal{C}_k(t)$.

(ii) Assume that we have already constructed $\mathcal{V}_{t_{l-1}} \cap \mathcal{C}_k(t)$ for some $1 \leq l \leq N$. We define $\mathcal{V}_{t_l} \cap \mathcal{C}_k(t)$ by deleting the connected component of i_l in the restriction of the graph G_{t_l} to the vertex set $\mathcal{V}_{t_{l-1}} \cap \mathcal{C}_k(t)$.

(iii) With this recursion we define $\mathcal{V}_{t_N} \cap \mathcal{C}_k(t)$. Since there are no lightnings hitting $\mathcal{C}_k(t)$ between t_N and t, let $\mathcal{V}_t \cap \mathcal{C}_k(t) = \mathcal{V}_{t_N} \cap \mathcal{C}_k(t)$.

Since $\mathcal{C}_k(t), k \in \mathbb{N}_+$ is a partition of $\mathbb{N}_+$, we define

$$\begin{aligned} &\mathcal{V}_t = \textstyle\bigcup_{k \geq 1} \mathcal{V}_t \cap \mathcal{C}_k(t) \text{ and} \\ &H_t \text{ to be the subgraph of } G_t \text{ spanned by } \mathcal{V}_t. \end{aligned} \tag{2.21}$$ {?}

Recalling Definition 2.1 we let

$$\mathbf{m}_t = \mathrm{ord}(\underline{m}, H_t). \tag{2.22}$$ def_bm_t_graphical

We define $S_2^{H_t}$ to be the sum of the squares of the weights of the components of H_t, that is $S_2^{H_t} = \|\mathbf{m}_t\|_2^2$.

⟨lemma_indeed_MCLD⟩? **Lemma 2.10.** *For any* $\underline{m} \in \ell_2^{\downarrow}$ *the graphical construction* (2.22) *of the process* $\mathbf{m}_t$ *gives an* $\mathrm{MCLD}(\lambda)$ *process with initial condition* $\underline{m}$, *i.e., an* $\ell_2^{\downarrow}$*-valued Markov process whose dynamics satisfy the informal definition given in* (1.2).

Proof. $\mathbf{m}_t$ is a random element of $\ell_2^{\downarrow}$, because we have

$$\|\mathbf{m}_t\|_2^2 = S_2^{H_t} \leq S_2^{G_t} \overset{(2.15)}{<} +\infty.$$

The fact that $\mathbf{m}_t$ is a Markov process with the prescribed transition rates follows from the memoryless property and independence of the random variables $(\xi_{i,j})_{i,j=1}^{\infty}$ and $(\lambda_i)_{i=1}^{\infty}$. □

raphical_rep_cadlag⟩ **Lemma 2.11.** *With probability* 1, *the function* $t \mapsto \mathrm{ord}(\underline{m}, H_t)$ *is c.à.d.l.à.g. with respect to the* $\mathrm{d}(\cdot,\cdot)$*-metric (defined in* (1.1)*).*

Proof. Let us fix some $T \geq 0$. We know that the event A defined in (2.17) almost surely holds. Denote by B the event that every connected component of G_T is exposed to only finitely many lightning strikes on $[0, T]$. By (2.20), the event B occurs almost surely. Assuming that $A \cap B$ holds, we will show that $t \mapsto \mathrm{ord}(\underline{m}, H_t)$ is c.à.d.l.à.g. on $[0, T)$.

For any $t \geq 0$ we define $\hat{H}_{t+\Delta t}$ to be the graph that we obtain by restricting G_t to $\mathcal{V}_{t+\Delta t}$. Define $\check{H}_{t+\Delta t}$ to be the graph that we obtain by restricting $G_{t+\Delta t}$ to $\mathcal{V}_t$. With these definitions the inclusions

$$\hat{H}_{t+\Delta t} \subseteq H_t \subseteq \check{H}_{t+\Delta t}, \qquad \hat{H}_{t+\Delta t} \subseteq H_{t+\Delta t} \subseteq \check{H}_{t+\Delta t}$$

hold, so we can apply Lemma 2.2 in order to reduce our task of proving right-continuity at t to showing that

$$\text{(a)} \lim_{\Delta t \to 0} S_2^{\check{H}_{t+\Delta t}} - S_2^{H_t} = 0, \qquad \text{(b)} \lim_{\Delta t \to 0} S_2^{H_t} - S_2^{\hat{H}_{t+\Delta t}} = 0.$$

Now (a) follows from the fact that the graphical representation of the multiplicative coalescent possesses the c.à.d.l.à.g. property (see Lemma 2.9).

In order to show (b) we observe that on the event B, for every connected component $\mathcal{C}$ of G_T, we have

$$\lim_{\Delta t \to 0} \mathbb{1}[\, \exists\, i \in \mathcal{C} \;:\; t m_i < \lambda_i \leq (t + \Delta t) m_i \,] = 0.$$

Given this observation, we see that for every connected component $\mathcal{C}$ of H_t we have $\lim_{\Delta t \to 0} \mathbb{1}[\, \mathcal{C} \subseteq \mathcal{V}_{t+\Delta t} \,] = 1$. Using this fact and the dominated convergence theorem we obtain (b).

The proof of the existence of left limits is similar and we omit it. □

2.3 Proof of Feller property

ller_coupling_proof⟩ **Definition 2.12.** The graphical construction of Section 2.2 gives a joint realization of all of the MLCD(λ) processes with different initial conditions by using the same collection of random variables $(\xi_{i,j})_{i,j\in\mathbb{N}}$ and $(\lambda_i)_{i\in\mathbb{N}}$ (see (2.18)). We call this coupling the (ξ,λ)-coupling.

⟨thm:feller⟩ **Theorem 2.13.** *Let $\underline{m}^{(n)}, n \in \mathbb{N}$ be a convergent sequence of elements of $\ell_2^{\downarrow}$ and let $\underline{m}^{(\infty)}$ denote their limit, i.e., $\lim_{n\to\infty} \mathrm{d}(\underline{m}^{(n)}, \underline{m}^{(\infty)}) = 0$. For any $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+ \cup \{\infty\}$, denote by $\mathbf{m}_t^{(n)}$ the* MCLD(λ) *process with initial condition $\underline{m}^{(n)}$ at time t. Under the (ξ,λ)-coupling, we have*

$$\mathrm{d}(\mathbf{m}_t^{(n)}, \mathbf{m}_t^{(\infty)}) \xrightarrow{p} 0, \quad n \to \infty. \tag{2.23}$$ feller_convergence_

This statement in particular implies that the MCLD(λ) Markov process indeed possesses the Feller property, i.e., Theorem 1.1 holds.

Proof. Let us fix $t \geq 0$, the sequence $\underline{m}^{(n)}, n \in \mathbb{N}$ and the limit $\underline{m}^{(\infty)}$. We prove Theorem 2.13 using truncation. For any $n \in \mathbb{N}_+ \cup \{\infty\}$, let $\mathbf{m}_t^{(n,m)}$ denote the realization under the (ξ,λ)-coupling of the MCLD(λ) with initial state

$$\underline{m}^{(n,m)} = (m_1^{(n)}, \dots, m_m^{(n)}, 0, 0, \dots), \text{ where } \underline{m}^{(n)} = (m_1^{(n)}, m_2^{(n)}, \dots). \tag{2.24}$$ truncation_def

We also define $\mathcal{V}_t^{(n,m)}$ to be the set of intact vertices of the graph $H_t^{(n,m)}$ of the MCLD(λ) with initial state $\underline{m}^{(n,m)}$ under the (ξ,λ)-coupling.

In order to prove (2.23) we only need to show that for every $\varepsilon > 0$ there exists $m, n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\mathbf{P}\left(\mathrm{d}(\mathbf{m}_t^{(n)}, \mathbf{m}_t^{(n,m)}) \geq \varepsilon\right) \leq 4\varepsilon, \tag{2.25}$$ X^n_and_X^nm_close

$$\mathbf{P}\left(\mathrm{d}(\mathbf{m}_t^{(n,m)}, \mathbf{m}_t^{(\infty,m)}) \geq \varepsilon\right) \leq \varepsilon, \tag{2.26}$$ X^nm_and_X^inftym_c

$$\mathbf{P}\left(\mathrm{d}(\mathbf{m}_t^{(\infty,m)}, \mathbf{m}_t^{(\infty)}) \geq \varepsilon\right) \leq 4\varepsilon. \tag{2.27}$$ X^inftym_and_X^inft

For the rest of Section 2.3, we will fix $t \geq 0$ and omit the dependence of random variables on t:

⟨def_G_truncated⟩ **Definition 2.14.** Let us fix $t \geq 0$. Given some $\underline{m} \in \ell_2^{\downarrow}$, denote by G, $G^{m\downarrow}$ and $G^{m\uparrow}$ the graphs spanned by the edges $\mathbb{1}[\xi_{i,j} \leq t m_i m_j]$ on the vertex set $\mathbb{N}_+$, $\{1, \dots, m\}$, and $\{m+1, m+2, \dots\}$, respectively.

Let $\underline{m}^{(m)}$ denote the vector truncated at index m. Denote by $\mathbf{m}$ the state of the MCLD(λ) process with initial state $\underline{m}$ at time t and by $\mathbf{m}^{(m)}$ the state of the MCLD(λ) process with initial state $\underline{m}^{(m)}$ at time t.

Denote by $\mathcal{V}$ and $\mathcal{V}^{(m)}$ the corresponding sets of intact vertices.
Denote by $H^{(m)}$ and H the subgraphs of G spanned by $\mathcal{V}^{(m)}$ and $\mathcal{V}$.

In order to prove (2.25) and (2.27) we will need the following result.

ry_graph_inclusions⟩ **Lemma 2.15.** *If $\hat{G}^{(m)}$ and $\check{G}^{(m)}$ are random graphs with vertex sets*

$$V(\hat{G}^{(m)}), V(\check{G}^{(m)}) \subseteq \mathbb{N}_+$$

and under the (ξ, λ)-coupling we have

$$\hat{G}^{(m)} \subseteq H^{(m)} \subseteq \check{G}^{(m)}, \quad \hat{G}^{(m)} \subseteq H \subseteq \check{G}^{(m)} \tag{2.28}$$

G_nm_H_nm_inclusion

then almost surely we have

$$\mathrm{d}(\mathbf{m}, \mathbf{m}^{(m)}) \leq 3 \cdot \sqrt{S_2^{\check{G}^{(m)}} - S_2^{\hat{G}^{(m)}}}. \tag{2.29}$$

sandwich

Proof. First note that it follows from (2.28) that

$$S_2^{\hat{G}^{(m)}} \leq S_2^{H^{(m)}} \leq S_2^{\check{G}^{(m)}}, \quad S_2^{\hat{G}^{(m)}} \leq S_2^{H} \leq S_2^{\check{G}^{(m)}}. \tag{2.30}$$

G_nm_H_nm_S_2_ineqs

Thus we have

$$\begin{aligned}
\mathrm{d}(\mathbf{m}, \mathbf{m}^{(m)}) &\overset{(2.2)}{\leq} \mathrm{d}(\mathbf{m}, \mathrm{ord}(\underline{m}, \check{G}^{(m)})) + \\
&\mathrm{d}(\mathrm{ord}(\underline{m}, \check{G}^{(m)}), \mathrm{ord}(\underline{m}^{(m)}, \hat{G}^{(m)})) + \mathrm{d}(\mathrm{ord}(\underline{m}^{(m)}, \hat{G}^{(m)}), \mathbf{m}^{(m)}) \overset{(*)}{\leq} \\
&\sqrt{S_2^{\check{G}^{(m)}} - S_2^{H}} + \sqrt{S_2^{\check{G}^{(m)}} - S_2^{\hat{G}^{(m)}}} + \sqrt{S_2^{H^{(m)}} - S_2^{\hat{G}^{(m)}}} \overset{(2.30)}{\leq} \\
&\qquad 3 \cdot \sqrt{S_2^{\check{G}^{(m)}} - S_2^{\hat{G}^{(m)}}},
\end{aligned}$$

where $(*)$ follows from (2.22), the inclusions (2.28) and Lemma 2.2. This concludes the proof of Lemma 2.15. □

We will now construct auxiliary graphs $\hat{G}^{(m)}$ and $\check{G}^{(m)}$ in such a way that (2.28) holds. Recall Definition 2.14. Note that $H^{(m)}$ is the subgraph of $G^{m\downarrow}$ spanned by the vertex set $\mathcal{V}^{(m)}$. In particular, every connected component of $H^{(m)}$ is a subset of a connected component of $G^{m\downarrow}$.

The next definition only involves the random variables $(\xi_{i,j})_{i,j=1}^{\infty}$ only (i.e., we don't have to look at $(\lambda_i)_{i=1}^{\infty}$).

_bipartite_parallel⟩ **Definition 2.16.** Given $G^{m\downarrow}$ and $G^{m\uparrow}$ denote the connected components of $G^{m\downarrow}$ by $\mathcal{C}_k^{m\downarrow}, k \in K$ and the connected components of $G^{m\uparrow}$ by $\mathcal{C}_l^{m\uparrow}, l \in L$.

Let us define an auxiliary bipartite graph $\mathcal{B}$ with vertex set $K \cup L$. Declare $k \in K$ and $l \in L$ connected in $\mathcal{B}$ if $\mathcal{C}_k^{m\downarrow}$ is connected to $\mathcal{C}_l^{m\uparrow}$ in G. We allow *parallel* edges to be present in $\mathcal{B}$: if $\mathcal{C}_k^{m\downarrow}$ is connected to $\mathcal{C}_l^{m\uparrow}$ by more than one edge in G, then we put an equal number of parallel edges between $k \in K$ and $l \in L$.

Now we define a subset $K^* \subseteq K$ indexing "bad" components of $G^{m\downarrow}$. This definition involves the random variables $(\xi_{i,j})_{i,j=1}^{\infty}$ as well as $(\lambda_i)_{i=1}^{\infty}$.

d_components_feller⟩ **Definition 2.17.** Declare $k \in K^*$ if there is a path in $\mathcal{B}$ with no repeated edges which consists of at least one edge and connects k to a vertex in $K \cup L$ which corresponds to a not entirely intact connected component of $G^{m\downarrow}$ or $G^{m\uparrow}$ at time t, i.e., a lightning hit a vertex of that component before time t.

Now we state some useful properties of the "good" components of $G^{m\downarrow}$ indexed by $K \setminus K^*$ whose straightforward proof we omit.

Claim 2.18.

(i) If $k \in K \setminus K^*$ and $\mathcal{C}_k^{m\downarrow}$ is hit by a lightning before t then there are no parallel edges connected to k in $\mathcal{B}$ and no circle of the graph $\mathcal{B}$ contains k as a vertex.

(ii) Recalling Definition 2.14, we have

$$\forall\, k \in K \setminus K^* \;:\; \mathcal{C}_k^{m\downarrow} \cap \mathcal{V}^{(m)} = \mathcal{C}_k^{m\downarrow} \cap \mathcal{V}. \tag{2.31}$$

good_components_ide

Now we define auxiliary random graphs $\hat{G}^{(m)}$ and $\check{G}^{(m)}$ that satisfy (2.28).

_sandwich_m_graphs⟩? **Definition 2.19.** Let $\check{G}^{(m)}$ be the subgraph of G spanned by the vertices

$$V(\check{G}^{(m)}) = \left(\bigcup_{k \in K \setminus K^*} \mathcal{C}_k^{m\downarrow} \cap \mathcal{V}^{(m)} \right) \cup \left(\bigcup_{k \in K^*} \mathcal{C}_k^{m\downarrow} \right) \cup \{m+1, m+2, \dots\}. \tag{2.32}$$

eq_def_major_G_m

Define $\hat{G}^{(m)}$ to be the subgraph of G spanned by the vertices

$$V(\hat{G}^{(m)}) = \bigcup_{k \in K \setminus K^*} \mathcal{C}_k^{\downarrow m} \cap \mathcal{V}^{(m)}. \tag{2.33}$$

eq_def_minor_G_m

We now show that with these definitions the inclusions (2.28) hold. The inclusions $V(\hat{G}^{(m)}) \subseteq \mathcal{V}^{(m)} \subseteq V(\check{G}^{(m)})$ follow from the definitions (2.32), (2.33). Thus $\hat{G}^{(m)} \subseteq H^{(m)} \subseteq \check{G}^{(m)}$ follows from the fact that $H^{(m)}$ is the subgraph of G spanned by the vertex set $\mathcal{V}^{(m)}$. The inclusions $\hat{G}^{(m)} \subseteq H \subseteq \check{G}^{(m)}$ follow from the observation (2.31) and the fact that H is the subgraph of G spanned by the vertex set $\mathcal{V}$.

The next lemma is quite similar to Lemma 2.6.

a_minor_major_close⟩ **Lemma 2.20.** *Given the above set-up let us condition on the graphs $G^{m\downarrow}$ and $G^{m\uparrow}$ and denote by*

$$\alpha = S_2^{G^{m\downarrow}}, \quad \beta = S_2^{G^{m\uparrow}}.$$

There exists a constant $C = C(\lambda, t)$ such that if $t^2\alpha\beta \le \frac{1}{2}$ holds then we have

$$\mathbf{E}\left(S_2^{\check{G}^{(m)}} - S_2^{\hat{G}^{(m)}} \,\middle|\, G^{m\downarrow},\, G^{m\uparrow} \right) \le C\cdot\beta\cdot\left((1+t\alpha)^2 + (1+t\alpha)\cdot\alpha^{3/2}\right). \quad (2.34)$$?ineq_minor_major_cl

We postpone the proof of Lemma 2.20 until Section 2.3.1. Now we finish the proof of Theorem 2.13 by showing that for every $\varepsilon > 0$ there exists $m, n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have (2.25),(2.26),(2.27).

Let us fix $t, \lambda \in \mathbb{R}_+$, $\varepsilon > 0$. We know from Lemma 2.8 that

$$\mathbf{P}\left(S_2^{G_t^{(\infty)}} < +\infty \right) = 1,$$

where $G_t^{(\infty)}$ denotes the random graph constructed from the exponential variables $(\xi_{i,j})_{i,j=1}^{\infty}$ and the initial state $\underline{m}^{(\infty)} \in \ell_2^{\downarrow}$ according to the rules described in Definition 2.3. Given $\varepsilon > 0$, we can find $M \in \mathbb{R}_+$ such that

$$\mathbf{P}\left(S_2^{G_t^{(\infty)}} \ge M - 1 \right) \le \varepsilon. \quad (2.35)$$ eq_def_M

Recal the notion of the constant $C = C(t, \lambda)$ from Lemma 2.20. Let us choose $\delta > 0$ such that

$$t^2 M\delta \le \frac{1}{2} \quad \text{and} \quad 9C\cdot\delta\cdot\left((1+tM)^2 + (1+tM)\cdot M^{3/2}\right) = \varepsilon^3 \quad (2.36)$$ choice_of_delta

holds. Now we choose the truncation threshold m. Since $\underline{m}^{(n)} \to \underline{m}^{(\infty)}$ in l_2, we can make

$$\sup_{n\in\mathbb{N}\cup\{\infty\}} \|\underline{m}^{(n)} - \underline{m}^{(n,m)}\|_2$$

(where $\underline{m}^{(n,m)}$ is defined in (2.24)) as small as we wish by making m large. Thus by (2.9) and the Markov inequality we can choose m such that

$$\forall\, n \in \mathbb{N}\cup\{\infty\} \;:\; \mathbf{P}\left(S_2^{G_t^{(n,m)\uparrow}} \ge \delta \right) \le \varepsilon. \quad (2.37)$$ remaining_small_S2

Having m fixed, we note that under the (ξ, λ)-coupling we have

$$\mathrm{d}(\mathbf{m}_t^{(n,m)}, \mathbf{m}_t^{(\infty,m)}) \xrightarrow{p} 0, \quad n \to \infty.$$

We also have

$$S_2^{G_t^{(n,m)\downarrow}} \xrightarrow{p} S_2^{G_t^{(\infty,m)\downarrow}} \le S_2^{G_t^{(\infty)}}, \tag{2.38}$$

limsup_truncated_S2

thus we can choose n_0 such that for all $n \ge n_0$ we have (2.26) and

$$\forall\, n \in \{n_0, n_0+1, \dots\} \cup \{\infty\} \;:\; \mathbf{P}\left(S_2^{G_t^{(n,m)\downarrow}} \ge M \right) \overset{(2.35),(2.38)}{\le} 2\varepsilon. \tag{2.39}$$

uniform_truncated_S

Now we are ready to show (2.25) and (2.27).

For any $n \in \{n_0, n_0+1, \dots\} \cup \{\infty\}$ we have

$$\begin{aligned}\mathbf{P}\left(\mathrm{d}(\mathbf{m}_t^{(n)}, \mathbf{m}_t^{(n,m)}) \ge \varepsilon \right) &\le \mathbf{P}\left(S_2^{G_t^{(n,m)\uparrow}} \ge \delta \right) + \mathbf{P}\left(S_2^{G_t^{(n,m)\downarrow}} \ge M \right) + \\ &\mathbf{P}\left(\mathrm{d}(\mathbf{m}_t^{(n)}, \mathbf{m}_t^{(n,m)}) \ge \varepsilon,\ S_2^{G_t^{(n,m)\uparrow}} \le \delta,\ S_2^{G_t^{(n,m)\downarrow}} \le M \right) \overset{(2.37),(2.39)}{\le} \\ &\qquad \varepsilon + 2\varepsilon + \mathbf{P}\left(\mathrm{d}(\mathbf{m}_t^{(n)}, \mathbf{m}_t^{(n,m)}) \ge \varepsilon,\ A \right),\end{aligned}$$

where $A = \{S_2^{G_t^{(n,m)\uparrow}} \le \delta,\ S_2^{G_t^{(n,m)\downarrow}} \le M\}$. We bound

$$\begin{aligned}\mathbf{P}\left(\mathrm{d}(\mathbf{m}_t^{(n)}, \mathbf{m}_t^{(n,m)}) \ge \varepsilon,\ A \right) &\overset{(2.29)}{\le} \mathbf{P}\left(9 \cdot \left(S_2^{\check{G}^{(n,m)}} - S_2^{\hat{G}^{(n,m)}} \right) \ge \varepsilon^2,\ A \right) = \\ &\mathbf{E}\left(\mathbf{P}\left(9 \cdot \left(S_2^{\check{G}^{(n,m)}} - S_2^{\hat{G}^{(n,m)}} \right) \ge \varepsilon^2 \,\middle|\, G^{(n,m)\downarrow},\ G^{(n,m)\uparrow} \right) \;;\; A \right) \\ &\overset{(*)}{\le} \frac{9C \cdot \delta \cdot \left((1+tM)^2 + (1+tM) \cdot M^{3/2} \right)}{\varepsilon^2} \overset{(2.36)}{\le} \varepsilon,\end{aligned}$$

where in the equation marked by $(*)$ we used Lemma 2.20 and the Markov inequality. This concludes the proof of (2.25),(2.26),(2.27) and Theorem 2.13, given the statement of Lemma 2.20. □

2.3.1 Proof of Lemma 2.20

proof_lemma_min_maj⟩ We fixed $t \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}_+$. Recall Definition 2.14. Note that $H^{(m)}$ is the subgraph of $G^{m\downarrow}$ spanned by the vertex set $\mathcal{V}^{(m)}$. In particular, every connected component of $H^{(m)}$ is subset of a connected component of $G^{m\downarrow}$.

Recall the bipartite graph $\mathcal{B}$ from Definition 2.16 and the set of "bad" vertices K^* from Definition 2.17. Recall the property (2.31) of "good" components.

$\check{G}^{(m)}$ is the subgraph of G spanned by the vertices $V(\check{G}^{(m)})$, see (2.32).

$\hat{G}^{(m)}$ is the subgraph of G spanned by the vertices $V(\hat{G}^{(m)})$, see (2.33).

Given the above set-up we conditioned on the graphs $G^{m\downarrow}$ and $G^{m\uparrow}$ and denoted by

$$\alpha = S_2^{G^{m\downarrow}}, \quad \beta = S_2^{G^{m\uparrow}}.$$

In order to prove Lemma 2.20 we need to show that there exists a constant $C = C(\lambda, t)$ such that if

$$t^2 \alpha \beta \leq \frac{1}{2} \tag{2.40}$$

t_2_alpha_beta_leq_

holds, then we have

$$\mathbf{E}\left(S_2^{\check{G}^{(m)}} - S_2^{\hat{G}^{(m)}} \,\middle|\, G^{m\downarrow},\ G^{m\uparrow} \right) \leq C \cdot \beta \cdot \left((1 + t\alpha)^2 + (1 + t\alpha) \cdot \alpha^{3/2} \right). \tag{2.41}$$

truncation_uniform_

For any subset $\mathcal{C}$ of $\mathbb{N}$, denote by

$$w(\mathcal{C}) = \sum_{i \in \mathcal{C}} m_i$$

the weight of the subset, where $\underline{m} = (m_1, m_2, \dots)$.

Definition 2.21. Define a bipartite weighted graph $\widetilde{\mathcal{B}}$ whose "left" vertices correspond to the connected components of the restriction of G to the vertex set

$$\widetilde{V}^{(m)} := V(\check{G}^{(m)}) \cap \{1, \dots, m\} = \left(\bigcup_{k \in K \setminus K^*} \mathcal{C}_k^{m\downarrow} \cap \mathcal{V}^{(m)} \right) \cup \left(\bigcup_{k \in K^*} \mathcal{C}_k^{m\downarrow} \right),$$

and the "right" vertices correspond to the components of $G^{m\uparrow}$. Define the weights of the vertices of $\widetilde{\mathcal{B}}$ to be the $w(\cdot)$-weight of the corresponding connected components. We declare two vertices in $\widetilde{\mathcal{B}}$ to be connected if the corresponding subsets are connected in $\check{G}^{(m)}$. Denote by $\widetilde{G}^{(m)}$ the subgraph of G spanned by $\widetilde{V}^{(m)}$.

With the above notation we have

$$S_2^{\check{G}^{(m)}} = S_2^{\widetilde{\mathcal{B}}}, \qquad S_2^{\widetilde{G}^{(m)}} = S_2^{\hat{G}^{(m)}} + \sum_{k \in K^*} w(\mathcal{C}_k^{m\downarrow})^2.$$

Thus we can start to rewrite the left-hand side of (2.41):

$$\mathbf{E}\left(S_2^{\check{G}^{(m)}} - S_2^{\hat{G}^{(m)}} \,\middle|\, G^{m\downarrow},\ G^{m\uparrow} \right) = \\ \mathbf{E}\left(S_2^{\widetilde{\mathcal{B}}} - S_2^{\widetilde{G}^{(m)}} \,\middle|\, G^{m\downarrow},\ G^{m\uparrow} \right) + \mathbf{E}\left(\sum_{k \in K^*} w(\mathcal{C}_k^{m\downarrow})^2 \,\middle|\, G^{m\downarrow},\ G^{m\uparrow} \right).$$

In order to show (2.41), it is enough to prove that $t^2\alpha\beta \leq \frac{1}{2}$ implies

$$\mathbf{E}\left(S_2^{\widetilde{\mathcal{B}}} - S_2^{\widetilde{G}(m)} \,\middle|\, G^{m\downarrow},\ G^{m\uparrow} \right) \leq 2\beta \cdot (1+t\alpha)^2, \tag{2.42}$$

truncation_uniform_

$$\mathbf{E}\left(\sum_{k \in K^*} w(\mathcal{C}_k^{m\downarrow})^2 \,\middle|\, G^{m\downarrow},\ G^{m\uparrow} \right) \leq 2t^2\lambda\beta \cdot (1+t\alpha) \cdot \alpha^{3/2}. \tag{2.43}$$

truncation_uniform_

First we deduce (2.42) from Lemma 2.6, with the underlying bipartite graph being $\widetilde{\mathcal{B}}$. Note that the condition (2.12) holds, because $S_2^{\widetilde{G}(m)} \leq S_2^{G^{m\downarrow}} = \alpha$ and $\beta = S_2^{G^{m\uparrow}}$. Thus we have

$$\mathbf{E}\left(S_2^{\widetilde{\mathcal{B}}} - S_2^{\widetilde{G}(m)} \,\middle|\, G^{m\downarrow},\ G^{m\uparrow},\ (\lambda_i)_{i=1}^m \right) \overset{(2.13)}{\leq} 2S_2^{G^{m\uparrow}} \cdot (1+tS_2^{\widetilde{G}(m)})^2 \leq 2\beta \cdot (1+t\alpha)^2.$$

Now (2.42) follows by averaging over the values of $(\lambda_i)_{i=1}^m$.

In order to prove (2.43), we first give an upper bound on the probability of the event $\{k \in K^*\}$. For $k \in K$, denote by $x_k' = w(\mathcal{C}_k^{m\downarrow})$ and for $l \in L$, denote $y_l' = w(\mathcal{C}_l^{m\uparrow})$. Note that we have

$$\alpha = \sum_{k \in K} (x_k')^2, \qquad \beta = \sum_{l \in L} (y_l')^2.$$

Recall the definition of K^* from Definition 2.17. The next calculation is similar to (2.7), so we omit the first few steps:

$$\begin{aligned}
&\mathbf{P}\left(k \in K^* \,\middle|\, G^{m\downarrow},\ G^{m\uparrow} \right) \leq \\
&\quad \sum_{l_1 \in L} \left(x_k' y_{l_1}' t\right) \left(\lambda y_{l_1}' t\right) + \sum_{l_1 \in L} \sum_{k_1 \in K} \left(x_k' y_{l_1}' t\right) \left(y_{l_1}' x_{k_1}' t\right) \left(\lambda x_{k_1}' t\right) + \\
&\quad \sum_{l_1 \in L} \sum_{k_1 \in K} \sum_{l_2 \in L} \left(x_k' y_{l_1}' t\right) \left(y_{l_1}' x_{k_1}' t\right) \left(x_{k_1}' y_{l_2}' t\right) \left(\lambda y_{l_2}' t\right) + \cdots = \\
&\quad x_k' t^2 \lambda\beta + x_k' t^3 \lambda\alpha\beta + x_k' t^4 \lambda\alpha\beta^2 + \cdots = \\
&\quad x_k' t^2 \lambda\beta \cdot (1+t\alpha) \cdot \sum_{n=0}^{\infty} \left(t^2\alpha\beta\right)^n \overset{(2.40)}{\leq} 2x_k' t^2 \lambda\beta \cdot (1+t\alpha)\,.
\end{aligned}$$

Now we are ready to prove (2.43):

$$\mathbf{E}\left(\sum_{k \in K^*} (x_k')^2 \,\middle|\, G^{m\downarrow},\ G^{m\uparrow} \right) \leq \sum_{k \in K} 2(x_k')^3 t^2 \lambda\beta \cdot (1+t\alpha) \overset{(*)}{\leq} 2t^2\lambda\beta \cdot (1+t\alpha) \cdot \alpha^{3/2},$$

where in $(*)$ we used the fact that $x_k' \leq \sqrt{\alpha}$ for any $k \in K$. This completes the proof of (2.41) and Lemma 2.20.

3 Rigid representation results: finite state space

_finite_state_space⟩

3.1 Construction of size-biased sequences using independent exponential random variables

⟨sec:expsizebiased⟩? Now we recall some useful definitions from [1, Section 3.3].

ze_biased_for_lzero⟩ **Definition 3.1.** Let $\underline{m} = (m_1, m_2, \dots, m_n) \in \ell_0^{\downarrow}$. A random total linear ordering $\prec$ on $[n]$ is a *size-biased ordering* (with respect to $\underline{m}$) if for each permutation $i_1, i_2, \dots, i_n$ of $[n]$,

$$\mathbb{P}(i_1 \prec i_2 \prec \cdots \prec i_n) = \prod_{r=1}^{n} \frac{m_{i_r}}{m_{i_r} + m_{i_{r+1}} + \cdots + m_{i_n}}. \tag{3.1}$$

sizebiased

d_with_exponentials⟩ **Definition 3.2.** Suppose $\underline{m} \in \ell_\infty^{\downarrow}$. Let $E_i \sim \mathrm{Exp}(m_i)$ independently for each i. Define a random linear ordering on $\mathbb{N}$ by $i \prec j$ if and only if $E_i < E_j$. Then the law of $\prec$ is size-biased (with respect to the sizes m_i), in the sense that equation (3.1) holds for any finite subset of the index set $\mathbb{N}$.

⟨dense_size_biased⟩ e_biased_extensions⟩ **Remark 3.3.** (i) There is a smallest element with respect to the order $\prec$ if and only if $\underline{m} \in \ell_1^{\downarrow}$. If $\underline{m} \in \ell_\infty^{\downarrow} \setminus \ell_1^{\downarrow}$ then the values E_i are dense in $\mathbb{R}_+$, see Lemma 5.1.

tial_size_biased_ii⟩

(ii) The excursions (see Definition 1.2) of the random function f_0 defined in Definition 1.6 appear in size-biased order.

3.2 A size-biased interval representation of $\mathrm{MCLD}(\lambda)$

val_coalescent_repr⟩ In this section we restrict the $\mathrm{MCLD}(\lambda)$ process $(\mathbf{m}_t)$ to the state space $\ell_0^{\downarrow}$. Recall that the dynamics of $(\mathbf{m}_t)$ consist of coalescence and deletion:

- if $\underline{m}, \underline{m}' \in \ell_0^{\downarrow}$ where $\underline{m}'$ arises from $\underline{m}$ by merging two blocks of size m_I and m_J then the rate of this transition is $\mathcal{R}_{MC}(\underline{m}, \underline{m}') = m_I m_J$
- if $\underline{m}, \underline{m}' \in \ell_0^{\downarrow}$ where $\underline{m}'$ arises from $\underline{m}$ by deleting a block of size m_I then the rate of this transition is $\mathcal{R}_{MC}(\underline{m}, \underline{m}') = \lambda m_I$

⟨def:ic_state_space⟩ **Definition 3.4.** Denote by $\Omega^b = \cup_n \mathbb{R}_+^n$ the space of non-negative finite sequences. Given $\underline{b} = (b_1, \dots, b_n) \in \Omega^b$, let $\Psi(\underline{b}) \in \ell_0^{\downarrow}$ denote the reordering of $\underline{b}$ into non-increasing order.

We now define a process $\mathbf{b}_t$, $t \geq 0$, where only neighbouring intervals are allowed to merge and only the leftmost interval is allowed to be deleted.

interval_coalescent⟩ **Definition 3.5** (Interval coalescent with linear deletion, ICLD(λ))**.** The state space of the continuous-time Markov process $(\mathbf{b}_t)$ is Ω^b. The dynamics consist of coalescence and deletion:

(i) If $\underline{b}, \underline{b}' \in \Omega^b$ where $\underline{b}' \in \mathbb{R}_+^{n-1}$ arises from $\underline{b} \in \mathbb{R}_+^n$ by merging the blocks b_k and b_{k+1} for some $1 \leq k < n$; that is

$$b'_i = \begin{cases} b_i, & i = 1, 2, \ldots, k-1 \\ b_k + b_{k+1}, & i = k \\ b_{i+1}, & i = k, \ldots, n-1 \end{cases} \tag{3.2}$$?icchange?

then the rate of the transition from $\underline{b}$ to $\underline{b}'$ is

$$\mathcal{R}_{IC}(\underline{b}, \underline{b}') = b_k \cdot \sum_{l>k} b_l. \tag{3.3}$$ eq:merge_rate_ic

(ii) If $\underline{b}, \underline{b}' \in \Omega^b$ where $\underline{b}' \in \mathbb{R}_+^{n-1}$ arises from $\underline{b} \in \mathbb{R}_+^n$ by deleting the leftmost block, i.e.,

$$b'_i = b_{i+1}, \qquad 1 \leq i \leq n-1$$

then the rate of this transition is

$$\mathcal{R}_{IC}(\underline{b}, \underline{b}') = \lambda \cdot \sum_{l} b_l. \tag{3.4}$$ eq:deletion_rate_ic

:intervalcoalescent⟩ **Theorem 3.6.** *Let $\underline{m} \in \ell_0^{\downarrow}$. Let $\mathbf{b}_0$ be a random size-biased reordering (see Definition 3.1) of $\underline{m}$. Let $\mathbf{b}_t, t \geq 0$ be an* ICLD(λ) *started from the initial state $\mathbf{b}_0$. Then the law of the process $\Psi(\mathbf{b}_t), t \geq 0$ is that of the* MCLD(λ) *process $\mathbf{m}_t, t \geq 0$ started from $\mathbf{m}_0 = \underline{m}$.*

We will prove Theorem 3.6 in Section 3.2.1.

rval_coalescent_l1⟩? **Remark 3.7.** Theorem 3.6 generalizes in a natural way to any initial condition $\underline{m} \in \ell_1^{\downarrow}$. For initial conditions with infinite total mass, the situation is more complicated since under the natural extension of the concept of size-biased order (see Definition 3.2) there is no smallest element of the order, and any two elements are separated by infinitely many other elements in the order, c.f. Remark 3.3(i).

3.2.1 Proof of Theorem 3.6

_intervalcoalescent⟩ For $\underline{m} \in \ell_0^{\downarrow}$ let $\pi_{\underline{m}}$ denote the probability distribution on Ω^b (see Definition 3.4) which arises from the size-biased reordering of $\underline{m}$:

$$\pi_{\underline{m}}(\underline{b}) = \mathbb{1}[\,\underline{b} \in \Psi^{-1}(\underline{m})\,] \cdot \prod_{i=1}^{n} \frac{b_i}{\sum_{j=i}^{n} b_j}, \quad \underline{b} = (b_1, \dots, b_n). \tag{3.5}$$ eq:def_pi_sizebiase

⟨lemma_mc_ic⟩ **Lemma 3.8.** *For every $\underline{m} \in \ell_0^{\downarrow}$ and $\underline{b}' \in \Omega^b$ we have*

$$\sum_{\underline{b} \in \Psi^{-1}(\underline{m})} \pi_{\underline{m}}(\underline{b}) \mathcal{R}_{IC}(\underline{b}, \underline{b}') = \mathcal{R}_{MC}(\underline{m}, \underline{m}') \pi_{\underline{m}'}(\underline{b}'), \text{ where } \underline{m}' = \Psi(\underline{b}'). \tag{3.6}$$ ic_mc_identity

lary_coal_induction⟩ **Corollary 3.9.** *If* $\mathbf{P}\left(\mathbf{b}_0 = \underline{b} \,\middle|\, \Psi(\mathbf{b}_0) = \underline{m}\right) = \pi_{\underline{m}}(\underline{b})$ *for any* $\underline{m} \in \ell_0^{\downarrow}$ *and* $\underline{b} \in \Omega^b$ *and* $(\mathbf{b}_t)$ *is an* ICLD(λ) *then the process* $(\Psi(\mathbf{b}_t))$ *is an* MCLD(λ)*, moreover for any* $t \geq 0$ *we have* $\mathbf{P}\left(\mathbf{b}_t = \underline{b} \,\middle|\, \Psi(\mathbf{b}_t) = \underline{m}\right) = \pi_{\underline{m}}(\underline{b})$ *for any* $\underline{m} \in \ell_0^{\downarrow}$ *and* $\underline{b} \in \Omega^b$*. In particular, Theorem 3.6 follows.*

In words: if the initial state of an ICLD(λ) is size-biased, then for any $t \geq 0$ it remains size-biased. First we derive Corollary 3.9 from Lemma 3.8, then we prove Lemma 3.8.

Proof of Corollary 3.9. Any possible initial state of $\Psi(\mathbf{b}_0)$ is in $\ell_0^{\downarrow}$, so if we fix the initial state then the set of reachable states of our Markov chain $(\mathbf{b}_t)$ becomes finite and we are allowed to prove Corollary 3.9 by induction on the number of jumps of our Markov process $(\mathbf{b}_t)$.

Note that if $\mathbf{b}_.$ jumps then $\Psi(\mathbf{b}_.)$ also jumps. We assumed that $\mathbf{b}_0$ is size-biased. Now assume that, before a jump, we have $\Psi(\mathbf{b}_.) = \underline{m}$, and also assume that the distribution of the state of $\mathbf{b}_.$ is a size-biased version of the state $\underline{m}$, i.e., we have $\mathbf{b}_. \sim \pi_{\underline{m}}$. We want to show that

⟨ic_induction_1⟩ (i) the holding time until the next jump of $\mathbf{b}_.$ is exponentially distributed with parameter $\widetilde{\mathcal{R}}_{MC}(\underline{m}) := \sum_{\underline{m}'} \mathcal{R}_{MC}(\underline{m}, \underline{m}')$,

⟨ic_induction_2⟩ (ii) the probability that $\Psi(\mathbf{b}_.)$ jumps to $\underline{m}'$ is $\mathcal{R}_{MC}(\underline{m}, \underline{m}')/\widetilde{\mathcal{R}}_{MC}(\underline{m})$,

⟨ic_induction_3⟩ (iii) after $\Psi(\mathbf{b}_.)$ jumps to $\underline{m}'$, the state of the process $\mathbf{b}_.$ is again a size-biased version of the state $\underline{m}'$, i.e., we have $\mathbf{b}_. \sim \pi_{\underline{m}'}$.

Now (i)-(ii) is exactly how MCLD(λ) is supposed to behave, moreover by (iii) we are allowed to inductively repeat this argument, so as soon as we show (i)-(iii), the proof of Corollary 3.9 will follow. We will now prove (i)-(iii).

Proof of (i): note that for any $\underline{b} \in \Psi^{-1}(\underline{m})$ the total jump rate is

$$\widetilde{\mathcal{R}}_{IC}(\underline{b}) := \sum_{\underline{b}'} \mathcal{R}_{IC}(\underline{b}, \underline{b}') \overset{(3.3),(3.4)}{=} \sum_k b_k \cdot \sum_{l>k} b_l + \lambda \cdot \sum_l b_l = \sum_{I<J} m_I m_J + \lambda \cdot \sum_I m_I = \sum_{\underline{m}'} \mathcal{R}_{MC}(\underline{m}, \underline{m}') = \widetilde{\mathcal{R}}_{MC}(\underline{m}), \quad (3.7) \ \texttt{total\_rate\_ic\_mc}$$

therefore (i) holds if $\mathbf{b}. \sim \pi_{\underline{m}}$, since $\pi_{\underline{m}}$ is supported on $\Psi^{-1}(\underline{m})$.

Proof of (ii): the probability that $\Psi(\mathbf{b}.)$ jumps from $\underline{m}$ to $\underline{m}'$ is

$$\sum_{\underline{b}' \in \Psi^{-1}(\underline{m}')} \sum_{\underline{b} \in \Psi^{-1}(\underline{m})} \pi_{\underline{m}}(\underline{b}) \frac{\mathcal{R}_{IC}(\underline{b}, \underline{b}')}{\widetilde{\mathcal{R}}_{IC}(\underline{b})} \overset{(3.6),(3.7)}{=} \frac{\mathcal{R}_{MC}(\underline{m}, \underline{m}')}{\widetilde{\mathcal{R}}_{MC}(\underline{m})}.$$

Proof of (iii): conditioned on the event that $\Psi(\mathbf{b}.)$ jumps from $\underline{m}$ to $\underline{m}'$, the probability that $\mathbf{b}.$ is in state $\underline{b}'$ is

$$\left(\sum_{\underline{b} \in \Psi^{-1}(\underline{m})} \pi_{\underline{m}}(\underline{b}) \frac{\mathcal{R}_{IC}(\underline{b}, \underline{b}')}{\widetilde{\mathcal{R}}_{IC}(\underline{b})} \right) \cdot \left(\frac{\mathcal{R}_{MC}(\underline{m}, \underline{m}')}{\widetilde{\mathcal{R}}_{MC}(\underline{m})} \right)^{-1} \overset{(3.6),(3.7)}{=} \pi_{\underline{m}'}(\underline{b}').$$

□

Proof of Lemma 3.8. There are two cases that we have to handle, corresponding to coalescence and deletion.

We first treat the case of coalescence, that is we assume that the state $\Psi(\underline{b}') = \underline{m}' \in \ell_0^{\downarrow}$ arises from $\underline{m} \in \ell_0^{\downarrow}$ by merging some blocks m_I and m_J of $\underline{m}$, where $I \neq J$. Now $\underline{b}'$ has an interval of size $m_I + m_J$, say $b'_k = m_I + m_J$. There are exactly two elements $\underline{b}$ of $\Psi^{-1}(\underline{m})$ for which $\mathcal{R}_{IC}(\underline{b}, \underline{b}') > 0$, namely

$$\underline{b}^1 = (b'_1, \ldots, b'_{k-1}, m_I, m_J, b'_{k+1}, \ldots, b'_n), \quad (3.8) \ \{?\}$$

$$\underline{b}^2 = (b'_1, \ldots, b'_{k-1}, m_J, m_I, b'_{k+1}, \ldots, b'_n). \quad (3.9) \ \{?\}$$

Let us rewrite the two sides of (3.6). The left-hand side is

$$\sum_{\underline{b}\in\Psi^{-1}(\underline{m})} \pi_{\underline{m}}(\underline{b})\mathcal{R}_{IC}(\underline{b},\underline{b}') = \pi_{\underline{m}}(\underline{b}^1)\mathcal{R}_{IC}(\underline{b}^1,\underline{b}') + \pi_{\underline{m}}(\underline{b}^2)\mathcal{R}_{IC}(\underline{b}^2,\underline{b}') \overset{(3.3),(3.5)}{=}$$

$$\left(\prod_{i=1}^{k-1}\frac{b'_i}{\sum_{j=i}^n b'_j}\right)\frac{m_I}{\sum_{j=k}^n b'_j}\frac{m_J}{\sum_{j=k}^n b'_j - m_I}\cdot$$
$$\left(\prod_{i=k+1}^{n}\frac{b'_i}{\sum_{j=i}^n b'_j}\right)\cdot m_I\cdot\left(\sum_{j=k}^n b'_j - m_I\right)+$$
$$\left(\prod_{i=1}^{k-1}\frac{b'_i}{\sum_{j=i}^n b'_j}\right)\frac{m_J}{\sum_{j=k}^n b'_j}\frac{m_I}{\sum_{j=k}^n b'_j - m_J}\cdot$$
$$\left(\prod_{i=k+1}^{n}\frac{b'_i}{\sum_{j=i}^n b'_j}\right)\cdot m_J\cdot\left(\sum_{j=k}^n b'_j - m_J\right).$$

The right-hand side is

$$\mathcal{R}_{MC}(\underline{m},\underline{m}')\pi_{\underline{m}'}(\underline{b}') \overset{(3.5)}{=} m_I m_J\cdot\left(\prod_{i=1}^{k-1}\frac{b'_i}{\sum_{j=i}^n b'_j}\right)\frac{m_I+m_J}{\sum_{j=k}^n b'_j}\left(\prod_{i=k+1}^{n}\frac{b'_i}{\sum_{j=i}^n b'_j}\right).$$

Therefore (3.6) holds in the case of coalescence.

We now turn to the case of deletion, that is we assume that the state $\Psi(\underline{b}') = \underline{m}' \in \ell_0^\downarrow$ arises from $\underline{m} \in \ell_0^\downarrow$ by deleting some block m_I of $\underline{m}$. There is one element $\underline{b}$ of $\Psi^{-1}(\underline{m})$ for which $\mathcal{R}_{IC}(\underline{b},\underline{b}') > 0$, namely

$$\underline{b}^0 = (m_I, b'_1,\ldots,b'_n). \tag{3.10}$$ {?}

Thus (3.6) holds in the case of deletion:

$$\sum_{\underline{b}\in\Psi^{-1}(\underline{m})} \pi_{\underline{m}}(\underline{b})\mathcal{R}_{IC}(\underline{b},\underline{b}') = \pi_{\underline{m}}(\underline{b}^0)\mathcal{R}_{IC}(\underline{b}^0,\underline{b}') \overset{(3.4),(3.5)}{=}$$

$$\frac{m_I}{m_I+\sum_{j=1}^n b'_j}\pi_{\underline{m}'}(\underline{b}')\cdot\lambda\cdot\left(m_I+\sum_{j=1}^n b'_j\right) =$$
$$\lambda m_I\cdot\pi_{\underline{m}'}(\underline{b}') = \mathcal{R}_{MC}(\underline{m},\underline{m}')\pi_{\underline{m}'}(\underline{b}')$$

This completes the proof of (3.6), and Lemma 3.8. □

3.3 Tilt and shift representation of ICLD(λ), finitely many blocks

?⟨tiltproofl1⟩? The representation of the multiplicative coalescent in Section 3.2 moved some of the randomness of the process into the choice of an initial condition (using a size-biased ordering). Thereafter the possible transitions of the process were restricted (only neighbouring blocks are allowed to merge).

In this section we take this to an extreme by giving natural constructions of the process in which *all* the randomness is in the initial condition; the evolution of the process thereafter is entirely deterministic, but nonetheless the process projects to the multiplicative coalescent. Such processes might be called "rigid".

Recall the notion of excursions from Definition 1.2 and the notion of $\Omega^b = \cup_n \mathbb{R}_+^n$ and $\Psi : \Omega^b \to \ell_0^{\downarrow}$ from Definition 3.4.

?⟨def:EX⟩? **Definition 3.10.** Assume $g : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ has only finitely many excursions. Denote by

$$\mathrm{EX}(g) \in \Omega^b$$

the sequence of the lengths of the excursions of g, in order of appearance. Denote by

$$\mu(g) = \sum_i m_i \cdot \delta_{Y_i} \tag{3.11}$$ mu_of_g

the point measure with a Dirac mass m_i at location Y_i where m_i is the length and the Y_i is the height of excursion i of g.

def_exp_measure_law⟩ **Definition 3.11.** Let $\underline{m} = (m_1, m_2, \dots) \in \ell_0^{\downarrow}$. We say that the random point measure $\mu = \sum_i m_i \cdot \delta_{Y_i}$ has $\mathcal{E}(\underline{m})$ distribution if $-Y_i = E_i \sim \mathrm{Exp}(m_i)$ and $E_1, E_2, \dots$ are independent.

We have $\mu_0 \sim \mathcal{E}(\underline{m})$, where μ_0 is defined in (1.8).

p:tilt_and_shift_ic⟩ **Proposition 3.12.** *Let $\underline{m} \in \ell_0^{\downarrow}$. Define the process $g_t(\cdot), t \geq 0$ by Definition 1.10 where $g_0(\cdot) = f_0(\cdot)$ is defined by Definition 1.6. Then*

⟨t_s_ic_a⟩ *(a) the process $\mathbf{b}_t := \mathrm{EX}(g_t), t \geq 0$ has the law of the* ICLD(λ) *process, as defined in Definition 3.5;*

⟨t_s_ic_b⟩ *(b) for any $t \geq 0$, the conditional distribution of $\mu(g_t)$ given $\mathbf{m}_t := \mathrm{ORDX}(g_t)$ is $\mathcal{E}(\mathbf{m}_t)$.*

Proof. Let $\underline{m} = (m_1, \dots, m_n)$. Recalling Remark 1.7(ii), the function g_0 is a non-increasing c.à.d.l.à.g. function with values $-E_1, \dots, -E_m$ and $-\infty$. The length of the interval on which it takes the value $-E_i$ is m_i. By (1.7), with probability 1 the values E_i are distinct, hence $\underline{m} = \mathrm{ORDX}(g_0)$.

Let $\sigma = (\sigma_1, \dots, \sigma_n)$ be the random permutation of $[n]$ expressing the ordering of the random variables E_i, so that $E_{\sigma_1} < \dots < E_{\sigma_n}$. Then

$$\mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_n) := (m_{\sigma_1}, \dots, m_{\sigma_n}) = \mathrm{EX}(g_0) \overset{(*)}{\sim} \pi_{\underline{m}},$$

where $(*)$ holds by Remark 3.3(ii), c.f. (3.5).

Recalling Definition 1.2, the minima of g_0 are located at the points

$$x_0 = 0, \quad x_1 = \mathbf{b}_1, \quad x_2 = \mathbf{b}_1 + \mathbf{b}_2, \quad \dots \quad x_n = \mathbf{b}_1 + \dots + \mathbf{b}_n, \tag{3.12}$$ eq:def:minima_x_k

and the values of g_0 at these points are

$$g_0(x_0) = -E_{\sigma_1}, \quad \dots \quad g_0(x_{n-1}) = -E_{\sigma_n}, \quad g_0(x_n) = -\infty.$$

For convenience we also formally write $g_0(x_{-1}) = 0$.

Let $\underline{b} \in \Psi^{-1}(\underline{m})$. Conditional on the event $\mathrm{EX}(g_0) = \underline{b}$, what is the joint distribution of $g_0(x_k), 0 \le k < n$?

By repeated use of the memoryless property of the exponential distribution, one finds that the conditional joint distribution of the height gaps between successive minima of g_0 is

$$g_0(x_{k-1}) - g_0(x_k) \sim \mathrm{Exp}(b_{k+1} + \dots + b_n), \text{ independently for } 0 \le k < n. \tag{3.13}$$ heightgap

We are now going to prove part (a) of Proposition 3.12. For the proof of (a), we will fix $\underline{b}$ and condition all probabilities on the event $\mathrm{EX}(g_0) = \underline{b}$. We will show that if g_0 has distribution (3.13) then the process

$$\mathbf{b}_t := \mathrm{EX}(g_t), t \ge 0$$

evolves as an ICLD(λ) process with $\mathbf{b}_0 = \underline{b}$, c.f. Definition 3.5. Denote by

$$\tau_k = \min\{\, t \;:\; g_t(x_{k-1}) - g_t(x_k) = 0 \,\}, \quad 0 \le k < n, \qquad \tau = \min_{0 \le k < n} \tau_k. \tag{3.14}$$ eq_tau_gaps_close

By the *Tilt* part of Definition 1.10, $\frac{\mathrm{d}}{\mathrm{d}t} g_t(x) = \lambda + x$. Hence, for any $t < \tau$,

⟨speed_1⟩ (i) $g_t(x_{k-1}) - g_t(x_k)$ decreases at speed $x_k - x_{k-1} = b_k$ if $0 < k < n$,

⟨speed_2⟩ (ii) $g_t(x_{-1}) - g_t(x_0) = -g_t(0)$ decreases at speed λ.

Also note that for any $t < \tau$ we have $\mathrm{EX}(g_t) = \mathrm{EX}(g_0) = \underline{b}$. If we define

$$\tau_k^* = \frac{g_0(x_{k-1}) - g_0(x_k)}{b_k}, \quad 0 < k < n, \quad \tau_0^* = \frac{g_t(x_{-1}) - g_t(x_0)}{\lambda}, \tag{3.15}$$ eq:def_tau_star

then by (i)-(ii) above and (3.14) we see that $\min_{0 \le k < n} \tau_k^* = \tau$.

Using (3.13) and (3.15) we obtain that

$$\begin{gathered} \tau_k^* \sim \mathrm{Exp}\left(b_k \textstyle\sum_{l>k} b_l\right), \quad 0 < k < n, \qquad \tau_0^* \sim \mathrm{Exp}\left(\lambda \textstyle\sum_l b_l\right), \\ \tau_0^*, \dots, \tau_{n-1}^* \quad \text{are independent.} \end{gathered} \tag{3.16}$$ eq:tau_star_exponen

Let us define K to be the random index for which $\tau_K = \tau_K^* = \tau$ holds.

(i) If $0 < K < n$ then at time $t = \tau$ the point x_K is no longer a minimum, therefore the excursions (x_{K-1}, x_K) and (x_K, x_{K+1}) merge into an excursion on the interval (x_{K-1}, x_{K+1}), thus $\mathrm{EX}(g_\tau) = \mathbf{b}'$ where

$$\mathbf{b}'_i = \begin{cases} b_i, & i = 1, 2, \dots, K-1 \\ b_K + b_{K+1}, & i = K \\ b_{i+1}, & i = K+1, \dots, n-1 \end{cases} \tag{3.17}$$ case1:excursions_me

(ii) If $K = 0$ then $g_{\tau-}(0) = 0$, moreover by (1.12) and (3.12) we have $x^*(\tau_-) = x_1 = b_1$, so by the *Shift* part of Definition 1.10, we obtain $g_\tau(x) = g_{\tau-}(x + b_1)$, thus $\mathrm{EX}(g_\tau) = \mathbf{b}'$ where

$$\mathbf{b}'_i = b_{i+1}, \qquad 1 \le i \le n-1. \tag{3.18}$$ case2:excursion_del

As a result of (3.16), (3.17) and (3.18) we see that

- $\tau = \min_k \tau_k^*$ and $K = \mathrm{argmin}_k \tau_k^*$ are independent,
- the rate of the exponential variable τ is exactly the same as the rate of the first jump in an ICLD(λ) process with $\mathbf{b}_0 = \underline{b}$ (see (3.7)),
- the probability distribution on the possible transitions from $\mathrm{EX}(g_{\tau-}) = \underline{b}$ to $\mathrm{EX}(g_\tau) = \mathbf{b}'$ arising from the probability distribution of K via (3.17) & (3.18) above exactly coincides with the probability distribution on the possible transitions at the time of the first jump in an ICLD(λ) process with $\mathbf{b}_0 = \underline{b}$ (see Definition 3.5).

Thus the process $\mathrm{EX}(g_t)$ behaves like an ICLD(λ) up to (and including) the first jump. Now we show that after τ the situation is very similar to (3.13), see (3.19) below.

Denoting $\mathrm{EX}(g_\tau) = \mathbf{b}'$, analogously to (3.12), we define

$$x'_0 = 0, \quad x'_1 = \mathbf{b}'_1, \quad x'_2 = \mathbf{b}'_1 + \mathbf{b}'_2, \quad \dots \quad x'_{n-1} = \mathbf{b}'_1 + \cdots + \mathbf{b}'_{n-1},$$

so that x'_i are the locations of the minima of g_τ.

Given τ and K, what is the conditional joint distribution of the height gaps between successive minima of g_τ?

Applying the memoryless property of the exponential distribution, the conditional distribution of the remaining height gaps (apart from the one which has already reached 0) is unchanged from what it was at time 0. As a result we have that

$$g_{\tau-}(x_{k-1}) - g_{\tau-}(x_k) \sim \mathrm{Exp}(b_{k+1} + \cdots + b_n)$$
$$\text{independently for } 0 \le k < n, \ k \ne K.$$

Then by considering the cases (3.17) and (3.18) separately and using the fact that $g_\tau(x_{K-1}) - g_\tau(x_K) = 0$ to translate from $\mathbf{b}, \mathbf{x}$ to $\mathbf{b}', \mathbf{x}'$, we obtain that conditional on the event $\mathbf{b}' = \underline{b}'$, the conditional joint distribution of the height gaps between successive minima of g_τ is

$$g_\tau(x'_{k-1}) - g_\tau(x'_k) \sim \mathrm{Exp}(b'_{k+1} + \cdots + b'_{n-1}) \text{ independently for } 0 \le k < n-1, \tag{3.19}$$

heightgap_after_jum

where we formally defined $g_\tau(x'_{-1}) = 0$ for convenience.

Continuing recursively the argument starting after (3.13), we find that the process $\mathrm{EX}(g_t), t \ge 0$ continues to evolve like an $\mathrm{ICLD}(\lambda)$. This completes the proof of Proposition 3.12(a).

Now we prove Proposition 3.12(b). First we show that (b) holds when t is a jump time. Let $\underline{m}' \in \ell_0^{\downarrow}$ and $\underline{b}' \in \Psi^{-1}(\underline{m}')$. Let μ' denote a measure-valued random variable with distribution $\mathcal{E}(\underline{m}')$ (see Definition 3.11).

⟨condexp_law_i⟩

(i) By Remark 3.3(ii), Proposition 3.12(a) and Corollary 3.9, the conditional distribution of $\mathrm{EX}(g_t)$ given $\mathrm{ORDX}(g_t) = \underline{m}'$ agrees with the distribution of $\mathrm{EX}(f_{\mu'})$ (see Definition 1.4), namely both have distribution $\pi_{\underline{m}'}$ (defined in (3.5)).

⟨condexp_law_ii⟩

(ii) By induction, (3.19) holds for all jump times, so the conditional law of $\mu(g_t)$ conditioned on $\mathrm{EX}(g_t) = \underline{b}'$ agrees with the conditional law of μ' conditioned on the event $\mathrm{EX}(f_{\mu'}) = \underline{b}'$.

Therefore (b) follows when t is a jump time from (i) & (ii) above by averaging out over the possible outcomes of $\mathrm{EX}(g_t)$.

The fact that (b) holds between jumps easily follows from the memoryless property of exponential random variables, so we omit the details. The proof of Proposition 3.12 is complete.

□

WE WILL INCLUDE A DIAGRAM HERE TO ILLUSTRATE $t = 0$ and $t = T_1$.

Proof of Proposition 1.11. Recalling the definition of ORDX from Definition 1.3 we see that $\Psi(\mathrm{EX}(g)) = \mathrm{ORDX}(g)$ for any $g : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ with finitely many excursions, thus Proposition 1.11 is an immediate corollary of Theorem 3.6 and Proposition 3.12. □

4 Particle representation: finite state space

ic⟨defeparticle_rep⟩ **Definition 4.1.** Let $\underline{m} = (m_1, \dots, m_n) \in \ell_0^{\downarrow}$. Let $Y_i(0) = -E_i$ be the initial height of particle i with mass m_i, $i = 1, \dots, n$. The heights of particles evolve over time; let us now describe the joint time evolution of the heights $Y_1(t), \dots, Y_n(t)$ using a system of ordinary differential equations.

Analogously to the definition of μ_0 in (1.8), we define

$$\mu_t = \sum_{i=1}^{n} m_i \cdot \delta_{Y_i(t)}. \tag{4.1}$$ def_eq_mu_t

The system of differential equations that $Y_1(t), \dots, Y_n(t)$ solves is

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_i(t) = \lambda \cdot \mathbb{1}[\, Y_i(t) < 0 \,] + \mu_t(Y_i(t), 0). \tag{4.2}$$ particle_dynamics_l

We say that the particle i "dies" at time t_i, where t_i is defined by

$$t_i := \min\{\, t \,:\, Y_i(t) = 0 \,\}. \tag{4.3}$$ def_eq_t_i_death_ti

A "time-t block" consists of the union of all the particles that share the same (strictly negative) height at time t.

In words, particles start at negative locations and move up. If a particle reaches zero then it stops there and dies. Before it dies, the speed of a particle is equal to λ plus the total weight of particles strictly above it and strictly below zero. If two particles merge they stay together forever.

The next lemma states that the above particle representation gives the same coalescence process as the tilt & shift representation. The proof is straightforward and we omit it.

ift_particle_equiv⟩? **Lemma 4.2.** *For any $\underline{m} = (m_1, \dots, m_n) \in \ell_0^{\downarrow}$, if $g_t(\cdot)$ is defined by Definition 1.10 and if $Y_1(t), \dots, Y_n(t)$ and μ_t are defined by Definition 4.1 then we have*

$$\bar{g}_t(\cdot) \equiv f_{\mu_t}(\cdot) \quad \text{and} \quad \mu(g_t) \equiv \mu_t, \tag{4.4}$$ eq_particle_tilt_sh

where $\bar{g}_t(\cdot)$ is defined in (1.4)*, f_{μ_t} is defined using Definition 1.4 and $\mu(g)$ is defined in* (3.11)*. In particular, we have*

$$\mathrm{ORDX}(g_t) \overset{(1.5)}{=} \mathrm{ORDX}(\bar{g}_t) \overset{(4.4)}{=} \mathrm{ORDX}(f_{\mu_t}), \qquad t \geq 0. \tag{4.5}$$

?eq_particle_tilt_sh

Recalling Definition 3.11, Propositions 1.11 and 3.12(b) *imply*

$$\mu_0 \sim \mathcal{E}(\underline{m}) \implies \begin{cases} \mathbf{m}_t := \mathrm{ORDX}(f_{\mu_t}), t \geq 0 \quad \textit{is a} \quad \mathrm{MCLD}(\lambda), \\ \mu_t \sim \mathcal{E}(\mathbf{m}_t) \ \textit{ given } \ \mathbf{m}_t \ \textit{ for any } \ t \geq 0. \end{cases} \tag{4.6}$$

particle_exp_for_al

Recalling the definition of ν from (1.13) *and t_i from* (4.3)*, we have*

$$\nu = \sum_{i=1}^{n} m_i \cdot \delta_{t_i}. \tag{4.7}$$

def_eq_death_time_m

In words, if a time-t block of particles dies at time t, then the total mass of this block is equal to the length $x^*(t)$ of the shift that occurs at the same time t in the tilt & shift representation.

Our next lemma quantifies the effect of the insertion of a new particle on the death times of other particles.

article_death_times⟩ **Lemma 4.3.** *For any $\underline{m} = (m_1, \dots, m_n) \in \ell_0^{\downarrow}$, let us define $Y_1(t), \dots, Y_n(t)$, μ_t and $t_1, \dots, t_n$ as in Definition 4.1.*

Let us initialize a new particle system $\widetilde{Y}_0(t), \dots, \widetilde{Y}_n(t)$ by letting $\widetilde{Y}_i(0) = Y_i(0)$ for any $i = 1, \dots, n$ and by adding a new particle with initial height $\widetilde{Y}_0(0)$ and mass m_0.

Let us then define $\widetilde{Y}_0(t), \dots, \widetilde{Y}_n(t)$, $\widetilde{\mu}_t$ and $\widetilde{t}_0, \dots, \widetilde{t}_n$ according to Definition 4.1. Then we have

$$|\widetilde{t}_i - t_i| \leq \mathbb{1}[\widetilde{Y}_0(0) > Y_i(0)] \frac{m_0 |Y_i(0)|}{\lambda^2} \exp\left(\frac{\mu_0(Y_i(0), 0)}{\lambda} \right), \quad i = 1, \dots, n. \tag{4.8}$$

eq_insertion_of_new

Remark 4.4. Lemma 4.3 is a deterministic statement, i.e., it holds for all possible choices of $\widetilde{Y}_0(0), Y_1(0), \dots, Y_n(0)$.

4.1 Proof of Lemma 4.3

on_proof_insertion⟩? It is enough to prove (4.8) for infinitesimally small $m_0 = \varepsilon$, because (4.8) is linear in m_0, so we can achieve (4.8) for any $m_0 > 0$ by stacking infinitesimally small weights on top of each other.

Without loss of generality we may assume $Y_1(0) > Y_2(0) > \cdots > Y_n(0)$, because if $Y_i(0) = Y_j(0)$ for some $i \neq j$ then we can replace our particle system by another one with fewer particles in which these two particles are merged.

This implies $t_1 \leq t_2 \leq \cdots \leq t_n$. Let us denote

$$\Delta t_i = \widetilde{t}_i - t_i, \qquad 1 \leq i \leq n.$$

Denote by

$$i^* = \inf\{\, i \;:\; Y_i(0) < \widetilde{Y}_0(0) \,\}. \tag{4.9}$$?def_eq_i_star?

In particular, we define $i^* = \infty$ if $Y_i(0) \geq \widetilde{Y}_0(0)$ for any $i \in [1, n]$.

By (4.2) the speed of $Y_i(t)$ only depends on the locations of particles strictly above it, so we have $\widetilde{Y}_i(t) \equiv Y_i(t)$ for any $1 \leq i < i^*$, thus $\Delta t_i = 0$ for any $1 \leq i < i^*$, and (4.8) trivially follows for these particles.

Denote by $Y_0(t)$ the position at time t of a particle with weight 0 and initial location $\widetilde{Y}_0(0)$, inserted into the particle system $Y_1(t), \ldots, Y_n(t)$. Denote by t_0 the death time of this particle.

Denote by $\widetilde{t}_0$ the death time of particle $\widetilde{Y}_0(t)$ in the particle system $\widetilde{Y}_0(t), \widetilde{Y}_1(t), \ldots, \widetilde{Y}_n(t)$. Note that we have $t_0 = \widetilde{t}_0$.

finitesimal_change⟩? **Lemma 4.5.** *If $m_0 = \varepsilon$ is infinitesimal then for any $i^* \leq i \leq n$ we have*

$$|\Delta t_i| \leq \frac{1}{\lambda}\left(\varepsilon \cdot t_0 + \sum_{j=i^*}^{i-1} m_j \cdot |\Delta t_j|\right). \tag{4.10}$$ Delta_t_i_recursive

Proof. We will prove (4.10) by induction on i. We begin with $i = i^*$. If $i^* = 1$, then by (4.2) we have

$$Y_1(0) < \widetilde{Y}_0(0), \quad t_0 = \frac{-\widetilde{Y}_0(0)}{\lambda}, \quad t_1 = \frac{-Y_1(0)}{\lambda}, \quad \widetilde{t}_1 = t_1 - \frac{\varepsilon \cdot t_0}{\lambda},$$

so (4.10) holds in this case.

If $i = i^* > 1$, then we will consider two sub-cases:

(a) If $t_{i^*-1} < t_{i^*}$ then $\Delta t_{i^*} = -\frac{\varepsilon \cdot t_0}{\lambda}$,

(b) If $t_{i^*-1} = t_{i^*}$ then $\Delta t_{i^*} = \Delta t_{i^*-1} = 0$.

In both cases, (4.10) holds.

Now we turn to the case $i > i_*$. We will again consider various sub-cases:

⟨case_no_merge⟩ (A) If $t_{i-1} < t_i$ then

$$\Delta t_i = -\frac{1}{\lambda}\left(\varepsilon \cdot t_0 + \sum_{j=i^*}^{i-1} m_j \cdot \Delta t_j\right), \tag{4.11}$$ exact_formula_on_in

because

(I) the insertion of the new particle with weight ε at initial location $\widetilde{Y}_0(0)$ increases the velocity of the particle with index i by ε on the time interval $[0, t_0]$;

(II) for any $j < i$, the change Δt_j in the death time of particle j changes the velocity of particle i by m_j for a (signed) time interval of length Δt_j;

(III) the speed of particle i just before it dies is λ if $t_{i-1} < t_i$.

Now (4.11) implies (4.10).

⟨case_merge⟩ (B) If $t_{i-1} = t_i$ and $\min\{\, t \,:\, Y_{i-1}(t) = Y_i(t)\,\} < t_i$, then $\Delta t_i = \Delta t_{i-1}$, so

$$|\Delta t_i| = |\Delta t_{i-1}| \overset{(*)}{\leq} \frac{1}{\lambda}\left(\varepsilon \cdot t_0 + \sum_{j=i^*}^{i-2} m_j \cdot |\Delta t_j|\right) \leq \frac{1}{\lambda}\left(\varepsilon \cdot t_0 + \sum_{j=i^*}^{i-1} m_j \cdot |\Delta t_j|\right), \tag{4.12}$$ merge_i_and_i_minus

where $(*)$ holds by our induction hypothesis. Now (4.12) implies (4.10).

(C) If $t_{i-1} = t_i = \min\{\, t \,:\, Y_{i-1}(t) = Y_i(t)\,\}$, then either we have $\Delta t_{i-1} \leq 0$ and then we can proceed like in case (A) and obtain

$$\Delta t_i = \Delta t_{i-1} - \frac{1}{\lambda} m_i \Delta t_{i-1},$$

or we have $\Delta t_{i-1} \geq 0$ and we can proceed like in case (B) and obtain $\Delta t_i = \Delta t_{i-1}$. As we have seen earlier, (4.10) holds in both cases.

□

From the recursive inequalities (4.10) one deduces by induction on i the following explicit bound:

$$|\Delta t_i| \leq \frac{\varepsilon \cdot t_0}{\lambda} \cdot \prod_{j=i^*}^{i-1}\left(1 + \frac{m_j}{\lambda}\right), \qquad i^* \leq i \leq n. \tag{4.13}$$ Delta_t_i_explicit_

Now we are ready to deduce (4.8) for any $i^* \le i \le n$:

$$|\widetilde{t}_i - t_i| \overset{(4.13)}{\le} \frac{\varepsilon \cdot t_i}{\lambda} \cdot \prod_{j=1}^{i-1} \left(1 + \frac{m_j}{\lambda}\right) \le \frac{\varepsilon \cdot t_i}{\lambda} \cdot \exp\left(\sum_{j=1}^{i-1} \frac{m_j}{\lambda}\right) \le \frac{\varepsilon \cdot |Y_i(0)|}{\lambda^2} \exp\left(\frac{\mu_0(Y_i(0), 0)}{\lambda}\right),$$

The proof of Lemma 4.3 is complete.

5 Some facts about μ_0

f_lemma:exponential⟩ *Proof of Lemma 1.5.* For any $0 \le a < b$,

$$\mathbb{E}\left(\mu_0[-b,-a]\right) \overset{(1.8)}{=} \sum_i m_i \mathbb{P}(E_i \in [a,b]) \overset{(1.7)}{=} \sum_i m_i e^{-am_i}\left(1 - e^{-(b-a)m_i}\right) < \sum_i m_i^2 (b-a) < \infty$$

since $\underline{m} \in \ell_2^{\downarrow}$, and this is already enough to give (i).

For (ii), we have

$$\mathbb{E}\left(\mu_0[-k-1,-k]\right) = \sum_i m_i e^{-km_i}\left(1 - e^{-m_i}\right) \le \sum_i m_i^2 e^{-km_i} \to 0 \quad \text{as } k \to \infty,$$

and also

$$\mathrm{Var}\left(\mu_0[-k-1,-k]\right) = \sum_i m_i^2 \,\mathrm{Var}\left(\mathbb{1}[k \le E_i \le k+1]\right).$$

Thus $\sum_k \mathrm{Var}\left(\mu_0[-k,-k-1]\right) \le \sum_i m_i^2 < \infty$.

Then let k be large enough such that $\mathbb{E}\left(\mu_0[-k-1,-k]\right) \le \delta/2$. Then by Chebyshev's inequality,

$$\mathbb{P}(\mu_0[-k,-k-1] > \delta) \le \frac{\mathrm{Var}\left(\mu_0[-k,-k-1]\right)}{\left(\delta - \delta/2\right)^2}.$$

Hence $\sum_k \mathbb{P}(\mu_0[-k,-k-1] > \delta) < \infty$ and the result in (ii) follows from Borel-Cantelli. □

⟨lemma:dense⟩ **Lemma 5.1.** *If* $\underline{m} \in \ell_2^{\downarrow} \setminus \ell_1^{\downarrow}$, *then with probability 1 we have* $\mu_0[-b,-a] > 0$ *for any* $0 < a < b$, *where* μ_0 *is defined by* (1.8).

Proof. It is enough to prove that for all pairs of rational numbers $0 < a < b$ we have $\mu_0[-b,-a] > 0$ with probability 1. This follows from the second Borel-Cantelli lemma and the fact that

$$\sum_{i=1}^{\infty} \mathbb{P}(\, a \le E_i \le b\,) \stackrel{(1.7)}{=} \sum_{i=1}^{\infty} e^{-am_i}\left(1 - e^{-(b-a)m_i}\right) \stackrel{(*)}{=} +\infty \quad \text{if} \quad \underline{m} \in \ell_2^{\downarrow} \setminus \ell_1^{\downarrow},$$

where $(*)$ follows from $e^{-am}\left(1 - e^{-(b-a)m}\right) \approx (b-a)m$ as $m \to 0$. □

6 Good functions

We define a set $\mathcal{G}$ of "good" functions. Recall the notion of excursions from Definition 1.2.

⟨def:good_functions⟩ **Definition 6.1.** If g is a function from $[0,\infty)$ to $\mathbb{R} \cup \{-\infty\}$, we say $g \in \mathcal{G}$ if:

⟨good_i_cadlag⟩ (i) g is c.à.d.l.à.g.

⟨good_ii_minusinf⟩ (ii) If $g(x) = -\infty$ then $g(x') = -\infty$ for all $x' > x$.

⟨good_iii_arrangable⟩ (iii) For any $\varepsilon > 0$, g has only finitely many excursions above its minimum with length greater than or equal to ε.

⟨good_iv_lebesgue⟩ (iv) Let $x_{\max} = \sup\{\, x : x > -\infty \,\} \le \infty$. The Lebesgue measure of the set of points in $(0, x_{\max})$ which are not contained in some excursion above the minimum is 0.

If $g \in \mathcal{G}$, then $\mathrm{ORDX}(g)$ (see Definition 1.3) is well-defined.

⟨lemma:ftgood⟩ **Lemma 6.2.** *Suppose* f_0 *is defined from* $\underline{m} \in \ell_2^{\downarrow}$ *by Definition 1.6. Define* f_t *by* (1.11). *Then with probability 1, we have* $f_t \in \mathcal{G}$ *for all* $t \ge 0$.

Proof. Properties (i), (ii) in Definition 6.1 can be deduced for f_t directly from the definitions (1.9) and (1.11). Property (iv) for f_t follows from the fact that (iv) holds for f_0 (see Remark 1.7(i)) and the observation that every excursion of f_0 is contained in an excursion of f_t.

It remains to justify property (iii). The function f_0 is non-increasing, and Lemma 1.5(ii) tells us that the length of the interval on which f_0 takes values in $[y, y+1]$ tends to 0 as $y \to -\infty$. Hence for every ε there exists a K_ε such that $f_0(x) - f_0(x-\varepsilon) < -1$ for all $x \geq K_\varepsilon$. As a result, $f_t(x) - f_t(x-\varepsilon) < -1+t\varepsilon$. If $\varepsilon < 1/t$, we find that $f_t(x) < f_t(x-\varepsilon)$, so all excursions intersecting (K_ε, ∞) must have length less than ε, as desired.

□

⟨lemma:cadlag_tilt⟩ **Lemma 6.3.** *Given some $f_0 \in \mathcal{G}$ define f_t by* (1.11) *and assume that $f_t \in \mathcal{G}$ for all $t \geq 0$. The function* $\mathrm{ORDX}(f_t)$ *from* $[0,\infty)$ *to* $\ell_\infty^\downarrow$ *is c.à.d.l.à.g.*

Proof. Let us write $\underline{m}(t) = (m_1(t), m_2(t), \dots) = \mathrm{ORDX}(f_t)$. Since we use the topology of coordinatewise convergence on $\ell_\infty^\downarrow$, it is enough to show that the function $t \mapsto m_i(t)$ is c.à.d.l.à.g. for all i.

Consider $0 \leq t' < t$. Since f_t is obtained from $f_{t'}$ by adding on an increasing function, any minimum of f_t is also a minimum of $f_{t'}$, and any excursion of $f_{t'}$ is a sub-interval of an excursion of f_t.

Fix t and suppose (l,r) is an excursion of f_t. Take ε with $0 < \varepsilon < 2l$. Recalling the notion of $\bar{f}$ from (1.4), we have $\bar{f}_t(l-\varepsilon/2) > f_t(l)$; hence if δ is sufficiently small, then

$$\bar{f}_{t+\delta}(l-\varepsilon/2) > f_t(l) + \delta l = f_{t+\delta}(l),$$

and so $f_{t+\delta}$ has a minimum in $[l-\varepsilon/2, l]$.

Also, there is some $x \in (r, r+\varepsilon/2)$ with $f_t(x) < f_t(l)$. Hence if δ is sufficiently small, then $f_{t+\delta}$ has a minimum in $[r, r+\varepsilon/2]$.

So for any ε, we can find δ such that the length of the excursion of $f_{t+\delta}$ which includes (l,r) is at most $r-l+\varepsilon$.

Now we will argue that for any $\varepsilon > 0$ there exists small enough δ such that the length $m_1(t+\delta)$ is at most $m_1(t)+\varepsilon$.

Fix any $T > t$ and consider $\delta \in (0, T-t)$. Since the excursions of $f_{t+\delta}$ are contained in the excursions of f_T, any excursion of $f_{t+\delta}$ of length more than $m_1(t)+\varepsilon$ must be contained in an excursion of f_T whose length also exceeds that. There are only finitely many such excursions of f_T since $f_T \in \mathcal{G}$. Let $\mathcal{U}$ be the union of those excursions, which has finite total length, say L.

Now let us look at all the excursions of f_t contained in $\mathcal{U}$. There are at most countably many. We can take a finite number of them whose total length is at least $L-\varepsilon$. Each of them has length no more than $m_1(t)$. From the property above, if we choose δ small enough, then at time $t+\delta$, none of them is contained in an excursion of length more than $m_1(t)+\varepsilon$. But also, since the remaining length of $\mathcal{U}$ outside this set is only ε, then also no other point in $\mathcal{U}$ is contained in an excursion of length more than $m_1(t)+\varepsilon$.

It follows that $m_1(t+\delta) \leq \varepsilon + m_1(t)$ as desired.

In similar fashion we can also obtain that $\sum_{i=1}^k m_i(t+\delta) \leq \varepsilon + \sum_{i=1}^k m_i(t)$ for sufficiently small δ, for any k. But note that $\sum_{i=1}^k m_i(t)$ is increasing in

t. So for each k, $\sum_{i=1}^{k} m_i(t)$ is right-continuous with left limits, and hence the same is true for $m_i(t)$ for each i. □

⟨def_uniformly_good⟩ **Definition 6.4.** A family of good functions $f^{(i)} \in \mathcal{G}, i \in \mathcal{I}$ is said to be *uniformly good* if for any ε there exists $K_\varepsilon \in \mathbb{R}$ such that for any $i \in \mathcal{I}$ the excursions of $f^{(i)}$ intersecting $[K_\varepsilon, \infty)$ are all shorter than ε.

unif_good_conv_ordx⟩ **Lemma 6.5.** *Let $f \in \mathcal{G}$ be continuous and assume that all of the excursions of f are strict (c.f. Definition 1.2). Let $f^{(n)} \in \mathcal{G}$, $n \in \mathbb{N}$ be a sequence of (not necessarily continuous) functions that converge to f uniformly on bounded intervals. Let us also assume that the family consisting of f and $f^{(n)}, n \in \mathbb{N}$ is uniformly good. Then $\mathrm{ORDX}(f^{(n)}) \to \mathrm{ORDX}(f)$ as $n \to \infty$ in the product topology on $\ell_\infty^{\downarrow}$.*

Proof. Suppose (l, r) is an excursion of f. For any given $\gamma > 0$ (with $\gamma < l$), there is a $\delta > 0$ such that the following properties hold:

(i) $f(x) \geq f(l) + \delta$ for all $x \in [0, l - \gamma]$;

(ii) $f(x) \geq f(l) + \delta$ for all $x \in [l + \gamma, r - \gamma]$;

(iii) $f(x) \leq f(l) - \delta$ for some $x \in [r, r + \gamma]$.

Here (i) holds since f has a minimum at l and, being continuous, must achieve its bounds on $[0, l - \gamma]$; (ii) holds since the excursion is strict, and (iii) holds since by the definition of excursions, there must be points arbitrarily close to the right of r which take value lower than $f(l)$.

Now suppose n is large enough that $|f^{(n)}(x) - f(x)| < \delta/2$ for all $x \in [0, r + \gamma]$. Then we obtain the following properties:

(i) $f^{(n)}(x) \geq f(l) + \delta/2$ for all $x \in [0, l - \gamma]$;

(ii) $f^{(n)}(x) \geq f(l) + \delta/2$ for all $x \in [l + \gamma, r - \gamma]$;

(iii) $g^{(n)}(x) \leq f(l) - \delta/2$ for some $x \in [r, r + \gamma]$;

(iv) $g^{(n)}(l) \in (f(l) - \delta/2, f(l) + \delta/2)$.

Then $f^{(n)}$ must have an excursion which starts somewhere in $[l - \gamma, l + \gamma]$ and ends somewhere in $[r - \gamma, r + \gamma]$.

Now let $\varepsilon > 0$ and choose K_ε such that the excursions of f and $f^{(n)}, n \in \mathbb{N}$ intersecting $[K_\varepsilon, \infty)$ are all shorter than ε.

Now by Definition 6.1(iv) there exists a finite collection of excursions (l_j, r_j) of f, whose union covers all of $[0, K_\varepsilon + \varepsilon]$ except for a set of total length less than $\varepsilon/2$. Let k be the total number of these excursions. Apply the above argument to all of these excursions with $\gamma = \varepsilon/4k$. Then if n is sufficiently large, we have that for each of these excursions of f, there is a corresponding excursion of $f^{(n)}$ whose length is within $\varepsilon/2k$; the remaining length in $[0, K_\varepsilon + \varepsilon]$ amounts to no more than ε; and we know that outside $[0, K_\varepsilon + \varepsilon]$, all excursions (either of $f^{(n)}$ or f) have length less than or equal to ε.

It follows that for any $i > 0$, the ith largest excursion of $f^{(n)}$ and the ith largest excursion of f differ by at most ε. Hence indeed $\mathrm{ORDX}(f^{(n)})$ converges componentwise to $\mathrm{ORDX}(f)$, as desired.

□

7 Extension of rigid representation to $\ell_2^\downarrow$

n:extension_to_ltwo⟩

7.1 Brief section about extension to $\ell_1^\downarrow$

ion_of_easy_l1_case⟩ So that we can later assume that we are in $\ell_2^\downarrow \setminus \ell_1^\downarrow$

$\sum m_i < \infty$. This case is much simpler. Since $E_i \sim \mathrm{Exp}(m_i)$ independently and $\sum m_i < \infty$, then for any $h < \infty$, there are only finitely many i such that $E_i < h$.

Let $x_{\max} = \sum m_i$. If $x \geq x_{\max}$ then $f_0(x) = -\infty$ and $f_0^{(n)}(x) = -\infty$ for all n. If $x < x_{\max}$, then $f_0(x) > -\infty$, and $f_0^{(n)}(x) > -\infty$ for all large enough n (namely, large enough that $\sum_{i=1}^n m_i > x$).

Fix any $\varepsilon > 0$. For sufficiently large n, the functions $f_0^{(n)}$ and f_0 are identical on $[0, x_{\max} - \varepsilon]$ (and hence so are the functions $f_t^{(n)}$ and f_t). So for large enough n, the kth longest excursion of $f_t^{(n)}$ is the same as the kth longest excursion of f_t, which, as before, leads to (7.1) as desired.

??? POSSIBLY NEED MORE HERE? DO WE NEED TO MENTION THAT WE ONLY NEED THE CASE WHERE INFINITELY MANY c_i ARE POSITIVE (SINCE THE OTHER CASE WAS ALREADY CONSIDERED) AND SO WE HAVE $f_t(x) \to -\infty$ as $x \to x_{\max}$?

7.2 Proof of extension of MC tilt representation to $\ell_2^\downarrow$

nsion_mc_tilt_to_l2⟩ The aim of this section is to prove Theorem 1.9.

Recalling Remark 1.7(i) the function f_0 has an excursion of length m_i at height $Y_i(0) = -E_i$. Let x_i denote the left endpoint of this excursion.

Since the function f_t is obtained by adding an increasing function to f_0, the set of left endpoints of excursions of f_t is a subset of the points $x_i, i \geq 1$. Set

$$Y_i(t) = f_t(x_i) = -Y_i(0) + x_i t.$$

⟨def_good_times⟩ **Definition 7.1.** With probability 1, for each pair i, j, there is at most one time t such that $Y_i(t) = Y_j(t)$; hence with probability 1, all the values $Y_i(t)$ are distinct for all except countably many t. Hence for all but countably many t, the probability that all the values $Y_i(t)$ are distinct at time t is 0. Let $\mathcal{T}$ be this set of "good" times t.

⟨strict_excursions_i⟩ ⟨remark_good_times⟩ **Remark 7.2.** (i) The function f_0 is piecewise constant; note that if x is not one of the points $x_i, i \geq 1$ then for $t > 0$, x is a point of increase of f_t (in fact, f_t is differentiable at x with derivative t). As a result, if $t \in \mathcal{T}$ then with probability 1, every excursion (l, r) of f_t is strict (see Definition 1.2).

(ii) $\mathcal{T}$ is a deterministic subset of $\mathbb{R}_+$ which might depend on the initial condition $\underline{m}$. We conjecture that $\mathcal{T} = \mathbb{R}_+$ for any $\underline{m} \in \ell_2^{\downarrow}$.

From Lemma 6.2 and Lemma 6.3 it follows that $t \mapsto \mathrm{ORDX}(f_t)$ is a c.à.d.l.à.g. process with respect to the product topology on $\ell_\infty^{\downarrow}$. The graphical representation of the multiplicative coalescent $\mathbf{m}_t$ is also a c.à.d.l.à.g. process with respect to the topology of the $\mathrm{d}(\cdot,\cdot)$-metric (see Lemma 2.9), thus it is c.à.d.l.à.g. with respect to the weaker product topology on $\ell_\infty^{\downarrow}$. Hence, since $\mathcal{T}$ is dense, if we can show that for any finite collection $t_1, \dots, t_r \in \mathcal{T}$, we have

$$(\mathrm{ORDX}(f_{t_i}), 1 \leq i \leq r) \stackrel{d}{=} (\mathbf{m}_{t_i}, 1 \leq i \leq r), \tag{7.1}$$ tiltconclusion

then indeed the law of $\mathrm{ORDX}(f_t)$ is that of the multiplicative coalescent.

For each n, let $\underline{m}^{(n)}$ be given by

$$m_i^{(n)} = \begin{cases} m_i, & i \leq n \\ 0, & i > n \end{cases}. \tag{7.2}$$ truncation_of_initi

For each n, $\underline{m}^{(n)} \in \ell_0^{\downarrow}$, and $\underline{m}^{(n)} \to \underline{m}$ in $\ell_2^{\downarrow}$ as $n \to \infty$.

We couple processes starting from $\underline{m}^{(n)}$, $n \geq 1$, by using the same height variables $Y_i = -E_i$ throughout. If we define

$$f_t^{(n)}(x) = f_0^{(n)}(x) + tx,$$

then by the $\lambda = 0$ case of Proposition 1.11 we have

$$(\mathrm{ORDX}(f_{t_i}^{(n)}, 1 \leq i \leq r) \stackrel{d}{=} (\mathbf{m}_{t_i}^{(n)}, 1 \leq i \leq r), \tag{7.3}$$ finite_dim_agree_fo

where $\mathbf{m}_t^{(n)}, t \geq 0$ is the multiplicative coalescent started from $\underline{m}^{(n)}$.

We will let $n \to \infty$, and show that

$$\mathrm{ORDX}(f_t^{(n)}) \to \mathrm{ORDX}(f_t) \quad \text{for all} \quad t \in \mathcal{T} \\ \text{coordinate-wise with probability 1.} \tag{7.4}$$ mc_convergence_ORDX

By the Feller property (i.e., the $\lambda = 0$ case of Theorem 1.1) we have

$$(\mathbf{m}_{t_i}^{(n)}, 1 \le i \le r) \xrightarrow{d} (\mathbf{m}_{t_i}, 1 \le i \le r), \qquad n \to \infty \tag{7.5}$$ eq_truncated_conv_f

(with respect to the topology of $\ell_2^{\downarrow}$ and hence also coordinatewise). Putting together (7.3),(7.4) and (7.5) we obtain (7.1) as required.

It remains to show (7.4). We will achieve this by checking that the conditions of Lemma 6.5 almost surely hold if $t \in \mathcal{T}$. We may assume that $\underline{m} \in \ell_2^{\downarrow} \setminus \ell_1^{\downarrow}$, as discussed in Section 7.1.

⟨lemma:tailsmall⟩ **Lemma 7.3.** *Fix $t > 0$. With probability 1 , the family of functions that consists of $f_t^{(n)}$, $n \ge 1$ and f_t is uniformly good (c.f. Definition 6.4).*

Proof. Recalling Definitions 1.4 and 1.6 we see that

$$f_0^{(n)} = f_{\mu_0^{(n)}}, \qquad \text{where} \qquad \mu_0^{(n)} = \sum_{i=1}^{n} m_i \cdot \delta_{Y_i}.$$

Now $\mu_0 - \mu_0^{(n)} = \sum_{n<i} m_i \cdot \delta_{Y_i}$ is a non-negative measure for each $n \in \mathbb{N}$, thus we obtain the proof of Lemma 7.3 by repeating the argument of proof Lemma 6.2, uniformly in n. □

⟨lemma:ucbi⟩ **Lemma 7.4.** *If $\underline{m} \in \ell_2^{\downarrow} \setminus \ell_1^{\downarrow}$ then for any $t \ge 0$ the function $f_t(\cdot)$ is continuous and $f_t^{(n)} \to f_t$ uniformly on bounded intervals.*

Proof. Since $f_t^{(n)}(x) = f_0^{(n)}(x) + tx$, and $f_t(x) = f_0(x) + tx$, it is enough to show the statements of the lemma for $t = 0$.

By Lemma 5.1 the values $E_i, i \ge 0$ are dense in $[0, \infty)$, thus the function f_0 is non-increasing and by Remark 1.7(i) the values it takes are dense in $(-\infty, 0)$. Hence f_0 is continuous.

Since f_0 is a continuous function on $[0, \infty)$, it is uniformly continuous on any bounded sub-interval.

Fix any $U < \infty$. Let us define $n_0 = \min\{\, n \,:\, \sum_{i=1}^{n} m_i > U \,\}$. For all $n \ge n_0$ we have $f_0^{(n)}(U) \ge f_0^{(n_0)}(U) =: -S$.

Consider $x \le U$, $n \ge n_0$. By (1.10) have $f_0^{(n)}(x) = Y_k$, where k is such that

$$x \in \left[\sum_{j \le n \,:\, E_j < E_k} m_j \,,\; m_k + \sum_{j \le n \,:\, E_j < E_k} m_j \right).$$

By considering the interval on which the function f_0 takes the same value Y_k, we have

$$f_0^{(n)}(x) = f_0(x+\delta) \qquad \text{where} \qquad \delta = \sum_{j>n\,:\,E_j<E_k} m_j.$$

For all $n \geq n_0$ we have $\delta \leq \sum_{j>n\,:\,E_j<S} m_j$. This goes to 0 as $n \to \infty$ by Lemma 1.5 and dominated convergence.

Then by the uniform continuity of f_0 on bounded intervals, we have that $f_0^{(n)} \to f_0$ uniformly on $[0, U]$, as desired. □

By Remark 7.2(i), for $t \in \mathcal{T}$, with probability 1, all excursions of f_t are strict. We can thus insert the properties derived in Lemmas 7.3, 7.4 into Lemma 6.5 to obtain (7.4). This completes the proof of Theorem 1.9.

7.3 Proof of extension of MCLD tilt and shift representation to $\ell_2^{\downarrow}$

⟨sion_by_truncation⟩? The aim of this section is to prove Theorem 1.12 using truncations and approximation.

Given some $\underline{m} \in \ell_2^{\downarrow}$, let us generate $E_1, E_2, \ldots$ as in (1.7). Define the truncation $\underline{m}^{(n)}$ by (7.2). We define $g_0^{(n)}(\cdot) \equiv f_0^{(n)}(\cdot)$ using $E_1, E_2, \ldots, E_n$ by (1.9). Note that we still denote by $f_t(\cdot)$ the function constructed from the un-truncated $\underline{m}$ by (1.9)&(1.11) and that we use the same sequence of random variables $E_1, E_2, \ldots$ to obtain a coupling of $g_0^{(n)}(\cdot), n \in \mathbb{N}$. We define $g_t^{(n)}(\cdot)$ from $g_0^{(n)}(\cdot)$ using the method of Definition 1.10 (this works, because $\underline{m}^{(n)} \in \ell_0^{\downarrow}$). This gives rise to the measure $\nu^{(n)}$ by (1.13) and the function $\Phi^{(n)}$ by (1.14). By (1.15) we have

$$g_t^{(n)}(x) = g_0^{(n)}(x + \Phi^{(n)}(t)) + \lambda t + \int_0^t \left(x + \Phi^{(n)}(t) - \Phi^{(n)}(s)\right) \mathrm{d}s. \qquad (7.6)$$ g_t_from_g_0_n

Our next result states that $\mathbb{P}$-almost surely $\nu^{(n)}$ vaguely converges to some ν as $n \to \infty$.

⟨lemma_vague_conv⟩ **Lemma 7.5.** *If $\nu^{(n)}$, $n \in \mathbb{N}$ is defined as above then $\mathbb{P}$-almost surely there exists a locally finite measure ν on $[0,\infty)$ such that for any compactly supported continuous function $h : [0,\infty) \to \mathbb{R}$ we have*

$$\lim_{n\to\infty} \int_0^\infty h(t)\, \mathrm{d}\nu^{(n)}(t) = \int_0^\infty h(t)\, \mathrm{d}\nu(t), \qquad \mathbb{P}-a.s. \qquad (7.7)$$ vague_conv_of_measu

ollary_portemanteau⟩ **Corollary 7.6.** *By the portemanteau theorem* (7.7) *implies*

$$\lim_{n\to\infty} \Phi^{(n)}(t) = \Phi(t) \quad \textit{if} \quad \nu(\{t\}) = 0, \quad \textit{where} \quad \Phi(t) := \nu([0,t]) \tag{7.8}$$ eq_Phi_n_conv_to_Ph

Proof of Lemma 7.5. We will use the particle representation (see Section 4)

$$Y_1^{(n)}(t), \dots, Y_n^{(n)}(t)$$

of $g_t^{(n)}(\cdot)$. We can then write $\nu^{(n)} = \sum_{i=1}^n m_i \cdot \delta_{t_i^{(n)}}$, see (4.7). Note that

$$Y_i^{(n)}(0) = Y_i(0) \qquad \text{for any} \qquad 1 \le i \le n.$$

Let us assume that the function $h : [0,\infty) \to \mathbb{R}$ for which we want to show (7.7) is supported on $[0,T]$.

Recalling the definition of $\mu_0 = \sum_{i=1}^\infty m_i \cdot \delta_{Y_i(0)}$ from (1.8), we analogously define $\mu_0^{(n)} = \sum_{i=1}^n m_i \cdot \delta_{Y_i(0)}$. Then by Lemma 1.5 there exists a $\mathbb{P}$-almost surely finite random variable K_0 such that

$$\sup_{n\ge 0} \frac{\mu_0^{(n)}[-K,0]}{K} = \frac{\mu_0[-K,0]}{K} \le \frac{1}{2T}, \qquad \text{for any} \quad K \ge K_0. \tag{7.9}$$ eq_sparse_weight

If $|Y_i(0)| = E_i > K_0$ then for any $t \ge 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_i^{(n)}(t) \overset{(4.2)}{\le} \lambda + \mu_t^{(n)}(Y_i^{(n)}(t), 0) \le \lambda + \mu_0^{(n)}(Y_i(0), 0) \overset{(7.9)}{\le} \lambda + \frac{|Y_i(0)|}{2T}. \tag{7.10}$$ speed_ineq

This implies that if $Y_i(0) < Y := -(K_0 \vee 2\lambda T)$, then

$$Y_i^{(n)}(T) \overset{(7.10)}{\le} Y_i(0) + \left(\lambda + \frac{|Y_i(0)|}{2T}\right) \cdot T < 0,$$

which implies that the time of death $t_i^{(n)}$ of particle i (see (4.7)) satisfies

$$h(t_i^{(n)}) = 0 \quad \text{if} \quad Y_i(0) < Y. \tag{7.11}$$ far_starting_point_

Our aim is to show that the sequence $\int_0^\infty h(t)\,\mathrm{d}\nu^{(n)}(t), n \in \mathbb{N}$ is Cauchy. In order to show this we let $n \le m$ and bound

$$\left| \int_0^\infty h(t)\,\mathrm{d}\nu^{(m)}(t) - \int_0^\infty h(t)\,\mathrm{d}\nu^{(n)}(t) \right| \overset{(4.7),(7.11)}{\le}$$

$$\sum_{i=1}^n m_i \cdot \left| h(t_i^{(m)}) - h(t_i^{(n)}) \right| \cdot \mathbb{1}[Y_i(0) \ge Y] + \|h\|_\infty \cdot \sum_{i=n+1}^m m_i \cdot \mathbb{1}[Y_i(0) \ge Y]. \tag{7.12}$$ bound_for_vague

In order to bound the first term on the right-hand side of (7.12) we observe that if $1 \le i \le n$ and $Y_i(0) \ge Y$ then

$$\left|t_i^{(m)} - t_i^{(n)}\right| \le \sum_{k=n}^{m-1} \left|t_i^{(k+1)} - t_i^{(k)}\right| \overset{(4.8)}{\le}$$
$$\sum_{k=n}^{m-1} \mathbb{1}[Y_{k+1}(0) \ge Y] \frac{m_{k+1}|Y_i(0)|}{\lambda^2} \exp\left(\frac{\mu_0^{(k)}(Y_i(0), 0)}{\lambda}\right) \le$$
$$\frac{|Y|}{\lambda^2} \exp\left(\frac{\mu_0(Y, 0)}{\lambda}\right) \sum_{k=n}^{\infty} m_{k+1} \cdot \mathbb{1}[Y_{k+1}(0) \ge Y]. \quad (7.13)$$ bound_for_death_tim

Note that Lemma 1.5 implies that with probability 1 we have

$$\sum_{i=n}^{\infty} m_i \cdot \mathbb{1}[Y_i(0) \ge Y] \to 0 \quad \text{as} \quad n \to \infty.$$

If we combine this with (7.12), (7.13) and the fact that $h(\cdot)$ is uniformly continuous, we can conclude that $\int_0^\infty h(t)\,\mathrm{d}\nu^{(n)}(t)$, $n \in \mathbb{N}$ is a Cauchy sequence for any $h \in \mathcal{C}_0(\mathbb{R})$, from which it follows that the exists ν for which (7.7) holds. □

ly_excursions_burn⟩? **Lemma 7.7.** *If ν is the random measure obtained in Lemma 7.5, then for every $t \ge 0$ there exists $y \in (-\infty, 0]$ such that*

$$\nu[0, t] = \mu_0[y, 0], \quad (7.14)$$ nu_mu_corresp

where the measure μ_0 is defined in (1.8).

Proof. Note that $\nu^{(n)} = \sum_{i=1}^n m_i \cdot \delta_{t_i^{(n)}}$ for every $n \in \mathbb{N}$, where $Y_i \ge Y_j$ implies $t_i^{(n)} \le t_j^{(n)}$ for every $1 \le i, j \le n$. Since $\nu^{(n)}$ is an atomic measure with masses $(m_i)_{i=1}^n$ located at $\left(t_i^{(n)}\right)_{i=1}^n$ and $\nu^{(n)} \to \nu$ vaguely, we can conclude that ν is also an atomic measure with masses $(m_i)_{i=1}^\infty$ located at $(t_i)_{i=1}^\infty$ where $\lim_{n\to\infty} t_i^{(n)} = t_i$, thus $Y_i \ge Y_j$ implies $t_i \le t_j$ for every $i, j \in \mathbb{N}$. From this (7.14) readily follows. □

Given some $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$ we defined $g_0(\cdot) = f_0(\cdot)$ by (1.9), ν by Lemma 7.5 and $\Phi(\cdot)$ by (7.8).

aim_g_t_n_unif_conv⟩ **Claim 7.8.** *Let $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$ and define $g_t(\cdot)$ using (1.15) and $g_t^{(n)}(\cdot)$ using (7.6). Then $g_t(\cdot)$ is continuous and if $t \in [0, \infty)$ satisfies $\nu(\{t\}) = 0$ then*

$$g_t^{(n)}(\cdot) \to g_t(\cdot) \quad \text{uniformly on compacts.} \quad (7.15)$$ g_n_t_converges_to_

Proof. (7.15) follows from Corollary 7.6, Lemma 7.4 and (1.15), (7.6). □

Recall from Definition 7.1 the notion of the set of good times $\mathcal{T}$. In particular, if $t \in \mathcal{T}$ then the function $f_t(\cdot)$ defined by (1.9) and (1.11) almost surely only has strict excursions, see Remark 7.2(i). Let us now define

$$\mathcal{T}^* = \mathcal{T} \cap \{ t \, : \, \mathbb{P}(\nu(\{t\}) = 0) = 1 \}. \tag{7.16}$$ good_times_for_mcld

Since ν is a random measure which only has countably many atoms, we see that the $\mathcal{T}^*$ is a deterministic set whose complement is at most countable. In particular, $\mathcal{T}^*$ is a dense subset of $\mathbb{R}_+$.

ion_ORDX_tilt_shift⟩ **Lemma 7.9.** *If* $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$, $t \in \mathcal{T}^*$ *then* $\mathrm{ORDX}(g_t^{(n)}) \to \mathrm{ORDX}(g_t)$ *in the product topology on* $\ell_\infty^\downarrow$.

Proof. First note that the functions $g_t(\cdot)$ and $g_t^{(n)}(\cdot), n \in \mathbb{N}$ are uniformly good (c.f. Definition 6.4): this follows from Lemma 7.3 and the fact that g_t is a "shifted" version of f_t:

$$g_t(x) \overset{(1.11),(1.15)}{=} f_t(x + \Phi(t)) + \lambda t - \int_0^t \Phi(s)\, \mathrm{d}s, \tag{7.17}$$ g_t_is_shifted_f_t

and similarly, $g_t^{(n)}$ is a left-shifted version of $f_t^{(n)}$.

It follows from Definition 7.1 and (7.17) that if $t \in \mathcal{T}$ then almost surely $g_t(\cdot)$ has no strict excursions. Now the claim of this lemma follows from Claim 7.8 and Lemma 6.5. □

mma_g_t_ORDX_cadlag⟩ **Lemma 7.10.** *If* $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$ *then the function* $t \mapsto \mathrm{ORDX}(g_t)$ *is c.à.d.l.à.g. with respect to the product topology on* $\ell_\infty^\downarrow$.

Proof. Let us fix $t \geq 0$ and define the auxiliary functions

$$g^*_{t+\Delta t}(x) = g_t(x + \Phi(t+\Delta t) - \Phi(t)), \qquad g^{**}_{t+\Delta t}(x) = g_t(x) + \Delta t \cdot x.$$

From (1.15) we obtain

$$g_{t+\Delta t}(x) = g^*_{t+\Delta t}(x) + \lambda \Delta t + \int_t^{t+\Delta t} (x + \Phi(t+\Delta t) - \Phi(s))\, \mathrm{d}s, \tag{7.18}$$ g_t_plus_delta_from

$$g_{t+\Delta t}(x) = g^{**}_{t+\Delta t}(x + \Phi(t+\Delta t) - \Phi(t)) + \lambda \Delta t - \int_t^{t+\Delta t} \Phi(s)\, \mathrm{d}s. \tag{7.19}$$ g_t_plus_delta_from

If $\underline{m}, \underline{m}' \in \ell_\infty^\downarrow$, we say that $\underline{m} \preceq \underline{m}'$ if $\sum_{j=1}^i m_j \leq \sum_{j=1}^i m'_j$ for any $i \in \mathbb{N}$.

We are going to show

$$\mathrm{ORDX}(g^*_{t+\Delta t}) \preceq \mathrm{ORDX}(g_{t+\Delta t}), \tag{7.20}$$
g_t_bound_lower

$$\mathrm{ORDX}(g_{t+\Delta t}) \preceq \mathrm{ORDX}(g^{**}_{t+\Delta t}), \tag{7.21}$$
g_t_bound_upper

$$\mathrm{ORDX}(g^*_{t+\Delta t}) \to \mathrm{ORDX}(g_t) \quad \text{in} \quad \ell_\infty^\downarrow \quad \text{as} \quad \Delta t \searrow 0, \tag{7.22}$$
g_t_conv_lower

$$\mathrm{ORDX}(g^{**}_{t+\Delta t}) \to \mathrm{ORDX}(g_t) \quad \text{in} \quad \ell_\infty^\downarrow \quad \text{as} \quad \Delta t \searrow 0. \tag{7.23}$$
g_t_conv_upper

As soon as we show (7.20)–(7.23), we immediately obtain

$$\mathrm{ORDX}(g_{t+\Delta t}) \to \mathrm{ORDX}(g_t) \quad \text{in} \quad \ell_\infty^\downarrow \quad \text{as} \quad \Delta t \searrow 0,$$

i.e., the right-continuity of $t \mapsto \mathrm{ORDX}(g_t)$ with respect to the $\ell_\infty^\downarrow$ topology. The proof of the existence of left limits is similar and we omit it.

(7.20) follows from the fact that $g_{t+\Delta t}$ is obtained from $g^*_{t+\Delta t}$ by adding an increasing function (see (7.18)), thus the collection of excursions of $g_{t+\Delta t}$ are obtained by merging some excursions of $g^*_{t+\Delta t}$.

(7.21) follows from the fact that $g_{t+\Delta t}$ is obtained from $g^{**}_{t+\Delta t}$ by a shift to the left plus an addition of a constant (see (7.19)), thus the excursions of $g_{t+\Delta t}$ are obtained by deleting/splitting some excursions of $g^*_{t+\Delta t}$.

From (7.14) it follows that for every $\Delta t \geq 0$ there exists some $y \in (-\infty, 0]$ such that $\Phi(t+\Delta t) - \Phi(t) = \mu_t[y, 0]$ (see (4.1)), thus the collection of excursions of $g^*_{t+\Delta t}$ is obtained by removing some excursions of g_t whose total length is $\Phi(t+\Delta t) - \Phi(t)$. From this (7.22) follows, since $\Phi(t+\Delta t) - \Phi(t) = \nu(t, t+\Delta t] \to 0$ as $\Delta t \searrow 0$.

From (1.15) and Lemma 6.2 it follows that $g^{**}_{t+\Delta t}(x) \in \mathcal{G}$ for any $\Delta t \geq 0$, thus Lemma 6.3 implies (7.23). □

proof of Theorem 1.12(i). Given $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$, we constructed two stochastic processes: the graphical construction of the MCLD(λ) process $\mathbf{m}_t = \mathrm{ord}(\underline{m}, H_t), t \geq 0$ was given in Section 2.2, the process $\mathrm{ORDX}(g_t), t \geq 0$ is defined by (1.15) using the control function $\Phi(\cdot)$ that appears in (7.8). We now want to show that these two $\ell_\infty^\downarrow$-valued processes have the same law. Both processes are c.à.d.l.à.g. with respect to the product topology on $\ell_\infty^\downarrow$ by Lemma 2.11 and Lemma 7.10.

Hence, since the set $\mathcal{T}^*$ defined in (7.16) is dense, if we can show that for any finite collection $t_1, \dots, t_r \in \mathcal{T}^*$, we have

$$(\mathrm{ORDX}(g_{t_i}), 1 \leq i \leq r) \stackrel{d}{=} (\mathbf{m}_{t_i}, 1 \leq i \leq r), \tag{7.24}$$
tilt_shift_conclusi

then indeed Theorem 1.12(i) will follow.

We prove (7.24) by replicating the argument given in Section 7.2. By Proposition 1.11 we have

$$(\mathrm{ORDX}(g^{(n)}_{t_i}, 1 \le i \le r) \stackrel{d}{=} (\mathbf{m}^{(n)}_{t_i}, 1 \le i \le r), \tag{7.25}$$

finite_dim_agree_fo

where $\mathbf{m}^{(n)}_t, t \ge 0$ is the MCLD(λ) process started from $\underline{m}^{(n)}$. By the Feller property (i.e., Theorem 1.1) we have

$$(\mathbf{m}^{(n)}_{t_i}, 1 \le i \le r) \stackrel{d}{\to} (\mathbf{m}_{t_i}, 1 \le i \le r), \qquad n \to \infty \tag{7.26}$$

eq_truncated_conv_f

(with respect to the topology of $\ell_2^{\downarrow}$ and hence also coordinatewise). Putting together (7.25), Lemma 7.9 and (7.26) we obtain (7.24). The proof of Theorem 1.12(i) is complete. □

Proof of Theorem 1.12(ii) . ??? Incomplete proof ??? It is easy to deduce that (1.16) holds if $\underline{m} \in \ell_0^{\downarrow}$ from Definition 1.10 by an induction on the number of “shift” events. Let us assume that $\underline{m} = (m_1, m_2, \dots) \in \ell_2^{\downarrow} \setminus \ell_1^{\downarrow}$.

Let us recursively define

$$n_1 = 1, \qquad n_k = \min\{\, i \, : \, m_i < m_{n_{k-1}} \,\}, \; k \ge 2, \qquad \widetilde{m}_k = m_{n_k}.$$

Thus we have $\{m_1, m_2, \dots\} = \{\widetilde{m}_1, \widetilde{m}_2, \dots\}$ and $\widetilde{m}_1 > \widetilde{m}_2 > \dots$.

We have already shown that the measure-valued random variable ν is almost surely the vague limit of the sequence of the measure-valued random variables $\nu^{(n)}$ as $n \to \infty$, so the same remains true for the sub-sequence $\nu^{(n_k-1)}$ as $k \to \infty$. Consequently, (1.16) will follow as soon as we show that $\nu^{(n_k-1)}$ is $\sigma\left(g_0(x), x \ge 0\right)$-measurable for any $k \ge 2$. This is indeed true, because $\nu^{(n_k-1)}$ is $\sigma\left(g_0^{(n_k-1)}(x), x \ge 0\right)$-measurable by Definition 1.10 and $g_0^{(n_k-1)}(\cdot)$ is $\sigma\left(g_0(x), x \ge 0\right)$-measurable, because $g_0^{(n_k-1)}(\cdot)$ is determined as soon as we know the set $\{\, Y_{n_i}, \dots, Y_{n_{i+1}-1} \,\}$ of initial heights of particles with mass $\widetilde{m}_i$ for each $i = 1, \dots, k-1$, but this information can be determined by looking at the heights of the excursions of $g_0(\cdot)$ if the exponential variables $E_i, i \ge 1$ all take different values, which happens almost surely. □

Proof of Theorem 1.12(iii). It is enough to show that for any $K > 0$ and $\varepsilon > 0$ we almost surely have $-\varepsilon \le g_t(0) \le 0$ for any $0 \le t \le K$. Let us fix $K, \varepsilon > 0$. Recall from (7.9)-(7.11) that there exists $Y < 0$ such that if $Y_i(0) < Y$ then $t_i^{(n)} > K$ for any $i \le n$. By Lemma 5.1 there exists an almost surely finite n_0 such that $\mu_0^{(n_0)}[y - \varepsilon, y] > 0$ for any $Y \le y \le 0$, thus for any $n \ge n_0$ and any $x \ge 0$ such that $Y \le g_0^{(n)}(x) \le 0$ we have $g_0^{(n)}(x_-) - g_0^{(n)}(x) \le \varepsilon$. In words: the gaps between consecutive particles initially located in $[Y, 0]$ are smaller than or equal to ε. By Definition 4.1, these gaps can only decrease with time, thus for any $n \geq n_0$ and $t \leq K$ there is a particle in $[-\varepsilon, 0]$, i.e., we have $-\varepsilon \leq g_t^{(n)}(0) \leq 0$. Now $g_t^{(n)}(0) \to g_t(0)$ as $n \to \infty$ for all except countably many values of $t \in [0, K]$ by Claim 7.8, moreover $g_t(0)$ is a c.à.d.l.à.g. function of t by (1.15), therefore $-\varepsilon \leq g_t(0) \leq 0$ holds for every $0 \leq t \leq K$.

□

8 Applications

8.1 Tilt representation for eternal MCs

8.2 Scaling limit of frozen percolation started from critical Erdős-Rényi graph

8.3 Particle representation of the forest fire model

This section contains a particle representation of the mean field forest fire model of [3], see Proposition 8.5. This representation is an adaptation of the one in Section 4 and we will briefly explain in Remark 8.6 how it sheds some new light on a certain controlled non-linear PDE problem (see (8.5) below) which played a central role in the theory developed in [3] and [2].

In [3] the authors modify the dynamical Erdős-Rényi model to obtain the mean field forest fire model:

f_forest_fire_model⟩ **Definition 8.1** ($\mathrm{FF}(n, \lambda(n))$)**.** We start with a graph on n vertices. Between each pair of unconnected vertices an edge appears with rate $1/n$, moreover a connected component of size k "burns" with rate $\lambda(n) \cdot k$, i.e., the component is replaced with k isolated vertices. The total number of vertices remains n.

Denote by $\mathcal{C}^n(i, t)$ the connected open component of vertex i at time t.

We define the empirical component size densities by

$$\mathbf{v}_k^n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}[|\mathcal{C}^n(i, t)| = k], \qquad \mathbf{v}^n(t) = (\mathbf{v}_k^n(t))_{k=1}^n . \tag{8.1}$$?vknt?

With the above definitions $\mathbf{v}^n(t), t \geq 0$ is a Markov process, let us call it here the forest fire component size density Markov process on n vertices with lightning rate $\lambda(n)$, or briefly $\mathrm{FF}(\mathrm{n}, \lambda(\mathrm{n}))$.

One investigates the model when $\frac{1}{n} \ll \lambda(n) \ll 1$ as $n \to \infty$. We assume that $\mathbf{v}_k^n(0) \to v_k(0)$ for all $k \in \mathbb{N}$ as $n \to \infty$, where $\sum_k k^3 v_k(0) < +\infty$.

Under these assumptions [3, Theorem 2] states that

$$\mathbf{v}_k^n(t) \to v_k(t) \quad \text{in probability as} \quad n \to \infty, \tag{8.2}$$
eq_forest_densities

where $(v_k(t))_{k=1}^{\infty}$ is the unique solution of the following system of ODE's:

$$\forall\, k \geq 2 \quad \frac{\partial}{\partial t} v_k(t) = \frac{k}{2} \sum_{l=1}^{k} v_l(t) v_{k-l}(t) - k v_k(t), \qquad \sum_{k=1}^{\infty} v_k(t) \equiv 1. \tag{8.3}$$
smolcontr

In order to prove that (8.3) is well-posed (c.f. [3, Theorem 1]), one looks at the Laplace transform

$$V(t,x) = \sum_{k=1}^{\infty} v_k(t) e^{-kx} - 1 \tag{8.4}$$
V_t_x_laplace_def

which satisfies the following controlled PDE (c.f. [3, (43)]):

$$\frac{\partial}{\partial t} V(t,x) = -V(t,x) \frac{\partial}{\partial x} V(t,x) + \varphi(t) e^{-x}, \qquad V(t,0) \equiv 0, \tag{8.5}$$
burgers_controlled

where the control function $\varphi(t)$ measures the intensity of fires at time t:

$$\varphi(t) = \frac{\partial}{\partial t} r(t), \quad r(t) = \lim_{n \to \infty} \mathbf{r}^n(t), \quad \mathbf{r}^n(t) = \frac{1}{n} \sum_{i=1}^{n} B^n(i,t), \tag{8.6}$$
def_eq_varphi_burni

and $B^n(i,t)$ denotes the number of times vertex i has burnt before time t. Given a solution $V(\cdot,\cdot)$ of (8.5) one defines the corresponding characteristic curves (c.f. [3, (66)]) as the solutions of the ODE

$$\frac{\mathrm{d}}{\mathrm{d}s} \xi(s) = V(s, \xi(s)). \tag{8.7}$$
characteristics_for

These curves are useful because by (8.5) they satisfy $\frac{\mathrm{d}^2}{\mathrm{d}s^2}\xi(s) = \varphi(s) e^{-\xi(s)}$, hence given $\varphi(\cdot)$ they can be constructed (c.f. [3, (65)]) without solving (8.5).

In Definition 8.4 below we are going to give a novel particle representation of $\mathrm{FF}(n, \lambda(n))$ by slightly modifying Definition 4.1.

?⟨def_dictionary⟩? **Definition 8.2.** If $n \in \mathbb{N}_+$ we let

$$\mathcal{V}^n = \left\{ \underline{v}^n = (v_k^n)_{k=1}^n \; : \; \sum_{k=1}^{n} v_k^n = 1 \; \text{ and } \; \frac{n}{k} v_k^n \in \mathbb{N} \; \text{ for all } \; k \right\},$$

$$\mathcal{M}^n = \left\{ \underline{m}^n = \left(m_j^n\right)_{j=1}^{N} \in \ell_0^{\downarrow} \; : \; \sum_{j=1}^{N} m_j^n = n \; \text{ and } \; m_j^n \in \mathbb{N}_+ \; \text{ for all } \; j \right\}.$$

We say that the component size density vector $\underline{v}^n \in \mathcal{V}^n$ and the ordered list of component sizes $\underline{m}^n \in \mathcal{M}^n$ correspond to each other if

$$v_k^n = \sum_{j=1}^{N} \frac{k}{n} \mathbb{1}[\, m_j^n = k \,] \quad \text{for all} \;\; k. \tag{8.8}$$ {?}

Note that this correspondence is one-to-one.

In plain words, $\underline{v}^n$ and $\underline{m}^n$ correspond to each other if there is a graph G on n vertices such that $\underline{v}^n$ and $\underline{m}^n$ both arise from G.

⟨def_dictionary2⟩ **Definition 8.3.** If $\widetilde{\mu}^n$ is a finite point measure on $\mathbb{R}_-$ such that $\widetilde{\mu}^n(\mathbb{R}_-) = 1$ and the masses of the atoms of $n\widetilde{\mu}^n$ are integers then we define $\mathbf{v}(\widetilde{\mu}^n)$ to be the element of $\mathcal{V}^n$ corresponding to the element of $\mathcal{M}^n$ which consists of the ordered list of masses of the atoms of $n\widetilde{\mu}^n$.

particle_rep_forest⟩ **Definition 8.4.** Given $\underline{v}^n(0) = (v_k^n(0))_{k=1}^n \in \mathcal{V}^n$ and the corresponding $\underline{m}^n = \big(m_j^n\big)_{j=1}^N \in \mathcal{M}^n$, we define the initial heights of the particles $\widetilde{Y}_i(t), 1 \leq i \leq n$ by letting $\widetilde{Y}_i(0) = -E_j$, where $E_j \sim \mathrm{Exp}(m_j^n)$, $1 \leq j \leq N$ are independent and vertex i initially belongs to component j in the forest fire model.

We define

$$\widetilde{\mu}_t^n = \sum_{i=1}^{n} \frac{1}{n} \delta_{\widetilde{Y}_i(t)} \quad (\text{ Note: } \mathbf{v}(\widetilde{\mu}_0^n) = \underline{v}^n(0) \text{ }). \tag{8.9}$$?mu_n_forest_particl

If $\widetilde{Y}_i(t_-) < 0$ then we let

$$\frac{\mathrm{d}}{\mathrm{d}t} \widetilde{Y}_i(t) = \lambda(n) + \widetilde{\mu}_t^n(\widetilde{Y}_i(t), 0), \tag{8.10}$$ particle_dynamics_l

and if $\widetilde{Y}_i(t_-) = 0$ then we say that vertex i burns and we let $-\widetilde{Y}_i(t)$ have $\mathrm{Exp}(1)$ distribution, independently from everything else.

In words, a clustered family of particles with mass $1/n$ start at negative locations, move up and merge with other particle clusters just like in Definition 4.1, but if a time-t block of particles with total mass k/n reaches 0, then this block burns and gets replaced by k particles of mass $1/n$ with i.i.d. locations with negative $\mathrm{Exp}(1)$ distribution.

forest_particle_rep⟩ **Proposition 8.5.** *(i) For any $n \in \mathbb{N}_+$ and any initial state $\underline{v}^n(0) \in \mathcal{V}^n$, the process $\mathbf{v}(\widetilde{\mu}_t^n), t \geq 0$ is a $\mathrm{FF}(n, \lambda(n))$ process with initial state $\underline{v}^n(0)$ (see Definitions 8.1, 8.4 and 8.3 for the definitions of $\mathrm{FF}(n, \lambda(n))$, $\widetilde{\mu}_t^n$ and $\mathbf{v}(\widetilde{\mu}^n)$, respectively).*

article_mu_exp_dist⟩ *(ii) For any $t \geq 0$, the conditional distribution of $n\widetilde{\mu}_t^n$ given $\mathbf{m}_t^n$ is $\mathcal{E}(\mathbf{m}_t^n)$, where $\mathbf{m}_t^n$ is the $\mathcal{M}^n$-valued random variable corresponding to the $\mathcal{V}^n$-values random variable $\mathbf{v}(\widetilde{\mu}_\tau^n)$.*

Proof. Recalling Definition 3.11 we observe that $n\widetilde{\mu}_0^n \sim \mathcal{E}(\underline{m}^n)$, where $\underline{m}^n \in \mathcal{M}^n$ corresponds to $\underline{v}^n(0) \in \mathcal{V}^n$.

Denote by τ the first burning time of the particle system $\widetilde{Y}_i(t), 1 \leq i \leq n$.

Denote by $Y_i(t) := \widetilde{Y}_i(nt)$ and $\mu_t := n\widetilde{\mu}_{nt}^n = \sum_{i=1}^n \delta_{Y_i(t)}$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_i(t) \overset{(8.10)}{=} n\lambda(n) + \mu_t(Y_i(t), 0), \quad 0 \leq t < \frac{\tau}{n},$$

thus the evolution of the particle system $Y_i(t), 1 \leq i \leq n$ satisfies Definition 4.1 (with $\lambda = n\lambda(n)$) up to time τ/n, including the time-τ/n block that burns.

Likewise, if $\mathbf{v}^n(t), t \geq 0$ is a $\mathrm{FF}(n, \lambda(n))$ process, then the $\mathcal{M}^n$-valued process corresponding to the $\mathcal{V}^n$-valued process $\mathbf{v}^n(nt), 0 \leq t \leq \tau/n$ satisfies the definition of a $\mathrm{MCLD}(n\lambda(n))$ process, including the time of the first deletion event and the component that gets deleted.

Therefore, by (4.6), Proposition 8.5 holds for $\mathbf{v}(\widetilde{\mu}_t^n), 0 \leq t \leq \tau$, because if a block of k particles burn at time τ, then after the insertion of k particles of mass $1/n$ with i.i.d. $-\mathrm{Exp}(1)$ distribution, we still have the property that the conditional distribution of $n\widetilde{\mu}_\tau^n$ given $\mathbf{m}_\tau^n$ is $\mathcal{E}(\mathbf{m}_\tau^n)$. Therefore we can inductively repeat this argument using (4.6) again and again to show that Proposition 8.5 holds for $\mathbf{v}(\widetilde{\mu}_t^n), 0 \leq t \leq \tau_i$, where τ_i is the i'th burning time. This completes the proof of Proposition 8.5. □

forest_particle_PDE⟩ **Remark 8.6.** Let us assume that $\widetilde{\mu}_t^n$ converges weakly in probability to some measure $\widetilde{\mu}_t$ as $n \to \infty$. Denote by

$$\widetilde{V}(t, y) = \widetilde{\mu}_t(y, 0), \qquad y \leq 0.$$

We will derive a PDE for $\widetilde{V}(t, y)$, see (8.11) below.

We have $\widetilde{\mu}_t[y - \mathrm{d}y, y] = -\frac{\partial}{\partial y}\widetilde{V}(t, y)\mathrm{d}y$, moreover by (8.10), each "particle" near the location y moves with speed $\widetilde{V}(t, y)$ (since $\lambda(n) \ll 1$), thus $\widetilde{V}(t, y)$ increases by $\widetilde{\mu}_t[y - \mathrm{d}y, y]$ on the time interval $[t, t + \mathrm{d}t]$, where $\mathrm{d}y = \widetilde{V}(t, y)\mathrm{d}t$. The mass $\widetilde{V}(t, y)$ also decreases by $\varphi(t)\mathrm{d}t$ because of burning (see (8.6)) and increases by $(1 - e^y)\varphi(t)\mathrm{d}t$ because of the re-insertion of burnt mass with distribution $-\mathrm{Exp}(1)$. Putting these effects together we obtain

$$\frac{\partial}{\partial t}\widetilde{V}(t, y) = -\widetilde{V}(t, y)\frac{\partial}{\partial y}\widetilde{V}(t, y) - e^y \varphi(t). \tag{8.11}$$

pde_for_particle_de

By comparing (8.5) and (8.11), we observe that $V(t,x)$ solves the same PDE as $-\widetilde{V}(t,-x)$. Indeed, by (8.2) and Proposition 8.5(ii) we have $\widetilde{V}(t,y) = \sum_k v_k(t)(1-e^{ky})$, which is equal to $-V(t,-y)$ by (8.4). Moreover, if $1 \ll n$ then $\widetilde{\mu}_t^n(y,0) \approx \widetilde{V}(t,y)$, thus by comparing (8.7) and (8.10) we see that the trajectories $-Y_i(s), s \geq 0$ of particles can be viewed as discrete approximations of characteristic curves.

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