A MESHFREE DLO FORMULATION FOR YIELD LINE ANALYSIS OF REINFORCED CONCRETE SLABS

Le Van Canh[1] - Vilakone Nouphaxay[2] - Luong Van Hai[2]

ABSTRACT

The yield-line method of analysis is a powerful tool that can be used to rapidly estimate the limit load sustainable by a reinforced concrete slab. In recent years, it has been limited in use due to its difficulties to computerise. Consequently, the Discontinuity Layout Optimization (DLO) procedure has been proposed to provide a systematic means of automating the method. In the DLO formulation, the size of the underlying optimization is highly affected by the number of the potential yield-lines generated. In this paper the concept of domain of influence in the framework of mesh-free methods will be introduced to the DLO method, resulting an efficient DLO method that can provide accurate solutions compared with the original DLO method while the problem size is very much smaller.

Keywords: Limit analysis, meshfree DLO, yield line analysis, reinforced concrete slabs.

1. INTRODUCTION

The yield-line method is a long-established and powerful means of estimating the ultimate capacity of concrete slabs. The method involves postulating a failure mechanism (or ‘yield-line pattern’) and then using work or equilibrium equations to calculate the loading required to cause collapse. Whilst benefits of the method are numerous, in use a perennial concern is that an overestimate of the true capacity will be obtained unless the correct mechanism has been identified. To address this, an automated method capable of reliably and systematically identifying critical yield-line patterns has long been sought. However, despite many attempts to automate the method (e.g. Chan, Munro & da Fonseca, Johnson etc) success has been limited. Consequently in recent years efforts have focussed principally on the development of numerical methods which use a continuous representation of the problem fields (e.g. using finite elements [1, 2] or meshless methods [3]).

The recently developed Discontinuity Layout Optimization’(DLO) procedure [4, 5] appears to provide an opportunity to return to the inherently discontinuous analysis approach of yield-line analysis, embodied in a rigorous numerical analysis procedure. DLO has recently been found to provide a simple yet systematic and completely general means of identifying critical yield-line patterns. In order to minimize the number of initial connectivity, [4] has proposed an adaptive nodal connection procedure. Alternatively, in this paper a procedure using the concept of domain of influence similar to the one in the framework of mesh-free methods [3, 6] will be proposed. The difference between the meshfree DLO method and the original DLO method is that the

[1]*Department of Civil Engineering, International University, VNU HCMC, Vietnam.*
[2]*Faculty of Civil Engineering, University of Technology, VNU HCMC, Vietnam.*

domain of influence can limit the number of elements so that the potential yield lines and the size of the resulting optimization problem is kept minimal.

2. KINEMATIC FORMULATION FOR SLABS USING DLO

The general discretized kinematic DLO problem formulation may be stated as follows (after [4, 5]):

$$\lambda^+ = \min \mathbf{g}^T\mathbf{p} \tag{1a}$$

$$s.t\ \mathbf{Bd} = \mathbf{0} \tag{1b}$$

$$\mathbf{Np} - \mathbf{d} = \mathbf{0} \tag{1c}$$

$$\mathbf{f}_L^T \mathrm{d} = 1 \tag{1d}$$

$$\mathbf{p} \geq \mathbf{0} \tag{1e}$$

Considering the kinematic problem formulation for slabs, the contributions of a given yield-line i to the global compatibility constraint equation (1b) can be written as:

$$\mathbf{B}_i\mathbf{d}_i = \begin{bmatrix} \alpha_i & -\beta_i & 0 \\ \beta_i & \alpha_i & 0 \\ 0 & \frac{l_i}{2} & 1 \\ -\alpha_i & \beta_i & 0 \\ -\beta_i & -\alpha_i & 0 \\ 0 & \frac{l_i}{2} & -1 \end{bmatrix} \begin{bmatrix} \theta_{\mathrm{n}i} \\ \theta_{\mathrm{t}i} \\ \delta_i \end{bmatrix} \tag{2}$$

where $\theta_{\mathrm{n}i}$, $\theta_{\mathrm{t}i}$ and δ_i are respectively the normal rotation along a potential yield-line, the twisting rotation, and the out-of-plane displacement, and where α_i and β_i are x-axis and y-axis direction cosines.

Suppose that there exists no coupling between normal and twisting rotations, and between the shear displacement along a yield-line. In this case the contributions of a given yield-line i to the global flow rule constraint (1c) can be written as:

$$\mathbf{N}_i\mathbf{p}_i - \mathbf{d}_i = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} p_i^1 \\ p_i^2 \\ p_i^3 \\ p_i^4 \\ p_i^5 \\ p_i^6 \end{bmatrix} - \begin{bmatrix} \theta_{\mathrm{n}i} \\ \theta_{\mathrm{t}i} \\ \delta_i \end{bmatrix} \tag{3}$$

However, at a typical yield-line it can generally be assumed that the torsional (twisting) and out-of-plane displacements, $\theta_{\mathrm{t}i}$ and δ_i respectively, will be zero, and hence these variables can be omitted from the formulation, along with their corresponding plastic multiplier variables, p_i^3, p_i^4, p_i^5 and p_i^6. This situation does not apply at free boundaries, however, where θ_{ti} and δ_i should be free to take on arbitrary values. i.e. such variables should be added to signal the presence of such a boundary. Similarly at a line of symmetry, δ_i should be free to take on an arbitrary value.

3. WORKED EXAMPLE

Consider a fixed square slab ABCD of unit area and subject initially to a single central unit point load (assume vertices: A[0,0], B[1,0], C[1,1] and D[0,1]). If this problem is discretized using $n = 4$ nodes then $\frac{n(n-1)}{2} = \frac{4(4-1)}{2} = 6$ potential yield-lines will interconnect the nodes and the problem matrices and vectors of (1) are compact enough to be written out in full as follows:

$$\min E = \begin{bmatrix} 1 & 1 & \sqrt{2} & \sqrt{2} & 1 & 1 & 1 & 1 & \sqrt{2} & \sqrt{2} & 1 & 1 \end{bmatrix} \begin{bmatrix} \theta_{AB}^{+} \\ \theta_{AB}^{-} \\ \theta_{AC}^{+} \\ \theta_{AC}^{-} \\ \theta_{AD}^{+} \\ \theta_{AD}^{-} \\ \theta_{BC}^{+} \\ \theta_{BC}^{-} \\ \theta_{DB}^{+} \\ \theta_{DB}^{-} \\ \theta_{DC}^{+} \\ \theta_{DC}^{-} \end{bmatrix} \tag{4}$$

subject to:

$$\begin{bmatrix} 1 & -1 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & & & & & & & & \\ & & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 1 & -1 & & & & & & \\ -1 & 1 & & & & & & & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & \\ & & & & & & 1 & -1 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & & \\ & & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & & & & & & -1 & 1 \\ & & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & & -1 & 1 & & & & \\ & & & & & & & & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 1 & -1 \\ & & & & -1 & 1 & & & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & \end{bmatrix} \begin{bmatrix} \theta_{AB}^{+} \\ \theta_{AB}^{-} \\ \theta_{AC}^{+} \\ \theta_{AC}^{-} \\ \theta_{AD}^{+} \\ \theta_{AD}^{-} \\ \theta_{BC}^{+} \\ \theta_{BC}^{-} \\ \theta_{DB}^{+} \\ \theta_{DB}^{-} \\ \theta_{DC}^{+} \\ \theta_{DC}^{-} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{AB}^{+} \\ \theta_{AB}^{-} \\ \theta_{AC}^{+} \\ \theta_{AC}^{-} \\ \theta_{AD}^{+} \\ \theta_{AD}^{-} \\ \theta_{BC}^{+} \\ \theta_{BC}^{-} \\ \theta_{DB}^{+} \\ \theta_{DB}^{-} \\ \theta_{DC}^{+} \\ \theta_{DC}^{-} \end{bmatrix} = 1 \qquad (6)$$

If the slab is instead subject to a uniform out-of-plane pressure loading of unit intensity, the only change necessary is to replace Equation 5 with Equation 6 below:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{6\sqrt{2}} & -\frac{1}{6\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{6\sqrt{2}} & -\frac{1}{6\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{AB}^{+} \\ \theta_{AB}^{-} \\ \theta_{AC}^{+} \\ \theta_{AC}^{-} \\ \theta_{AD}^{+} \\ \theta_{AD}^{-} \\ \theta_{BC}^{+} \\ \theta_{BC}^{-} \\ \theta_{DB}^{+} \\ \theta_{DB}^{-} \\ \theta_{DC}^{+} \\ \theta_{DC}^{-} \end{bmatrix} = 1 \qquad (7)$$

Once the appropriate LP problems are solved, the resulting load factors at collapse can be found to be 16 and 48 for the point load and distributed load problems defined by (6) and (7) respectively. Other methods can of course be used to identify the same values for this very coarse numerical discretization, but the novel feature of the formulation described here is that *there has been no need to explicitly add a node at the centre of the slab*, something that is clearly not the case with the element-based methods put forward by workers such as [7] and [8].

4. THE CONCEPT OF DOMAIN OF INFLUENCE IN DLO

In order to minimize the number of initial connectivity, [4] has proposed an adaptive nodal connection procedure. Alternatively, in this section we will propose a procedure using the concept of domain of influence similar to the one in the framework of mesh-free methods [3, 6]. The radius of the domain of influence may be determined by

$$R_I = \beta \cdot h_I \tag{8}$$

where β is the dimensionless size of influence domain, ranging from 1 to max(n_x,n_y), and h_I is the diagonal of a cell which can be calculated by

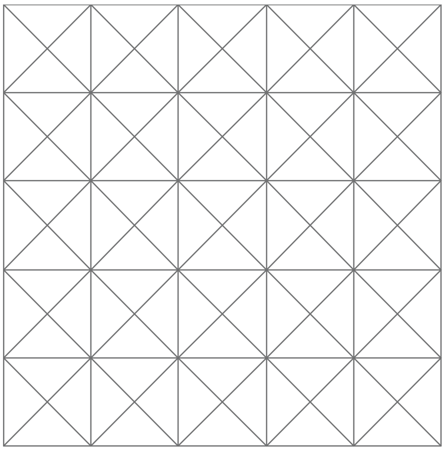
$$h_I = \sqrt{\left(\frac{L_x}{n_x}\right)^2 + \left(\frac{L_y}{n_y}\right)^2} \tag{9}$$

where L_x and L_y are the sizes of a slab in x and y direction; n_x and n_y are the number of elements in x and y direction, respectively.

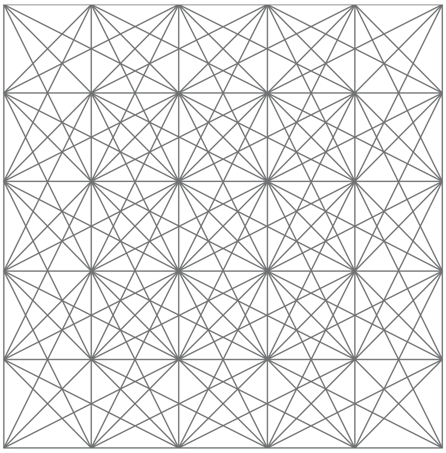
The influence of the size of domain of influence on the potential yield-lines is illustrated by Figure 1.

Figure 1. Potential yield-lines for different size of domain of influence

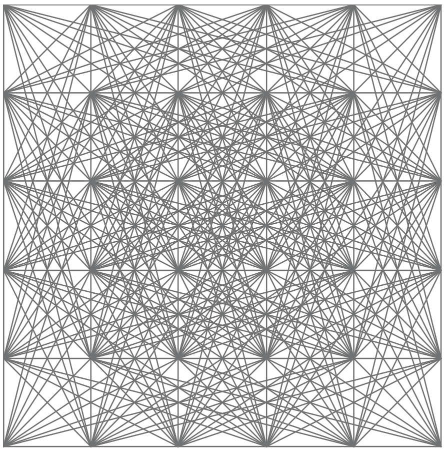
a) β = 1, 110 elements



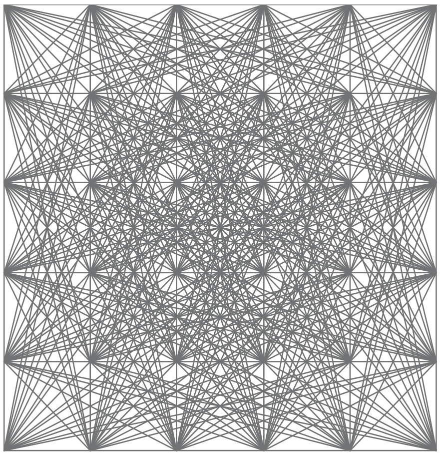
b) β = 2, 190 elements



c) β = 1, 338 elements



d) β = 1, 398 elements



5. NUMERICAL EXAMPLES

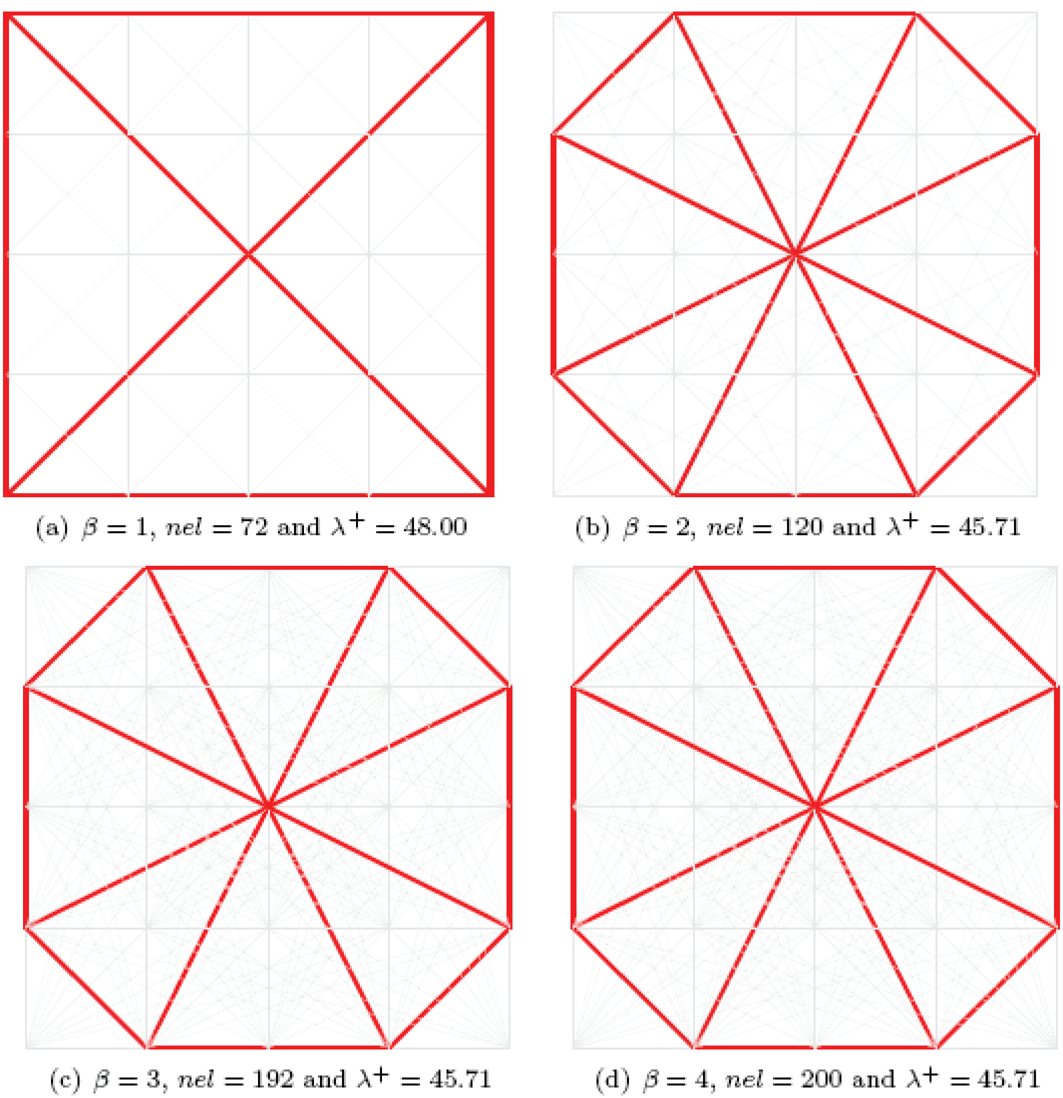
The procedure will now be applied to a range of isotropically reinforced slab problems which have previously been studied in the literature, including some which have known analytical solutions. The plastic moment is set to be unit, and Mosek optimization package will be used to obtain solutions.

The first example comprises a square slab with clamped supports on all edges. The exact solution has been identified by [9] as

$$\lambda_p = 42.851\frac{m_p}{qL^2} \tag{10}$$

Taking advantage of symmetric geometry, the slab was solved by the upper-right quarter. The influence of the size of domain of influence is first studied and illustrated in Figure 2. It can be observed that collapse multiplier and mechanism obtained using β values of 2, 3 and 4 are identical, while the number of elements (potential yield lines) in the case when β is equal to 2 is much smaller than the other cases ($\beta = 3,4$).

Figure 2. The influence of the size of domain of influence



(a) $\beta = 1$, $nel = 72$ and $\lambda^+ = 48.00$

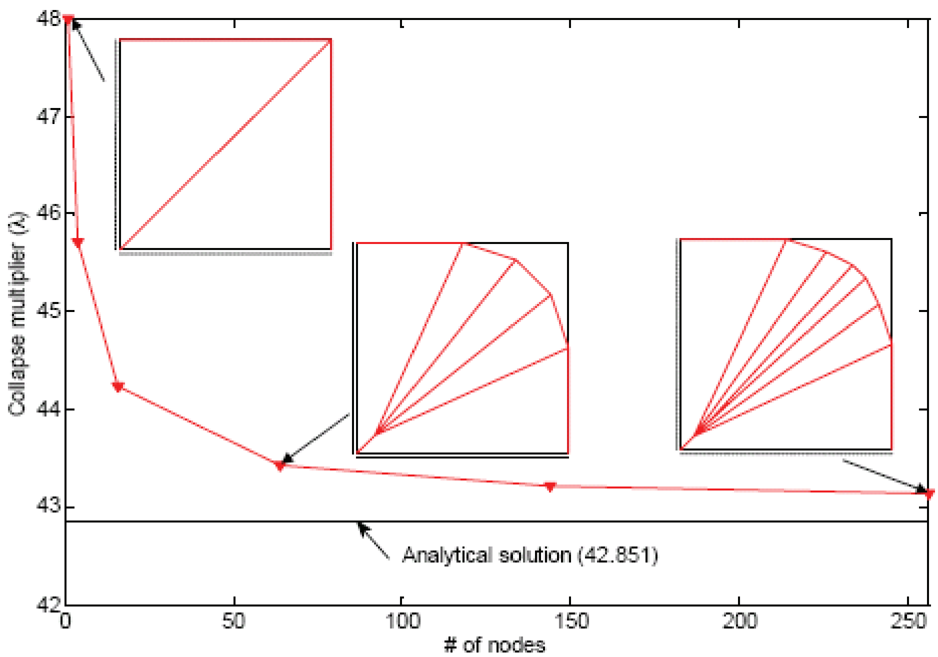
(b) $\beta = 2$, $nel = 120$ and $\lambda^+ = 45.71$

(c) $\beta = 3$, $nel = 192$ and $\lambda^+ = 45.71$

(d) $\beta = 4$, $nel = 200$ and $\lambda^+ = 45.71$

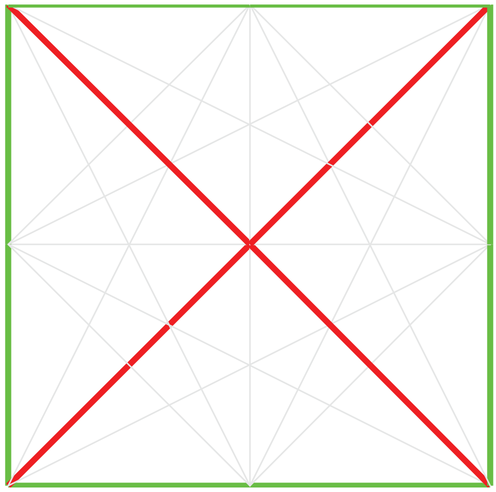
Collapse load multipliers and associated mechanisms for different number of nodes are reported in Figure 3.

Figure 3. Fixed square slab: collapse load vs number of nodes



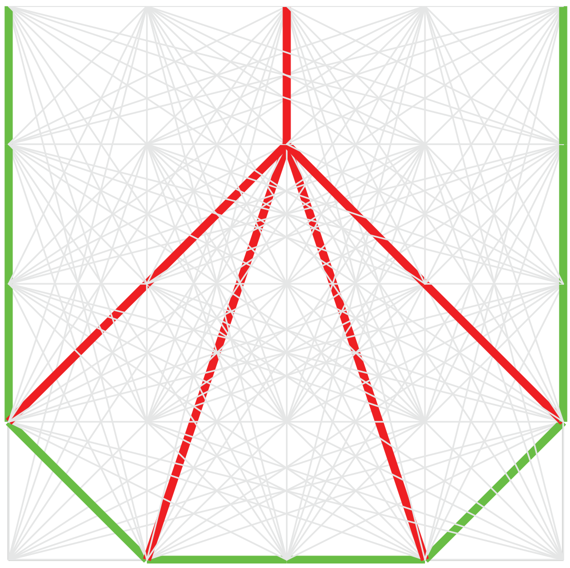
The second example is a square slab with simply supports on all edges. It is interesting to point out that the exact solution of (24.0 $\frac{m_p}{qL^2}$) can be obtained when only 4 nodes are employed, and the associated collapse mechanism is shown in Figure 4. In this case, the solution does not change when increasing the total number of nodes.

Figure 4. Square slab with simply supports: collapse mechanism (green line without plastic dissipation)



Finally, consider a square slab with 3 clamped and 1 free edges, subjected to a uniform load. The collapse load of 26.35 was obtained, and the associated collapse mechanism is plotted in Figure 5.

Figure 5. A square slab with 3 clamped and 1 free edges: collapse mechanism



6. CONCLUSION

A meshfree DLO method for yield-line analysis of reinforced concrete slabs has been described. The concept of domain of influence in the framework of mesh-free methods is introduced to the DLO method, ensuring that the size of the underlying optimization is reduced by minimizing the number of the potential yield-lines generated. Various examples were examined to show that the meshfree DLO based procedure can provide accurate solutions for engineering practice problems.

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