COMPUTATION OF LIMIT AND SHAKEDOWN USING THE NS-FEM AND SECOND-ORDER CONE PROGRAMMING

by Tran Trung Dung[1] - Le Van Canh[2] - Nguyen Xuan Hung[3]

ABSTRACT

This paper presents a novel numerical procedure for computation of limit and shakedown using node-based smoothed finite element method (NS-FEM) in combination with second-order cone programming (SOCP). The obtained discretization formulation is then cast in a form which involves second-order cone constraints, ensuring that the underlying optimization problem can be solved by highly efficient primal-dual interior point algorithm. Furthermore, in the NS-FEM, the system stiffness matrix is computed using the smoothed strains over the smoothing domains associated with nodes. This ensures that the size of the resulting optimization problem is kept to a minimum. The efficiency of the present approach is illustrated by examining several numerical examples.

Keywords: Limit and shakedown analysis (LSA), node-based smoothed finite element method (NS-FEM), second-order cone programming (SOCP).

1. INTRODUCTION

Limit and shakedown analysis (LSA) plays an important role in structural design and safety assessment of many engineering components and structures, from simple metal forming problems to large-scale engineering structures and nuclear power plants. Current research in the field of LSA is focussing on the development of numerical tools which are sufficiently efficient and robust to be of use to engineers working in practice [1].

Recently, a class of smoothed element methods (SFEMs) has been developed and applied to a variety of practical problems. The strain smoothing technique, which was originally proposed by Chen et al. [2] to stabilize a direct nodal integration in mesh-free methods, has been applied to the framework of FEM to formulate various smoothed finite element methods (SFEM) [3], including a cell-based SFEM, a node-based SFEM, an edge-based SFEM and a face-based SFEM [3]. Each of four new smoothing methods has different characters and advantages. In limit and shakedown, Tran *et al*. [4] have applied the ES-FEM and primal–dual algorithm based on a Newton iterative method to the problems using Koiter's theorem, in which fictitious elastic stresses are assumed.

Following this line of research, the main objective of this paper is to further develop NS-FEM for LSA of structures made of elastic-perfectly plastic material. However, in this paper the underlying optimization problem is transformed into the form of a SOCP problem with a large number of variables and nonlinear constraints so that it can be solved using the state-of-the-art primal-dual interior point algorithm. Moreover, in the NS-FEM the

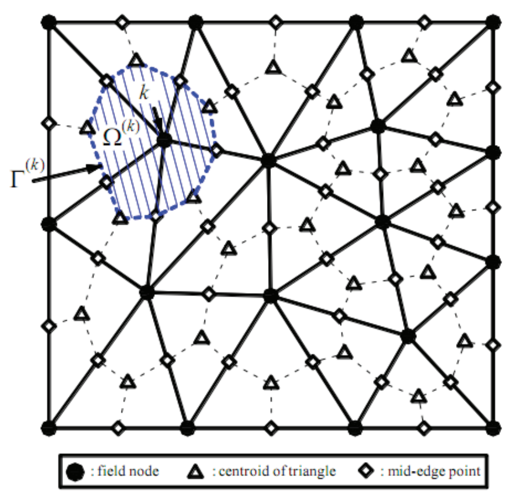
[1]*Faculty of Civil & Electrical Engineering, Ho Chi Minh City Open University, Vietnam.*
[2]*Department of Civil Engineering, International University, VNU HCMC, Vietnam.*
[3]*Department of Mechanics, Faculty of Mathematics & Computer Science, University of Science HCMC, Vietnam.*

system stiffness matrix is computed using the smoothed strains over the smoothing domains associated with nodes of element mesh. This ensures that the size of the resulting optimization problem is kept to a minimum. Numerical examples are presented to demonstrate the accuracy and effectiveness of the proposed method.

2. BRIEF OF THE NS-FEM

In NS-FEM, using the mesh of elements we further discretize the problem domain into smoothing domains based on nodes of the elements such that $\Omega \approx \sum_{k=1}^{N_n} \Omega^{(k)}$ and $\Omega^{(i)} \cap \Omega^{(j)} = \varnothing,\ i \neq j$, in which N_n is the total number of nodes of all elements in the entire problem domain. Moreover, NS-FEM shape functions are identical to those in the FEM. However, instead of using compatible strains, the NS-FEM uses strains smoothed over local smoothing domains. These local smoothing domains are constructed based on nodes of elements as shown in Fig.1. A strain smoothing formulation is now defined by the following operation:

Fig 1. Three-node triangular mesh and smoothing domains



$$\bar{\boldsymbol{\varepsilon}}_k = \int_{\Omega^{(k)}} \boldsymbol{\varepsilon}^h(\mathbf{x})\Phi_k(\mathbf{x})\mathrm{d}\Omega = \int_{\Omega^{(k)}} \nabla_s \mathbf{u}^h(\mathbf{x})\Phi_k(\mathbf{x})\mathrm{d}\Omega \tag{1}$$

where Φ(x) is a given smoothing function that satisfies at least unity property

$$\int_{\Omega^{(k)}} \Phi(\mathbf{x})\mathrm{d}\Omega = 1 \tag{2}$$

and in this work Φ(x) is assumed to be a step function given by

$$\Phi(\mathbf{x}) = \begin{cases} 1/A^{(k)} & \mathbf{x} \in \Omega^{(k)} \\ 0 & \mathbf{x} \notin \Omega^{(k)} \end{cases} \tag{3}$$

where $A^{(k)} = \int_{\Omega^{(k)}} \mathrm{d}\Omega$ is the area of the smoothing domain Ω_k and computed by $A^{(k)} = \int_{\Omega^{(k)}} \mathrm{d}\Omega = \frac{1}{3}\sum_{i=1}^{N_e^{(k)}} A_i^e$ in which $N_e^{(k)}$ is the number of elements connected to the node *k* and A_i^e is the area of the i^{th} element around the node *k*.

In term of nodal displacement vectors d_I, the smoothing strains $\bar{\boldsymbol{\varepsilon}}_k$ can be written as:

$$\bar{\boldsymbol{\varepsilon}}_k = \sum_{I \in N_n^{(k)}} \bar{\mathbf{B}}_I(x_k)\mathbf{d}_I \tag{4}$$

where $N_n^{(k)}$ is the number of nodes that are directly connected to node k, and $\overline{\mathbf{B}}_I(x_k)$ is the smoothed strain-displacement matrix on $\Omega^{(k)}$ the domain which is calculated numerically by an assembly process similarly as in the standard FEM

$$\overline{\mathbf{B}}_I(x_k)=\frac{1}{A^{(k)}}\sum_{j=1}^{N_e^{(k)}}\frac{1}{3}A_e^{(j)}\mathbf{B}_j^e \tag{5}$$

in which matrix $\mathbf{B}_j^e=\sum_{I\in S_e^j}\mathrm{B}_I$ is the compatible strain-displacement matrix for the j^{th} element around the node k. It is assembled from the compatible strain-displacement matrices $\mathbf{B}_I(\mathrm{x})$ of nodes in the set S_e^j which contains *nnel* nodes of the j^{th} linear element. Since linear shape functions are used, the entries of $\mathbf{B}_j^e$ are constants and therefore of $\overline{\mathbf{B}}_I(x_k)$ are also constants.

The smoothed domain stiffness matrix is then calculated by

$$\overline{\mathbf{K}}^{(k)}=\int_{\Omega^{(k)}}\overline{\mathbf{B}}_I^T\mathbf{C}\overline{\mathbf{B}}_I d\Omega=A^{(k)}\overline{\mathbf{B}}_I^T\mathbf{C}\overline{\mathbf{B}}_I \tag{6}$$

where $\mathbf{C}$ is the matrix of material constants, note that because the smoothed strains $\overline{\boldsymbol{\varepsilon}}_k$ in Eq. (1) are constants, the stresses $\overline{\boldsymbol{\sigma}}_k=\mathbf{C}\overline{\boldsymbol{\varepsilon}}_k$ are also constants in the smoothing domain $\Omega^{(k)}$.

3. LIMIT AND SHAKEDOWN ANALYSIS BASED ON ES-FEM

3.1. Kinematic formulation

Considering a continuum body is subjected to variable loads which may assume any value inside a bounded load domain. Let us now assume that the load domain is a convex hyperpolyhedron with m vertices $\hat{P}_i$ ($i=1,..,m$). At each load vertex, the kinematical condition may not be satisfied, however the accumulated strains over a load cycle $\Delta\overline{\boldsymbol{\varepsilon}}$ must be kinematically compatible. Let the fictitious elastic stress vector be σ^E. According to Koiter's theorem [5], the upper bound shakedown limit α^+ may be found by the solution of the following optimization problem [6].

$$\text{Min } \alpha^+=\sum_{i=1}^{m}\int_\Omega D^p\left(\dot{\overline{\boldsymbol{\varepsilon}}}_k\right)d\Omega \qquad (a) \tag{7}$$

$$\text{s.t.}\begin{cases}\Delta\overline{\boldsymbol{\varepsilon}}_k=\sum_{i=1}^{m}\dot{\overline{\boldsymbol{\varepsilon}}}_{ik} & \text{in } \Omega \qquad (b)\\ \Delta\mathbf{u}_k=0 & \text{on } \Gamma \qquad (c)\\ \sum_{i=1}^{m}\int_\Omega\dot{\overline{\boldsymbol{\varepsilon}}}_{ik}^T\boldsymbol{\sigma}_k^E\left(\mathbf{x},\hat{P}_i\right)d\Omega=1 & \qquad (d)\end{cases}$$

in which $D^p\left(\dot{\overline{\boldsymbol{\varepsilon}}}_k\right)$ is the plastic dissipation power per unit domain. Equation (7) implies the normalized condition, i.e. the external load power is equal to one. By discretizing the entire problem domain into smoothing domains, applying the strain smoothing technique and using Von Mises yield criterion, formulations (7) can be rewritten in the following form:

$$\min \qquad \alpha^+=\sum_{i=1}^{m}\sum_{k=1}^{N_n}A^{(k)}\sigma_0\sqrt{\dot{\overline{\boldsymbol{\varepsilon}}}_{ik}^T\mathbf{D}\dot{\overline{\boldsymbol{\varepsilon}}}_{ik}}$$

$$\text{s.t.} \quad \begin{cases} \sum_{i=1}^{m} \dot{\bar{\boldsymbol{\varepsilon}}}_{ik} = \overline{\mathbf{B}}_k \dot{\mathbf{u}} & \forall k = \overline{1, N^n}\ ,\ \forall i = \overline{1, m} \\ \sum_{i=1}^{m} \sum_{k=1}^{N_n} A^{(k)} \dot{\bar{\boldsymbol{\varepsilon}}}_{ik}^T \boldsymbol{\sigma}_{ik}^E = 1 & \end{cases} \tag{8}$$

where σ_0 is yield stress and $\mathbf{D} = \frac{1}{3}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for plane stress problem. It is worth noting that when m = 1, formulation (8) reduces to a limit analysis problem.

3.2. Solution procedure with second order cone programming

The above LSA problem is a non-linear optimization problem with equality constraints. The plastic dissipation, i.e. the objective function can now be written in the form:

$$\alpha^+ = \sum_{i=1}^{m} \sum_{k=1}^{N_n} A^{(k)} \sigma_0 \left\| \boldsymbol{\rho}_i^k \right\| \tag{9}$$

where $\left\| \boldsymbol{\rho}_i^k \right\|$ are additional variables defined by

$$\boldsymbol{\rho}_i^k = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\bar{\boldsymbol{\varepsilon}}}_{ik} \tag{10}$$

Introducing auxiliary variables $t_1, t_2, \ldots, t_{m*N_n}$, optimization problem (8) can be cast as a SOCP problem:

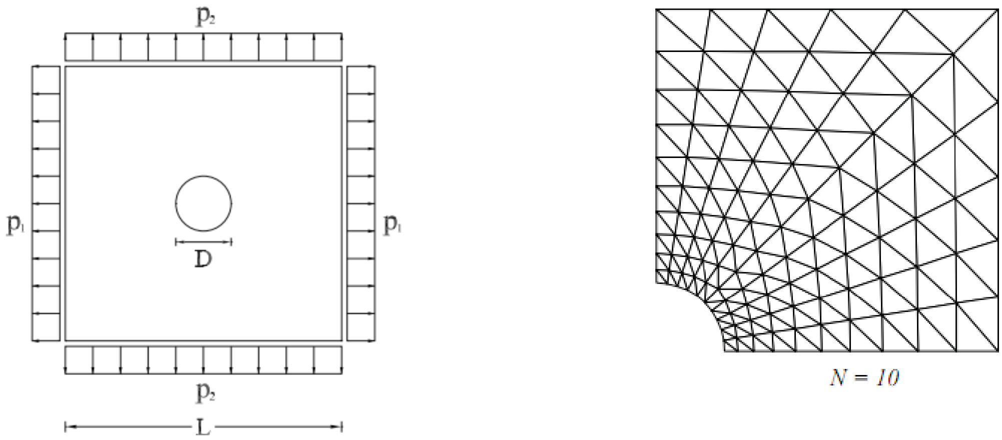
$$\min \quad \alpha^+ = \sum_{i=1}^{m} \sum_{k=1}^{N_n} A^{(k)} \sigma_0 t_{ik} \tag{11}$$

$$\text{s.t.} \quad \begin{cases} \sum_{i=1}^{m} \dot{\bar{\boldsymbol{\varepsilon}}}_{ik} = \overline{\mathbf{B}}_k \dot{\mathbf{u}} & \forall k = \overline{1, N^n}\ ,\ \forall i = \overline{1, m} \\ \sum_{i=1}^{m} \sum_{k=1}^{N_n} A^{(k)} \dot{\bar{\boldsymbol{\varepsilon}}}_{ik}^T \boldsymbol{\sigma}_{ik}^E = 1 & \\ \left\| \boldsymbol{\rho}_i^k \right\|_j \le t_j & j = 1, 2, \ldots, m * N_n \end{cases}$$

4. NUMERICAL EXAMPLE

In this section, the performance of the proposed solution procedure is illustrated via a benchmark problem in which analytical and other numerical solutions are available. The example deals with a square plate with a central circular hole with constant modulus of elasticity and thickness under independently varying pressure loads p_1 and p_2 as in Fig.2(a). The limit load factor was obtained analytically by Gaydon and McCrum [7] using plane stress hypothesis and von Mises yield criterion. Numerical limit and shakedown analyses were also investigated by some authors, e.g. Garcea et al. [8] for the case of *D/L*=0.2 and Heitzer [9], Vu [10], Tran [11] for different ratios of *D/L* to evaluate the elastic–plastic behaviour of the structure. Owing to its symmetry, only the upper-right quarter of the plate is modeled, see Fig2.(b). Symmetry conditions are enforced on the left and bottom edges.

Fig 2. A square plate with a circular hole:
(a) geometry and loading, (b) finite element mesh



Limit analysis

The procedure is first applied to the case of D/L=0.2 to verify the convergence of NS-FEM solutions in comparison with those of the FEM-Q4 and ES-FEM. Numerical solutions obtained for different models with variation of N are shown in Table 1. From these results, it is observed that all numerical limit load factors converge to the exact one, and NS-FEM can produce more accurate solutions than FEM and ES-FEM. Moreover, the present solution procedure with the use of second-order cone programming (SOCP) is more effcient and robust.

Table 1: Collapse load multiplier of the plate with variation of *N* (p2 = 0)

Formulation	N x N				Analytical solution
	6 x 6	12 x 12	24 x 24	48 x 48	
FEMQ4	0.8238	0.8090	0.8041	0.8021	
ES-FEM	0.8217	0.8077	0.8030	0.8013	0.8
Present method	0.8103	0.8035	0.8012	0.8004	

Next, in order to examine the geometric effect of the circular hole, various values of ratio D/L are considered. The obtained solutions were reported in Table 2. It can be observed that in case when $N = 12$ are used, $p_1 = 1$ and $p_2 = 0$, the present solutions are in good agreement with those obtained previously.

Table 2: Collapse multiplier when $p_1 = 1$ and $p_2 = 0$

D/L	Heitzer [9]	Tran [12]	Tran et al. [11]	ES-FEM	Present method
0.1	0.8951	0.9017	0.8932	0.9054	0.9021
0.2	0.7879	0.8015	0.7967	0.8077	0.8035
0.3	0.6910	0.7022	0.6930	0.7084	0.7039
0.4	0.5720	0.5914	0.5760	0.5961	0.5820
0.5	0.4409	0.4012	0.4011	0.4120	0.4022
0.6	0.2556	0.2425	0.2429	0.2524	0.2448
0.7	0.1378	0.1254	0.1277	0.1343	0.1288
0.8	0.0565	0.0523	0.0521	0.0573	0.0535
0.9	0.0193	0.0123	0.0133	0.0155	0.0131

Furthermore, Table 3 compares the best solutions obtained using the present method with solutions obtained previously by different limit analysis approaches (kinematic or static) using other FEM and meshfree models for case. Again, it can be seen that the NS-FEM solutions agree well with published ones. Specially, the NS-FEM produces the solutions that are more accurate (lower) compared with those obtained by Zouain et al. [15] using the mixed formulation (when $p_2 = 0$), despite the fact that our model uses only 544 degrees of freedom (DOFs) compared with 2014 DOFs used in [15]. The plastic dissipation distribution for the case D/L=0.2 is shown in Figure 3.

Table 3: Collapse load multiplier with different loading cases and N = 16 compared with previously obtained solutions D/L=0.2

Approach	Authors	Loading cases		
		$p_2 = p_1$	$p_2 = p_1/2$	$p_2 = 0$
Kinematic (upper bound)	da Silva and Antao [13]	0.899	0.915	0.807
	Le et al. [14]	0.895	0.911	0.801
	ES-FEM [4]	0.896	0.911	0.801
	Present method	0.894	0.911	0.802
Mixed formulation	Zouain et al. [15]	0.894	0.911	0.803
Analytical solution	Gaydon and McCrum [7]	–	–	0.800
Static (lower bound)	Chen et al. [16]	0.874	0.899	0.798
	Gross-Weege [17]	0.882	0.891	0.782

Shakedown analysis

The exact solution of this problem is not available and the first numerical study on this problem was shown by Belytschko. Table 4 compares the present solution with those in the literature for three special load combinations of p_1 and p_2. It can be seen that solutions obtained using the proposed numerical procedure are close to those obtained previously.

Table 4: Elastic shakedown load factors for *D/L* = 0.2

Approach and Methods	Authors	Loading cases		
		$p_1 = p_2$	$p_2 = p_1/2$	$p_2 = 0$
Lower bound	Zhang [18]	0.431	0.514	0.596
Nonlinear inequality approach (LB)	Genna [19]	0.478	0.566	0.653
BEM (LB)	Liu et al. [20]	0.477	0.549	0.647
Reduced basis technique (LB)	Gross-Weege [17]	0.446	0.524	0.614
Mixed approach	Zouain et al. [15]	0.429	0.500	0.594
Adaptive approach	Krabbenhøft et al. [23]	0.430	0.499	0.595
Iterative method	Garcea et al. [8]	0.438	0.508	0.604
Dual algorithm	Tran et al. [11]	0.444	0.514	0.610
Linear programming approach (UB)	Corradi and Zavelani [21]	0.504	0.579	0.654
Upper bound	Carvelli et al. [22]	0.518	0.607	0.696
Present method		0.518	0.593	0.694

5. CONCLUSION

A novel numerical LSA procedure that uses the node-based smoothed finite element method (NS-FEM) and second-order cone programming has been proposed. Advantages of applying the NS-FEM to LSA problems are that the size of optimization problem is reduced and accurate and stable solutions can be obtained with minimal computational effort. Numerical examples are given to demonstrate the efficiency and accuracy of the present method. It is shown that the proposed procedure is able to solve large-scale problems in engineering practice.

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