Electron. Commun. Probab. **20** (2015), no. 3, 1–11.
DOI: 10.1214/ECP.v20-3734
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

A short proof of the phase transition for the vacant set of random interlacements*

Balázs Ráth†

Abstract

The vacant set of random interlacements at level $u > 0$, introduced in [8], is a percolation model on $\mathbb{Z}^d$, $d \geq 3$ which arises as the set of sites avoided by a Poissonian cloud of doubly infinite trajectories, where u is a parameter controlling the density of the cloud. It was proved in [6, 8] that for any $d \geq 3$ there exists a positive and finite threshold u_* such that if $u < u_*$ then the vacant set percolates and if $u > u_*$ then the vacant set does not percolate. We give an elementary proof of these facts. Our method also gives simple upper and lower bounds on the value of u_* for any $d \geq 3$.

Keywords: Percolation; Random Interlacements.
AMS MSC 2010: 60K35; 82B43.
Submitted to ECP on August 12, 2014, final version accepted on January 5, 2015.
Supersedes arXiv:1407.8460v1.

1 Introduction

The model of random interlacements was introduced in [8]. The interlacement $\mathcal{I}^u$ at level $u > 0$ is a random subset of $\mathbb{Z}^d$, $d \geq 3$ that arises as the local limit as $N \to \infty$ of the range of the first $\lfloor uN^d \rfloor$ steps of a simple random walk on the discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$, $d \geq 3$, see [14]. The law of $\mathcal{I}^u$ is characterized by

$$\mathbb{P}[\mathcal{I}^u \cap K = \emptyset] = e^{-u \cdot \mathrm{cap}(K)}, \quad \text{for any finite } K \subseteq \mathbb{Z}^d, \tag{1.1}$$

where $\mathrm{cap}(K)$ denotes the discrete capacity of K, see (2.5). The vacant set of random interlacements $\mathcal{V}^u$ at level u is defined as the complement of $\mathcal{I}^u$ at level u:

$$\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u, \quad u > 0. \tag{1.2}$$

By [8, (1.68)] the correlations of $\mathcal{V}^u$ decay polynomially for any $u > 0$:

$$\mathbb{P}[x, y \in \mathcal{V}^u] - \mathbb{P}[x \in \mathcal{V}^u] \cdot \mathbb{P}[y \in \mathcal{V}^u] \asymp (|x-y| \vee 1)^{2-d}, \qquad x, y \in \mathbb{Z}^d. \tag{1.3}$$

One is interested in the connectivity properties of the subgraphs of the nearest-neighbour lattice $\mathbb{Z}^d$ spanned by the above random sets. For any $u > 0$, $\mathcal{I}^u$ is a $\mathbb{P}$-a.s. connected random subset of $\mathbb{Z}^d$ (see [8, (2.21)]), but $\mathcal{V}^u$ exhibits a percolation phase transition: there exists $u_* \in (0, \infty)$ such that

*Support: OTKA (Hungarian National Research Fund) grant K100473; Postdoctoral Fellowship of the Hungarian Academy of Sciences; Bolyai Research Scholarship of the Hungarian Academy of Sciences.
†Budapest University of Technology, Hungary. E-mail: rathb@math.bme.hu

(i) for any $u > u_*$, $\mathbb{P}$-a.s. all connected components of $\mathcal{V}^u$ are finite, and

(ii) for any $u < u_*$, $\mathbb{P}$-a.s. $\mathcal{V}^u$ contains an infinite connected component.

The fact that $u_* < \infty$ was proved in [8, Section 3], and the positivity of u_* was established in [8, Section 4] when $d \geq 7$, and later in [6] for all $d \geq 3$.

There is no reason to believe that an exact formula for the value of the critical threshold $u_* = u_*(d)$ exists. However, it is proved in [9, 10] that

$$\lim_{d\to\infty} \frac{u_*(d)}{\ln(d)} = 1, \tag{1.4}$$

in agreement with the principal asymptotic behaviour of the critical threshold of random interlacements on $2d$-regular trees, which is explicitly computed in [12, Proposition 5.2].

The aim of this paper is to give a short proof of the non-triviality of phase transition of $\mathcal{V}^u$ and to provide simple explicit upper and lower bounds on the value of $u_* = u_*(d), d \geq 3$.

For any $d \geq 3$ let us denote by $0 < c_g = c_g(d)$ and $C_g = C_g(d) < +\infty$ the best constants such that the inequalities

$$c_g \cdot (|x-y| \vee 1)^{2-d} \leq g(x,y) \leq C_g \cdot (|x-y| \vee 1)^{2-d}, \qquad x, y \in \mathbb{Z}^d \tag{1.5}$$

hold, where $|\cdot|$ is the ℓ^∞-norm on $\mathbb{Z}^d$ and $g(\cdot,\cdot)$ is the Green function of simple random walk on $\mathbb{Z}^d$, see (2.3). The positivity of c_g and $C_g < +\infty$ follow from [4, Theorem 1.5.4].

Theorem 1.1. *For any $d \geq 3$, we have*

$$\frac{c_g}{L_0} \frac{1}{\mathcal{C}_2} 2^{-(d+5)} \leq u_* \leq \frac{5}{2} C_g \ln(\mathcal{C}_d), \tag{1.6}$$

where

$$\mathcal{C}_d = (13^d - 11^d)(25^d - 23^d), \quad d \geq 2, \tag{1.7}$$

and

$$L_0 = \begin{cases} \left\lceil \exp\left(48 \frac{C_g}{c_g} \mathcal{C}_2\right) \right\rceil & \text{if} \quad d = 3, \\ \left\lceil \left(48 \frac{C_g}{c_g} \mathcal{C}_2\right)^{\frac{1}{d-3}} \right\rceil & \text{if} \quad d \geq 4. \end{cases} \tag{1.8}$$

The bounds (1.6) are not at all sharp, especially if we compare them with (1.4) as $d \to \infty$. This shortcoming of Theorem 1.1 is counterbalanced by the fact that its proof is very simple. In particular, our self-contained proof does not use the *"sprinkling" technique* and *decoupling inequalities* usually applied in order to overcome the long-range correlations (1.3) present in the model. The proof of $u_*(d) > 0$ for $d \geq 7$ in [8, Section 4] does not use "sprinkling", but the proof of $u_*(d) < +\infty$ for any $d \geq 3$ in [8, Section 3] and the proof of $u_*(d) > 0$ for $3 \leq d \leq 7$ in [6] does. Various forms of decoupling inequalities have been subsequently developed to study the connectivity properties of $\mathcal{V}^u$ in the subcritical [5, 7, 11] and supercritical [2, 13] phases. These techniques are very useful once they are available, but the elementary method of our paper seems to be easier to adapt to other percolation models with long-range correlations, e.g., *branching interlacements* [1].

Let us briefly describe the idea of the proof of Theorem 1.1. We employ multi-scale renormalization. In order to prove $u_* < +\infty$ we show that if $\mathcal{V}^u$ crosses an annulus at scale $L_n = 6^n$ then this vacant crossing contains a set $\mathcal{X}_\mathcal{T}$ of 2^n well-separated vertices which arises as the image of leaves under an embedding $\mathcal{T}$ of the dyadic tree of depth n (this method already appears in [11]). By construction, the number of possible embeddings is less than $\mathcal{C}_d^{2^n}$ (c.f. (1.7)), so we only need to show that $\mathrm{cap}(\mathcal{X}_\mathcal{T}) \asymp 2^n$ if we want to use (1.1) to to show that crossing of the annulus by $\mathcal{V}^u$ is unlikely when u is big enough. This is indeed the case, because by construction the embedding $\mathcal{T}$ is “spread-out on all scales", thus the cardinality and the capacity of $\mathcal{X}_\mathcal{T}$ are comparable.

In order to prove $u_* > 0$, we restrict our attention to a plane inside $\mathbb{Z}^d$. By planar duality we only need to show that a $*$-connected crossing of a planar annulus at scale $L_n = L_0 \cdot 6^n$ by $\mathcal{I}^u$ is unlikely. We show that such a crossing must intersect 2^n “frames”, where each frame is the union of four “sticks” of length $2L_0 - 1$. Such a collection of frames again arises from a spread-out embedding of the dyadic tree of depth n. We use that $\mathcal{I}^u$ can be written as the union of the ranges of a Poissonian cloud of independent random walks and the fact that random walks tend to avoid sticks if L_0 is large enough (c.f. (1.8)) to arrive at a large deviation estimate on the probability that the number of frames that intersect $\mathcal{I}^u$ is 2^n which is strong enough to beat the combinatorial complexity term $\mathcal{C}_2^{2^n}$. This stick-based approach to $u_* > 0$ is already present in [6, Section 3] and our large deviation estimate resembles the one in the proof of [8, Theorem 2.4].

The rest of this paper is organized as follows.

In Section 2 we introduce further notation and recall some useful facts related to the notion of capacity and random interlacements. In Section 3 we define the notion of a *proper embedding* of a dyadic tree into $\mathbb{Z}^d$ and derive some facts about such embeddings. In Sections 4 and 5 we prove the upper and lower bounds on u_* stated in Theorem 1.1.

2 Preliminaries

For a set K, we denote by $|K|$ its cardinality. We denote by $K \subset\subset \mathbb{Z}^d$ the fact that K is a finite subset of $\mathbb{Z}^d$. We denote by $|x|$ the ℓ^∞-norm of $x \in \mathbb{Z}^d$ and by $S(x,R)$ the ℓ^∞-sphere of radius R about x in $\mathbb{Z}^d$:

$$S(x,R) = \{y \in \mathbb{Z}^d \,:\, |y-x| = R\}. \tag{2.1}$$

For $x \in \mathbb{Z}^d$, denote by P_x the law of simple random walk $(X_n)_{n=0}^\infty$ on $\mathbb{Z}^d$ starting at $X_0 = x$. If m is a probability measure on $\mathbb{Z}^d$, we denote by

$$P_m = \sum_{x \in \mathbb{Z}^d} m(x) P_x \tag{2.2}$$

the law of simple random walk with initial distribution m and by E_m the corresponding expectation. The Green function of simple random walk on $\mathbb{Z}^d$ is defined by

$$g(x,y) = \sum_{n=0}^\infty P_x[X_n = y], \quad x,y \in \mathbb{Z}^d. \tag{2.3}$$

Let us denote by $\{X\} \subseteq \mathbb{Z}^d$ the range of the random walk:

$$\{X\} = \cup_{n=0}^\infty \{X_n\} \tag{2.4}$$

2.1 Potential theory

If $K \subset\subset \mathbb{Z}^d$, we define the equilibrium measure $e_K(\cdot)$ of K by

$$e_K(x) = P_x[\, X_n \notin K \text{ for any } n \geq 1\,], \qquad x \in K.$$

The total mass of the equilibrium measure is called the capacity of K:

$$\mathrm{cap}(K) = \sum_{x \in K} e_K(x). \tag{2.5}$$

One defines the normalized equilibrium measure $\widetilde{e}_K(\cdot)$ of K by

$$\widetilde{e}_K(x) = \frac{e_K(x)}{\mathrm{cap}(K)}. \tag{2.6}$$

Let us now collect some facts about capacity that we will use in the sequel. The proofs of the properties (2.7)-(2.10) below can be found in, e.g., [3, Section 1.3].

For any $x \in \mathbb{Z}^d$ and any $K \subset\subset \mathbb{Z}^d$,

$$P_x[\{X\} \cap K \neq \emptyset] = \sum_{y \in K} g(x,y) e_K(y) \overset{(2.5)}{\leq} \mathrm{cap}(K) \max_{y \in K} g(x,y). \tag{2.7}$$

For any $K_1, K_2 \subset\subset \mathbb{Z}^d$,

$$\mathrm{cap}(K_1 \cup K_2) \leq \mathrm{cap}(K_1) + \mathrm{cap}(K_2). \tag{2.8}$$

For any $K \subseteq K' \subset\subset \mathbb{Z}^d$,

$$\mathrm{cap}(K) \leq \mathrm{cap}(K'). \tag{2.9}$$

For any $K \subset\subset \mathbb{Z}^d$,

$$\frac{|K|}{\max_{x \in K} \sum_{y \in K} g(x,y)} \leq \mathrm{cap}(K) \leq \frac{|K|}{\min_{x \in K} \sum_{y \in K} g(x,y)}. \tag{2.10}$$

Let us denote by F the plane

$$F = \mathbb{Z}^2 \times \{0\}^{d-2} \subseteq \mathbb{Z}^d. \tag{2.11}$$

For any $y \in F$ and $L \geq 1$ let us define the frame $\square_y^L \subseteq F$ by

$$\square_y^L \overset{(2.1)}{=} S(y, L-1) \cap F. \tag{2.12}$$

The next lemma gives an explicit upper bound on the capacity of a frame. The bounds of (2.13) are actually sharp up to a dimension-dependent constant factor, but we will only use the upper bounds. The stronger bound for $d = 3$ is crucial to showing that random walks tend to avoid frames in $\mathbb{Z}^3$. The extra $\ln(L_0)$ makes the parameter p defined in (5.6) small, which is necessary for our proof of $u_*(3) > 0$. Recall the notion of c_g from (1.5).

Lemma 2.1. *For any $L \geq 1$ we have*

$$\mathrm{cap}\left(\square_y^L\right) \leq \begin{cases} 8\frac{L}{c_g} & \textit{if} \quad d \geq 4, \\ 8\frac{L}{c_g \cdot (1+\ln(L))} & \textit{if} \quad d = 3. \end{cases} \tag{2.13}$$

Proof. Denote by $\mathcal{S}_\ell = \{1, \ldots, \ell\} \times \{0\}^{d-1} \subseteq \mathbb{Z}^d$ the stick of length ℓ. We will use (2.10) to bound $\mathrm{cap}(\mathcal{S}_\ell)$. If $x \in \mathcal{S}_\ell$ then $x = \{i\} \times \{0\}^{d-1}$ for some $1 \leq i \leq \ell$ and

$$\sum_{y \in \mathcal{S}_\ell} g(x,y) \overset{(1.5)}{\geq} \sum_{j=1}^{\ell} c_g \cdot (|j-i| \vee 1)^{2-d} \geq \sum_{j=1}^{\ell} c_g \cdot (|j-1| \vee 1)^{2-d} =$$
$$c_g \cdot \left(1 + \sum_{k=1}^{\ell-1} k^{2-d}\right) \geq \begin{cases} c_g & \text{if} \quad d \geq 4, \\ c_g \cdot \left(1 + \int_1^\ell \frac{1}{s}\, \mathrm{d}s\right) = c_g \cdot (1 + \ln(\ell)) & \text{if} \quad d = 3. \end{cases}$$

Using these bounds, (2.10) and $|\mathcal{S}_\ell| = \ell$ we obtain that $\mathrm{cap}\,(\mathcal{S}_\ell) \leq \ell / c_g$ if $d \geq 4$ and $\mathrm{cap}\,(\mathcal{S}_\ell) \leq \ell/(c_g \cdot (1 + \ln(\ell)))$ if $d = 3$. Now the frame $\square_y^L$ is the union of four sticks of length $2L - 1$, thus (2.13) follows from the above bounds and (2.8), (2.9).

□

2.2 Constructive definition of random interlacements

The definition of the interlacement $\mathcal{I}^u$ at level u by the formula (1.1) is short, but it is not constructive. The construction of [8, Section 1] involves a Poisson point process with intensity measure $u \cdot \nu$, where ν is a sigma-finite measure on the space of equivalence classes of doubly infinite trajectories modulo time-shift. The union of the ranges of trajectories which are contained in the support of this Poisson point process is denoted by $\mathcal{I}^u$, and this random subset of $\mathbb{Z}^d$ indeed satisfies (1.1).

We will not use the full definition of random interlacements, only a corollary of it, which allows one to construct a set with the same law as $\mathcal{I}^u \cap K$ for any $K \subset\subset \mathbb{Z}^d$.

Recall the notion of P_m from (2.2), $\{X\}$ from (2.4) and $\widetilde{e}_K(\cdot)$ from (2.6).

Claim 2.2. *Let $d \geq 3$, $K \subset\subset \mathbb{Z}^d$, N_K be a Poisson random variable with parameter $u \cdot \mathrm{cap}(K)$, and $(X^j)_{j \geq 1}$ i.i.d. simple random walks with distribution $P_{\widetilde{e}_K}$ and independent from N_K. Then $K \cap \cup_{j=1}^{N_K} \{X^j\}$ has the same distribution as $\mathcal{I}^u \cap K$.*

This explicit "local representation" of $\mathcal{I}^u$ follows from the very construction of the sigma-finite measure ν, which is obtained by patching together certain explicit measures Q_K, $K \subset\subset \mathbb{Z}^d$ in a consistent manner in [8, Theorem 1.1]. The above representation of $\mathcal{I}^u \cap K$ is obtained from the Poisson point process with intensity measure uQ_K.

3 Renormalization

For $n \geq 0$, let $T_{(n)} = \{1,2\}^n$ (in particular, $T_{(0)} = \emptyset$). Denote by

$$T_n = \bigcup_{k=0}^{n} T_{(k)}$$

the dyadic tree of depth n. For $0 \leq k < n$ and $m \in T_{(k)}$, $m = (\xi_1, \ldots, \xi_k)$, we denote by

$$m_1 = (\xi_1, \ldots, \xi_k, 1) \qquad \text{and} \qquad m_2 = (\xi_1, \ldots, \xi_k, 2) \tag{3.1}$$

the two children of m in $T_{(k+1)}$. Given some $L_0 \geq 1$ we define the sequence of scales

$$L_n := L_0 \cdot 6^n, \quad n \geq 0. \tag{3.2}$$

For $n \geq 0$, we denote by $\mathcal{L}_n = L_n \mathbb{Z}^d$ the lattice $\mathbb{Z}^d$ renormalized by L_n.

Definition 3.1. *$\mathcal{T} : T_n \to \mathbb{Z}^d$ is a proper embedding of T_n with root at $x \in \mathcal{L}_n$ if*

1. *$\mathcal{T}(\emptyset) = x$;*
2. *for all $0 \leq k \leq n$ and $m \in T_{(k)}$ we have $\mathcal{T}(m) \in \mathcal{L}_{n-k}$;*
3. *for all $0 \leq k < n$ and $m \in T_{(k)}$ we have*

$$|\mathcal{T}(m_1) - \mathcal{T}(m)| = L_{n-k}, \qquad |\mathcal{T}(m_2) - \mathcal{T}(m)| = 2L_{n-k}. \tag{3.3}$$

We denote by $\Lambda_{n,x}$ the set of proper embeddings of T_n into $\mathbb{Z}^d$ with root at x.

Lemma 3.2. *For any $L_0 \geq 1$, $n \geq 0$ and $x \in \mathcal{L}_n$ the number of proper embeddings of T_n into $\mathbb{Z}^d$ with root at x is equal to*

$$|\Lambda_{n,x}| \stackrel{(1.7)}{=} \mathcal{C}_d^{2^n - 1}. \tag{3.4}$$

Proof. The claim is trivially true for $n = 0$. If $n \geq 1$, $x \in \mathcal{L}_n$ and $\mathcal{T} \in \Lambda_{n,x}$, we denote by $\mathcal{T}_1$ and $\mathcal{T}_2$ the two embeddings of T_{n-1} which arise from $\mathcal{T}$ as the embeddings of the descendants of the two children of the root, i.e., for any $0 \leq k \leq n-1$ and $m = (\xi_1, \dots, \xi_k) \in T_{(k)}$ let $\mathcal{T}_\xi(m) = \mathcal{T}(\xi, \xi_1, \xi_2, \dots, \xi_k)$ for $\xi \in \{1, 2\}$. By Definition 3.1 we have $\mathcal{T}_\xi \in \Lambda_{n-1,\mathcal{T}(\xi)}$ for $\xi \in \{1, 2\}$, thus we obtain (3.4) by induction on n:

$$|\Lambda_{n,x}| \stackrel{(3.3)}{=} |S(x, L_n) \cap \mathcal{L}_{n-1}| \cdot |S(x, 2L_n) \cap \mathcal{L}_{n-1}| \cdot |\Lambda_{n-1,\mathcal{T}(1)}| \cdot |\Lambda_{n-1,\mathcal{T}(2)}| \stackrel{(3.2)}{=}$$
$$|S(0, 6)| \cdot |S(0, 12)| \cdot |\Lambda_{n-1,\mathcal{T}(1)}| \cdot |\Lambda_{n-1,\mathcal{T}(2)}| \stackrel{(*)}{=} \mathcal{C}_d \cdot \mathcal{C}_d^{2^{n-1}-1} \cdot \mathcal{C}_d^{2^{n-1}-1} = \mathcal{C}_d^{2^n-1},$$

where in $(*)$ we used the induction hypothesis. ☐

We say that $\gamma : \{0, \dots, l\} \to \mathbb{Z}^d$ is a $*$-connected path if $|\gamma(i) - \gamma(i-1)| = 1$ for any $1 \leq i \leq l$. For such a path we denote by $\{\gamma\} = \{\gamma(1), \dots, \gamma(l)\}$ the range of γ.

Recall the notion of $S(x, R)$ from (2.1) and note that $S(x, 0) = \{x\}$.

Lemma 3.3. *If γ is a $*$-connected path in $\mathbb{Z}^d$, $d \geq 2$ and $x \in \mathcal{L}_n$ such that*

$$\{\gamma\} \cap S(x, L_n - 1) \neq \emptyset \quad \textit{and} \quad \{\gamma\} \cap S(x, 2L_n) \neq \emptyset \tag{3.5}$$

then there exists $\mathcal{T} \in \Lambda_{n,x}$ such that

$$\{\gamma\} \cap S(\mathcal{T}(m), L_0 - 1) \neq \emptyset \quad \textit{for all} \quad m \in T_{(n)}. \tag{3.6}$$

Proof. We will prove that (3.5) implies that there exists $\mathcal{T} \in \Lambda_{n,x}$ such that for all $0 \leq k \leq n$ we have

$$\begin{aligned} \{\gamma\} \cap S(\mathcal{T}(m), L_{n-k} - 1) \neq \emptyset \\ \{\gamma\} \cap S(\mathcal{T}(m), 2L_{n-k}) \neq \emptyset \end{aligned} \quad \text{for all} \quad m \in T_{(k)}. \tag{3.7}$$

We will construct such a $\mathcal{T} \in \Lambda_{n,x}$ by induction on k. By $\mathcal{T}(\emptyset) = x$ we see that the case $k = 0$ of (3.7) is just (3.5). Assuming that (3.7) holds for some $0 \leq k \leq n-1$ we now show that it also holds for $k+1$. If $m \in T_{(k)}$ then our induction hypothesis (3.7) and the fact that γ is a $*$-connected path imply

$$\{\gamma\} \cap S(\mathcal{T}(m), L_{n-k} + L_{n-k-1} - 1) \neq \emptyset,$$
$$\{\gamma\} \cap S(\mathcal{T}(m), 2L_{n-k} - L_{n-k-1} + 1) \neq \emptyset.$$

We also have

$$S(\mathcal{T}(m), L_{n-k} + L_{n-k-1} - 1) \subseteq \bigcup_{y \in S(\mathcal{T}(m), L_{n-k}) \cap \mathcal{L}_{n-k-1}} S(y, L_{n-k-1} - 1),$$
$$S(\mathcal{T}(m), 2L_{n-k} - L_{n-k-1} + 1) \subseteq \bigcup_{z \in S(\mathcal{T}(m), 2L_{n-k}) \cap \mathcal{L}_{n-k-1}} S(z, L_{n-k-1} - 1),$$

thus we can choose

$$\mathcal{T}(m_1) \in S(\mathcal{T}(m), L_{n-k}) \cap \mathcal{L}_{n-k-1} \quad \text{and} \quad \mathcal{T}(m_2) \in S(\mathcal{T}(m), 2L_{n-k}) \cap \mathcal{L}_{n-k-1}$$

such that

$$\{\gamma\} \cap S(\mathcal{T}(m_1), L_{n-(k+1)} - 1) \neq \emptyset, \qquad \{\gamma\} \cap S(\mathcal{T}(m_2), L_{n-(k+1)} - 1) \neq \emptyset.$$

It follows from this, $|\mathcal{T}(m_1) - \mathcal{T}(m_2)| \geq L_{n-k} = 6L_{n-(k+1)}$ and the fact that γ is a $*$-connected path that we also have

$$\{\gamma\} \cap S(\mathcal{T}(m_1), 2L_{n-(k+1)}) \neq \emptyset, \qquad \{\gamma\} \cap S(\mathcal{T}(m_2), 2L_{n-(k+1)}) \neq \emptyset.$$

We have thus constructed the embedding $\mathcal{T}$ up to depth $k+1$ so that Definition 3.1 is satisfied up to depth $k+1$ and (3.7) also holds for $k+1$. Therefore by induction we have constructed $\mathcal{T} \in \Lambda_{n,x}$ such that (3.7) holds for all $0 \leq k \leq n$, which implies (3.6). The proof of Lemma 3.3 is complete. ☐

For $0 \le k \le n$ and $m = (\xi_1, \dots, \xi_n) \in T_{(n)}$ we denote $m|_k = (\xi_1, \dots, \xi_k) \in T_{(k)}$. Let us denote the lexicographic distance of $m, m' \in T_{(n)}$ by

$$\rho(m, m') = \min\{k \ge 0 \ : \ m|_{n-k} = m'|_{n-k}\}.$$

For any $m \in T_{(n)}$ and $0 \le k \le n$ we define

$$T_{(n)}^{m,k} = \{m' \in T_{(n)} \ : \ \rho(m, m') = k\}, \tag{3.8}$$

see Figure 1 for an illustration. Note that

$$|T_{(n)}^{m,k}| = 2^{k-1}, \qquad 1 \le k \le n. \tag{3.9}$$

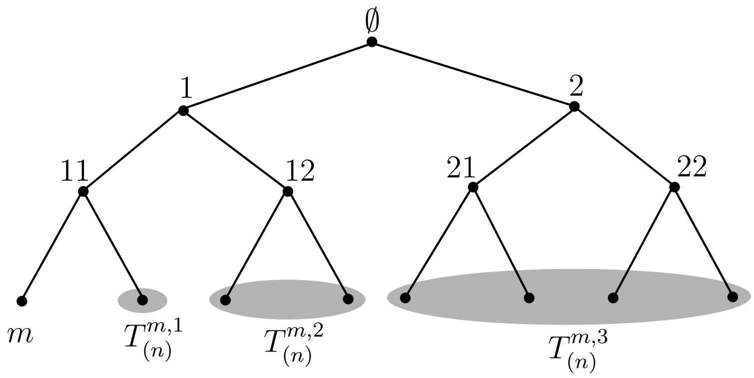

Figure 1: An illustration of the subsets $T_{(n)}^{m,k}$ of leaves of T_n defined in (3.8). The dyadic tree on the picture is of depth $n = 3$ and the leaf denoted by m is $111 \in T_{(n)}$.

The next lemma shows that a proper embedding is “spread-out on all scales."

Lemma 3.4.

$$\begin{gathered} \forall\, n \ge 1,\ x \in \mathcal{L}_n,\ \mathcal{T} \in \Lambda_{n,x},\ m \in T_{(n)},\ k \ge 1, \\ \forall\, m' \in T_{(n)}^{m,k},\ y \in S(\mathcal{T}(m), L_0 - 1),\ z \in S(\mathcal{T}(m'), L_0 - 1): \\ |y - z| \ge L_{k-1}. \end{gathered} \tag{3.10}$$

Proof. Let $m'' = m|_{n-k} = m'|_{n-k} \in T_{(n-k)}$. Recalling (3.1) we may assume w.l.o.g. that $m|_{n-k+1} = m''_1 \in T_{(n-k+1)}$ and $m'|_{n-k+1} = m''_2 \in T_{(n-k+1)}$. We have

$$|\mathcal{T}(m''_1) - \mathcal{T}(m''_2)| \overset{(3.3)}{\ge} L_k \overset{(3.2)}{=} 6L_{k-1},$$

moreover

$$|\mathcal{T}(m''_1) - y| \le |\mathcal{T}(m) - y| + \sum_{j=1}^{k-1} \left| \mathcal{T}(m|_{n-j}) - \mathcal{T}(m|_{n-j+1}) \right| \overset{(3.3)}{\le}$$
$$L_0 - 1 + \sum_{j=1}^{k-1} 2L_j \overset{(3.2)}{\le} 2L_{k-1} \sum_{i=0}^{\infty} 6^{-i} = \frac{12}{5} L_{k-1},$$

and similarly $|\mathcal{T}(m''_2) - z| \le \frac{12}{5} L_{k-1}$. Putting these bounds together we obtain (3.10). $\square$

4 Upper bound on u_*

Let us choose $L_0 = 1$ in (3.2). For $n \geq 1$ let us denote by A_n^u the event

$$A_n^u = \left\{ \begin{array}{c} \text{there exists a nearest-neighbour path in } \mathcal{V}^u \\ \text{that connects } S(0, L_n - 1) \text{ to } S(0, 2L_n) \end{array} \right\}.$$

Recall the definitions of C_g from (1.5) and $\mathcal{C}_d$ from (1.7).

Proposition 4.1. *For any $d \geq 3$ and*

$$u > \frac{5}{2} C_g \ln(\mathcal{C}_d) \tag{4.1}$$

there exists $q = q(d, u) \in (0, 1)$ such that for any $n \geq 1$ we have

$$\mathbb{P}[A_n^u] \leq q^{2^n}. \tag{4.2}$$

Corollary 4.2. *Proposition 4.1 implies the upper bound of Theorem 1.1, as we now explain. Let us denote by $\widetilde{A}_n^u$ the event that there exists a nearest-neighbour path in $\mathcal{V}^u$ that connects $S(0, L_n - 1)$ to infinity and by $\widetilde{A}_\infty^u$ the event that $\mathcal{V}^u$ has an infinite connected component. If* (4.1) *holds, then*

$$\mathbb{P}[\widetilde{A}_\infty^u] \overset{(*)}{=} \lim_{n\to\infty} \mathbb{P}[\widetilde{A}_n^u] \leq \lim_{n\to\infty} \mathbb{P}[A_n^u] \overset{(4.2)}{=} 0,$$

where $(*)$ holds by monotone convergence. Therefore we have $u_* \leq \frac{5}{2} C_g \ln(\mathcal{C}_d)$.

Proof of Proposition 4.1. For any $n \geq 1$ and $\mathcal{T} \in \Lambda_{n,0}$ we denote $\mathcal{X}_\mathcal{T} = \bigcup_{m \in T_{(n)}} \mathcal{T}(m)$. Noting that $S(\mathcal{T}(m), L_0 - 1) = S(\mathcal{T}(m), 0) = \{\mathcal{T}(m)\}$ for any $m \in T_{(n)}$ and that every nearest-neighbour path is also a $*$-connected path we can apply Lemma 3.3 to infer

$$\mathbb{P}[A_n^u] \overset{(3.6)}{\leq} \mathbb{P}\left[\bigcup_{\mathcal{T} \in \Lambda_{n,0}} \{\mathcal{X}_\mathcal{T} \subseteq \mathcal{V}^u\} \right] \overset{(1.1),(1.2)}{\leq}$$

$$\sum_{\mathcal{T} \in \Lambda_{n,0}} \exp\left(-u \cdot \mathrm{cap}(\mathcal{X}_\mathcal{T})\right) \overset{(3.4)}{\leq} \mathcal{C}_d^{2^n} \cdot \max_{\mathcal{T} \in \Lambda_{n,0}} \exp\left(-u \cdot \mathrm{cap}(\mathcal{X}_\mathcal{T})\right). \tag{4.3}$$

In order to finish the proof of Proposition 4.1 we only need to show that for any $\mathcal{T} \in \Lambda_{n,0}$ we have

$$\mathrm{cap}(\mathcal{X}_\mathcal{T}) \geq \frac{2}{5} \frac{1}{C_g} 2^n, \tag{4.4}$$

because then we indeed obtain

$$\mathbb{P}[A_n^u] \overset{(4.3),(4.4)}{\leq} \mathcal{C}_d^{2^n} \exp\left(-u \frac{2}{5} \frac{1}{C_g} 2^n\right) = \left(\mathcal{C}_d \exp\left(-u \frac{2}{5} \frac{1}{C_g}\right)\right)^{2^n} = q^{2^n}, \qquad q \overset{(4.1)}{<} 1.$$

We will show (4.4) using (2.10). For any $\mathcal{T} \in \Lambda_{n,0}$ and any $m \in T_{(n)}$ we have

$$\sum_{m' \in T_{(n)}} g(\mathcal{T}(m), \mathcal{T}(m')) \overset{(3.8)}{=} \sum_{k=0}^{n} \sum_{m' \in T_{(n)}^{m,k}} g(\mathcal{T}(m), \mathcal{T}(m')) \overset{(1.5),(3.10)}{\leq}$$

$$C_g + \sum_{k=1}^{n} C_g L_{k-1}^{2-d} \left| T_{(n)}^{m,k} \right| \overset{(3.2),(3.9)}{=} C_g \cdot \left(1 + \sum_{k=1}^{n} 6^{(k-1)(2-d)} 2^{(k-1)}\right) \overset{d\geq 3}{\leq}$$

$$C_g \cdot \left(1 + \sum_{k=1}^{\infty} 3^{1-k}\right) = \frac{5}{2} C_g. \tag{4.5}$$

Now (4.4) follows from (2.10), (4.5) and the fact that $|\mathcal{X}_\mathcal{T}| = 2^n$. The proof of Proposition 4.1 is complete. □

5 Lower bound on u_*

Let us choose L_0 according to (1.8) in (3.2). Recall the notion of the plane F from (2.11). For $n \geq 1$ and $x \in \mathcal{L}_n \cap F$ let us denote by $B^u_{n,x}$ the event

$$B^u_{n,x} = \left\{ \begin{array}{c} \text{there exists a } *\text{-connected path in } \mathcal{I}^u \cap F \\ \text{that connects } S(x, L_n - 1) \text{ to } S(x, 2L_n) \end{array} \right\}.$$

Recall the definitions of c_g, C_g from (1.5) and $\mathcal{C}_d$ from (1.7).

Proposition 5.1. *For any $d \geq 3$ and*

$$u < \frac{c_g}{L_0} \frac{1}{\mathcal{C}_2} 2^{-(d+5)}, \tag{5.1}$$

for any $n \geq 1$ and $x \in \mathcal{L}_n \cap F$ we have

$$\mathbb{P}[B^u_{n,x}] \leq \left(\frac{3}{4}\right)^{2^n}. \tag{5.2}$$

Corollary 5.2. *Proposition 5.1 implies the lower bound of Theorem 1.1, as we now explain. Let us denote by $\widehat{A}^u_n$ the event that there exists a nearest-neighbour path in $\mathcal{V}^u \cap F$ that connects $S(0, L_n)$ to infinity and by $\widehat{A}^u_\infty$ the event that $\mathcal{V}^u \cap F$ has an infinite connected component. By planar duality the event $(\widehat{A}^u_n)^c$ is equal to the event that there exists a $*$-connected path in $\mathcal{I}^u \cap F$ that surrounds $S(0, L_n - 1)$, thus if* (5.1) *holds, then*

$$\mathbb{P}[\widehat{A}^u_n] \geq 1 - \mathbb{P}\left[\bigcup_{k=n}^{\infty} \bigcup_{x \in \mathcal{L}_k,\, |x| \leq 2L_{k+1}} B^u_{k,x}\right] \overset{(3.2),(5.2)}{\geq} 1 - \sum_{k=n}^{\infty} 25^d \cdot \left(\frac{3}{4}\right)^{2^k},$$

which in turn implies $\mathbb{P}[\widehat{A}^u_\infty] = \lim_{n\to\infty} \mathbb{P}[\widehat{A}^u_n] = 1$. Therefore we have $u_* \geq \frac{c_g}{L_0}\frac{1}{\mathcal{C}_2} 2^{-(d+5)}$.

Proof of Proposition 5.1. We say that $\mathcal{T} : T_n \to F$ is a proper embedding of the dyadic tree T_n with root at $x \in \mathcal{L}_n \cap F$ into F if $\mathcal{T} \in \Lambda_{n,x}$ (see Definition 3.1). We denote by $\Lambda^F_{n,x}$ the set of proper embeddings of T_n into F.

For any $y \in \mathcal{L}_0 \cap F$ let us define the frame $\square_y \subseteq F$ by

$$\square_y \overset{(2.12)}{=} \square^{L_0}_y = S(y, L_0 - 1) \cap F.$$

For any $n \geq 1$, $x \in \mathcal{L}_n \cap F$ and $\mathcal{T} \in \Lambda^F_{n,x}$ let us denote by

$$\mathcal{X}^\square_\mathcal{T} = \bigcup_{m \in T_{(n)}} \square_{\mathcal{T}(m)}. \tag{5.3}$$

We start the proof of Proposition 5.1 by an application of Lemma 3.3 with $d = 2$:

$$\mathbb{P}\left[B^u_{n,x}\right] \overset{(3.6)}{\leq} \mathbb{P}\left[\bigcup_{\mathcal{T} \in \Lambda^F_{n,x}} \bigcap_{m \in T_{(n)}} \{\square_{\mathcal{T}(m)} \cap \mathcal{I}^u \neq \emptyset\}\right] \overset{(*)}{\leq}$$
$$\mathcal{C}_2^{2^n} \cdot \max_{\mathcal{T} \in \Lambda^F_{n,x}} \mathbb{P}\left[\bigcap_{m \in T_{(n)}} \{\square_{\mathcal{T}(m)} \cap \mathcal{I}^u \neq \emptyset\}\right], \tag{5.4}$$

where in $(*)$ we used Lemma 3.2 to infer $|\Lambda^F_{n,x}| \leq \mathcal{C}_2^{2^n}$.

In order to bound the probability on the right-hand side of (5.4) let us fix some $\mathcal{T} \in \Lambda^F_{n,x}$, recall the constructive definition of random interlacements from Claim 2.2 and denote the probability underlying the random objects (i.e., N_K and $(X^j)_{j\geq 1}$) introduced in that claim by P when $K = \mathcal{X}_{\mathcal{T}}^{\square}$. For a simple random walk X let us denote by

$$\mathcal{N}(X) = \sum_{m \in T_{(n)}} \mathbb{1}[\{X\} \cap \square_{\mathcal{T}(m)} \neq \emptyset]$$

the number of frames of form $\square_{\mathcal{T}(m)}$, $m \in T_{(n)}$ that X visits. We can bound

$$\mathbb{P}\left[\bigcap_{m \in T_{(n)}} \{\square_{\mathcal{T}(m)} \cap \mathcal{I}^u \neq \emptyset\}\right] \leq \mathrm{P}\left[\sum_{j=1}^{N_K} \mathcal{N}(X^j) \geq 2^n\right]. \tag{5.5}$$

Our next goal is to stochastically bound $\mathcal{N}(X)$. Recall the definitions of c_g, C_g from (1.5) and L_0 from (1.8). Let us define

$$p = \begin{cases} 12C_g/c_g \cdot L_0^{3-d} & \text{if} \quad d \geq 4, \\ 12C_g/c_g \cdot \frac{1}{1+\ln(L_0)} & \text{if} \quad d = 3. \end{cases} \tag{5.6}$$

For any $m \in T_{(n)}$, $y \in \square_{\mathcal{T}(m)}$ we have

$$P_y[\{X\} \cap \mathcal{X}_{\mathcal{T}}^{\square} \setminus \square_{\mathcal{T}(m)} \neq \emptyset] \overset{(3.8),(5.3)}{\leq} \sum_{k=1}^{n} \sum_{m' \in T_{(n)}^{m,k}} P_y[\{X\} \cap \square_{\mathcal{T}(m')} \neq \emptyset] \overset{(1.5),(2.7),(3.10)}{\leq}$$

$$\sum_{k=1}^{n} \sum_{m' \in T_{(n)}^{m,k}} C_g L_{k-1}^{2-d} \mathrm{cap}(\square_{\mathcal{T}(m')}) \overset{(3.2),(3.9)}{=} \sum_{k=1}^{n} 2^{k-1} C_g L_0^{2-d} 6^{(k-1)(2-d)} \mathrm{cap}(\square_0) \overset{d\geq 3}{\leq}$$

$$C_g L_0^{2-d} \mathrm{cap}(\square_0) \sum_{k=1}^{\infty} 3^{1-k} \overset{(2.13),(5.6)}{\leq} p. \tag{5.7}$$

The bound (5.7) together with the strong Markov property of simple random walk imply that $P_{\widetilde{e}_K}[\mathcal{N}(X) \geq k] \leq p^{k-1}$ for any $k \geq 1$. In other words, $\mathcal{N}(X)$ is stochastically dominated by a geometric random variable with parameter $1-p$, which implies $E_{\widetilde{e}_K}\left[z^{\mathcal{N}(X)}\right] \leq \frac{(1-p)z}{1-pz}$ for any $1 \leq z < \frac{1}{p}$. Recalling from Claim 2.2 that N_K is Poisson with parameter $u \cdot \mathrm{cap}(K) = u \cdot \mathrm{cap}(\mathcal{X}_{\mathcal{T}}^{\square})$, for any $1 \leq z < \frac{1}{p}$ we obtain

$$\mathrm{E}\left[z^{\sum_{j=1}^{N_K} \mathcal{N}(X^j)}\right] = \exp\left(u \cdot \mathrm{cap}(\mathcal{X}_{\mathcal{T}}^{\square}) \left(E_{\widetilde{e}_K}\left[z^{\mathcal{N}(X)}\right] - 1\right)\right) \leq$$

$$\exp\left(u \cdot \mathrm{cap}(\mathcal{X}_{\mathcal{T}}^{\square}) \left(\frac{z-1}{1-pz}\right)\right).$$

We can thus apply the exponential Chebyshev inequality with $z = \frac{1}{2p}$ to bound

$$\mathbb{P}\left[B_{n,x}^u\right] \overset{(5.4),(5.5)}{\leq} \mathcal{C}_2^{2^n} \mathrm{E}\left[\left(\frac{1}{2p}\right)^{\sum_{j=1}^{N_K} \mathcal{N}(X^j)}\right] (2p)^{2^n} \leq$$

$$\exp\left(u \cdot \mathrm{cap}(\mathcal{X}_{\mathcal{T}}^{\square}) \left(\frac{\frac{1}{2p} - 1}{1/2}\right)\right) (2p\mathcal{C}_2)^{2^n} \overset{(2.8)}{\leq} \exp\left(u \cdot \frac{\mathrm{cap}(\square_0)}{p}\right)^{2^n} (2p\mathcal{C}_2)^{2^n} \overset{(1.8),(5.6)}{\leq}$$

$$\exp\left(u \cdot \mathrm{cap}(\square_0) 2^d \mathcal{C}_2\right)^{2^n} 2^{-2^n} \overset{(2.13)}{\leq} \exp\left(u \frac{L_0}{c_g} 2^{d+3} \mathcal{C}_2\right)^{2^n} 2^{-2^n} \overset{(5.1)}{\leq} \left(\frac{3}{4}\right)^{2^n}.$$

This completes the proof of Proposition 5.1. ◻

References

[1] O. Angel, B. Ráth, Q. Zhu. Branching interlacement. *(work in progress)*

[2] A. Drewitz, B. Ráth and A. Sapozhnikov (2014) Local percolative properties of the vacant set of random interlacements with small intensity. *Annales de l'Institut Henri Poincaré* **50**(4), 1165–1197. MR-3269990

[3] A. Drewitz, B. Ráth and A. Sapozhnikov (2014) *An Introduction to Random Interlacements.* SpringerBriefs in Mathematics, Springer.

[4] G.F. Lawler. *Intersections of random walks.* Probability and its Applications, Birkhäuser Boston Inc., 1991. MR-1117680

[5] S. Popov and A. Teixeira (2012) Soft local times and decoupling of random interlacements. *(to appear in the J. of the Eur. Math. Soc.)* arXiv:1212.1605.

[6] V. Sidoravicius and A.-S. Sznitman (2009) Percolation for the vacant set of random interlacements. *Comm. Pure Appl. Math.* **62** (6), 831–858. MR-2512613

[7] V. Sidoravicius and A.-S. Sznitman Connectivity bounds for the vacant set of random interlacements. *Ann. Inst. Henri Poincaré., Prob. et Stat.* **46(4)**, 976–990., 2010. MR-2744881

[8] A.-S. Sznitman (2010) Vacant set of random interlacements and percolation. *Ann. Math.* **171** (2), 2039–2087. MR-2680403

[9] A.-S. Sznitman (2011) A lower bound on the critical parameter of interlacement percolation in high dimension. *Probab. Theory Relat. Fields.* **150**, 575–611. MR-2824867

[10] A.-S. Sznitman (2011) On the critical parameter of interlacement percolation in high dimension. *Ann. Probab.* **39** (1), 70–103. MR-2778797

[11] A.-S. Sznitman (2012) Decoupling inequalities and interlacement percolation on $G \times \mathbb{Z}$. *Invent. Math.* **187** (3), 645–706. MR-2891880

[12] A. Teixeira (2009) Interlacement percolation on transient weighted graphs. *Electron. J. Probab.* **14**, 1604–1627. MR-2525105

[13] A. Teixeira (2011) On the size of a finite vacant cluster of random interlacements with small intensity. *Probab. Theory Related Fields* **150** (3–4), 529–574. MR-2824866

[14] D. Windisch (2008) Random walk on a discrete torus and random interlacements. *Electron. Commun. Probab.* **13**, 140–150. MR-2386070

Acknowledgments. The author thanks Daniel Valesin and Qingsan Zhu for inspiring discussions and an anonymous referee for useful comments on the manuscript. This paper was written while the author was a postdoctoral fellow of the University of British Columbia.