# ANALYSES OF STIFFENED PLATES RESTING ON THE VISCOELASTIC FOUNDATION SUBJECTED TO A MOVING VEHICLE BY A CELL-BASED SMOOTHED TRIANGULAR PLATE ELEMENT

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**ABSTRACT**

*Recently, a cell-based smoothed discrete shear gap method (CS-FEM-DSG3) based on the first-order shear deformation theory (FSDT) was proposed for static and free vibration analyses of Mindlin plates. The CS-FEM-DSG3 uses three-node triangular elements that can be easily generated automatically for arbitrary complicated geometric domains. This paper further extends the CS-FEM-DSG3 for static, free vibration, and dynamic response of the stiffened plate resting on viscoelastic foundation subjected to a moving vehicle. The viscoelastic foundation is modeled by discrete springs and dampers whereas the stiffened plate can be considered as the combination between the Mindlin plate and the Timoshenko beam elements. The moving vehicle is transformed into one concentrated load at its central point. Some numerical examples are investigated and numerical results show that the CS-FEM-DSG3 overcomes shear-locking phenomena and has a fast convergence. The results also illustrate the good agreement of the CS-FEM-DSG3 for static and free vibration analyses of un-stiffened plate compared with the previous published methods. In addition, the numerical results for dynamic analysis of stiffened plates by the CS-FEM-DSG3 also show the expected property in which the deflection of the stiffened plate is much smaller than those of the un-stiffened plate.*

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**1. Introduction**

Studying behaviors of the plates resting on the foundation subjected to moving loads is important in civil structures and the real life applications. Modeling plate resting on the foundation may be used to analyze the behavior of building foundations, reinforced concrete pavements of highways, airport runways etc. The foundation is usually classified into two models: (1) the Winkler's foundation and (2) the two parameters elastic foundation. In addition, the model of plate is based on two kinds of assumption namely: (1) the Kirchhoff plate and (2) the Mindlin plate. In the beginning, the analytical solutions were adopted to solve the problem **Error! Reference source not found.**], **Error! Reference source not found.**]; however, this method was difficult to apply for the general cases of the problem because it required the problem have to be simplified such as the behavior of road was assumed by the 1D beam instead of by the 2D plate. The numerical methods were also adopted to solve the problems such as: the finite element method (FEM) **Error! Reference source not found.**], the boundary element method (BEM) **Error! Reference source not found.**], **Error! Reference source not found.**] and other methods **Error! Reference source not found.**], **Error! Reference source not found.**].

The numerical methods are superior to the analytical methods because of their simplicity and ability to be easily applied to several problems whose boundary conditions are complex. One of the most superior numerical methods is the finite element method (FEM). Nevertheless, the FEM also has some drawbacks when it is applied to specific problems. Especially, when analyzing behavior of plates, the isoparametric elements and high order conforming elements are usually adopted but simultaneously they also exist some different disadvantages. The isoparametric elements are simplest; conversely, they encounter "shear-locking" phenomena and get a low convergence rate. While the high order conforming elements usually are very complex and require high computational cost in spite of their higher convergence rate. In the trend of developing the FEM for plate/shell, new kinds of the isoparameter element were introduced. These elements not only overcome shear-locking but also improve the convergence rate of the original elements. Along with these trends, Nguyen-Thoi et al. **Error! Reference source not found.**] recently proposed a new modified isoparametric element by combining the cell-based smoothed technique with the discrete shear gap method, namely the cell-based smoothed discrete shear gap method (CS-FEM-DSG3). The CS-FEM-DSG3 shows that it is very suitable to apply for analyses of both thin and thick plates. Furthermore, it can be extended to several kinds of plate such as stiffened plate **Error! Reference source not found.**], FGM plate **Error! Reference source not found.**], composite plate **Error! Reference source not found.**], etc.

In this paper, a study on the static and dynamic analyses of plates resting on the homogeneous Winkler viscoelastic foundation subjected to a moving vehicle is presented. In addition, an analysis to interpret the effect of stiffeners to the plate is performed. After reinforced by beams, the stiffness of structure is increased significantly. In the paper, the triangular Mindlin plate element, CS-FEM-DSG3, is incorporated with the linear two-node Timoshenko beam element. The displacement compatible condition between the plate and the stiffener is imposed so that displacement fields of the stiffener can be expressed in terms of the midsurface displacement of the plate. Some numerical results by CS-FEM-DSG3 are performed and compared with those by other methods to illustrate its accuracy and reliability. Moreover, some comments and discussions about the effect of stiffened plate on viscoelastic foundation, which have not been mentioned much before, are presented.

**2. Weak-form for stiffened plates resting on viscoelastic foundation**

The model of a plate resting on viscoelastic foundation is shown as in *Figure 1* where the viscoelastic foundation is considered as a system including discrete springs and dampers. The foundation has the parameters given by: (1) foundation stiffness $k_f$; and (2) damping coefficients $c_f$. The plate resting on foundation based on Mindlin plate theory is stiffened by the beams based on Timoshenko beam theory as shown in *Fig 2*. Some assumptions for the stiffened plate on viscoelastic foundation are followed as: (1) the displacements of plate and stiffeners at the contact positions is the same, (2) the total strain energy of the stiffened plate is the sum of that of plate and stiffeners, and (3) there are no splitting between stiffened plate and foundation **Error! Reference source not found.**].

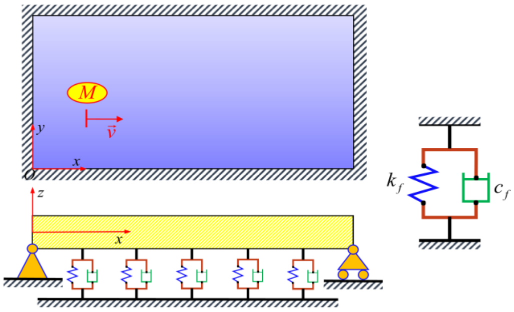



**Figure 1. Model of a plate resting on viscoelastic foundation subject to a moving load**

According to the second assumption, the total strain energy of stiffened plate resting on viscoelastic foundation can be expressed as

$$U = U_p + \sum_{i=1}^{N_{st}} U_{st} + U_k + U_d ; \quad (1)$$

where $N_{st}$ is the number of stiffeners and $U_p$, $U_{st}$, $U_k$, $U_d$ are the energies of plate, stiffeners, springs, and dampers, respectively. Particularly, the components of the total strain energy in of structures have formulations as following **Error! Reference source not found.**], **Error! Reference source not found.**], **Error! Reference source not found.**].

$$U = \frac{1}{2}\int_{\Omega_p} \boldsymbol{\varepsilon}^T \mathbf{D}\, \boldsymbol{\varepsilon}\, d\Omega + \frac{1}{2}\int_{\Omega_p} \boldsymbol{\kappa}^T \mathbf{D}\, \boldsymbol{\kappa} d\Omega + \frac{1}{2}\int_{\Omega_p} \boldsymbol{\gamma}^T \mathbf{D}\, \boldsymbol{\gamma} d\Omega_p ; \quad (2)$$

$$U_{st} = \frac{1}{2}\int_{\Omega_{st}} \left(\boldsymbol{\varepsilon}^{st}\right)^T \mathbf{D}^{st} \boldsymbol{\varepsilon}^{st} d\Omega_{st} ; \quad (3)$$

$$U_k = \int_{\Omega} \delta w^T k_f w d\Omega ; \quad (4)$$

$$U_d = \int_{\Omega} \delta w^T c_f \dot{w} d\Omega . \quad (5)$$

In Equation (2), $\boldsymbol{\varepsilon}_m = \mathbf{L}_m \mathbf{u}$, $\boldsymbol{\kappa} = \mathbf{L}_b \mathbf{u}$, and $\boldsymbol{\gamma} = \mathbf{L}_s \mathbf{u}$ are the membrane, bending, and shear strains of the plate, respectively, where $\mathbf{u} = \left\{u_0, v_0, w_0, \beta_x, \beta_y\right\}^T$ is the displacement field of the plate and $\mathbf{L}_i, i = m, b, s$, the gradient operator matrices, are given in **Error! Reference source not found.Error! Reference source not found.**] and **Error! Reference source not found.**]. The displacement field $\mathbf{u}$is defined by five components $u_0$, $v_0$, $w$, $\beta_x$, and $\beta_y$ which are the displacements in the plane of the plate, the out-of-plane deflection, and the rotations of the normal to the un-deformed middle surface in the *x*-*z* and *y*-*z*, respectively, with positive direction are defined as *Fig 2*. Lastly, the integral domain $\Omega_p \subset R^2$ is middle surface of plate as shown in *Fig 2* and $\mathbf{D}_i, i = m, b, s$ which are the material matrices can be found in **Error! Reference source not found.**].

In Equation (3), $\boldsymbol{\varepsilon}^{st} = \mathbf{L}^{st}\mathbf{u}_{st}$ is strain matrix of a stiffener where $\mathbf{u}_{st}$ is the displacement field of stiffener in local coordinate; and $\mathbf{L}^{st}$, the gradient operator matrix, is also given in **Error! Reference source not found.**]. $\mathbf{u}_{st} = \{u_r, u_s, u_z, \beta_r, \beta_s\}^T$ where $u_r$ is the axial displacement; $u_s$ is the lateral displacements in the plane of the plating; $u_z$ is the transversal deflection; $\beta_r$ and $\beta_s$ are the rotations made by the normal of the middle surface about the *s* and *r* axes, respectively. The relationship between stiffener's displacements in the local and global coordinate can be found in **Error! Reference source not found.**]. Furthermore, the displacements of stiffeners can be linked with those of plate by using the assumptions as discussed in **Error! Reference source not found.**]. Finally, the integral domain $\Omega_{st} \subset R$ is middle line of stiffener *Fig 2* and $\mathbf{D}^{st}$, the material matrix of stiffener, have the formulation as in **Error! Reference source not found.**], **Error! Reference source not found.**].

In Equation (5), $\dot{w}$ is the time derivative of the deflection of plate.

Next, in dynamic analysis of the problem, it is necessary to find the total potential energy of the stiffened plate subject to a moving vehicle. Similar to the way to find the total strain energy, the total potential energy has the form of

$$T = T_p + \sum_{i=1}^{N_{st}} T_{st}; \qquad (6)$$

where $T_p$ is the potential energy of plate given by

$$T_p = \frac{1}{2}\int_{\Omega_p} \dot{\mathbf{u}}^T \mathbf{m}_p \dot{\mathbf{u}} \mathrm{d}\Omega_p \qquad (7)$$

and $T_{st}$ is the potential energy of a stiffener given by

$$T_{st} = \frac{1}{2}\int_{\Omega_{st}} (\mathbf{T}\dot{\mathbf{u}})^T \mathbf{m}_{st} \mathbf{T}\dot{\mathbf{u}} \mathrm{d}\Omega_{st} \qquad (8)$$

in which $\dot{\mathbf{u}}$ denotes derivative with respect to time of the displacement field and $\mathbf{m}_p$, $\mathbf{m}_{st}$ are mass matrices of the plate and stiffener, respectively, defined by

$$\mathbf{m}_p = \rho \mathrm{diag}\left(h, h, h, h^3/12, h^3/12\right) \qquad (9)$$

and

$$\mathbf{m}_{st} = \rho A \begin{bmatrix} 1 & 0 & 0 & e & 0 \\ 0 & 1 & 0 & 0 & e \\ 0 & 0 & 1 & 0 & 0 \\ e & 0 & 0 & e^2 + I_s/A & 0 \\ 0 & e & 0 & 0 & e^2 + (I_s + I_z)/A \end{bmatrix} \qquad (10)$$

in which $\rho$ is the density of the plate; *h* is the thickness of plate; *A* is section area of the stiffener; $I_s$ is second moment of the stiffener cross-sectional area about an axis parallel with the *s*-axis and touching the centroid of the stiffener; $I_z$ is second moment of the stiffener cross-sectional area about the *z*-axis; and *e* is the eccentricity between plate and stiffener as in *Fig 2*.

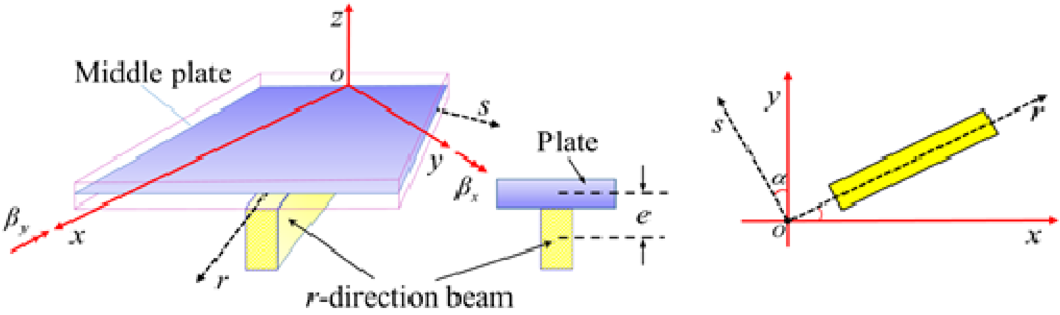



**Fig 2. Model of a stiffened plate and the global and local coordinate systems of stiffener.**

Finally, the standard Galerkin weak-form of the transient analysis of stiffened plate resting on viscoelastic foundation can now be written as

$$\begin{aligned}&\int_{\Omega}\delta\boldsymbol{\varepsilon}_m^T\mathbf{D}_m\boldsymbol{\varepsilon}_m d\Omega+\int_{\Omega}\delta\boldsymbol{\kappa}^T\mathbf{D}_b\boldsymbol{\kappa}d\Omega+\int_{\Omega}\delta\boldsymbol{\gamma}^T\mathbf{D}_s\boldsymbol{\gamma}d\Omega+\int_{\Omega}\delta\mathbf{u}^T\mathbf{m}_p\ddot{\mathbf{u}}d\Omega\\&+\sum_{N_{st}}\int_{\Omega_{st}}\delta\left(\boldsymbol{\varepsilon}^{st}\right)^T\mathbf{D}^{st}\boldsymbol{\varepsilon}^{st}d\Omega_{st}+\sum_{N_{st}}\int_{\Omega_{st}}\delta\mathbf{u}^T\mathbf{T}^T\mathbf{m}_p\mathbf{T}\ddot{\mathbf{u}}d\Omega_{st}\\&+\int_{\Omega}\delta w^Tk_f w d\Omega+\int_{\Omega}\delta w^Tc_f\dot{w}d\Omega=\int_{\Omega}\delta\mathbf{u}^T\mathbf{b}d\Omega\end{aligned}\tag{11}$$

where $\mathbf{b}=\{0\ 0\ p(x,y)\ 0\ 0\}^T$ in which $p(x,y)$ is the distributed load applied on the plate. The discrete system of equations by FEM will be obtained by replacing the displacement field in Equation (11) by approximation of it as in the following section.

### 3. Brief of the CS-FEM-DSG3 formulation

In this paper, the middle plane of the plate is divided to a set of the triangular elements. In each triangular element, the CS-FEM-DSG3 is used to analyze behavior of plate resting on elastic foundation. The CS-FEM-DSG3 is a combination of the discrete shear gap method using three-node triangular element (DSG3) with the cell-based gradient smoothing technique (CS-FEM). In the formulation of CS-FEM-DSG3, each of the original elements is divided to three sub-triangular elements, and then in each sub-triangle, the stabilized DSG3 is used to compute the strain field of plate. Finally the strain smoothing technique on the whole triangular element is used to smooth the strains on these three sub-triangles. Details of the formulation of the CS-FEM-DSG3 can be found in **Error! Reference source not found.**], **Error! Reference source not found.**].

After above-mention process, we obtained the smoothed membrane, bending and shear strains as, respectively

$$\tilde{\boldsymbol{\varepsilon}}_m=\tilde{\mathbf{B}}_m\mathbf{d},\ \boldsymbol{\kappa}=\tilde{\mathbf{B}}_b\mathbf{d},\ \boldsymbol{\gamma}=\tilde{\mathbf{B}}_s\mathbf{d}\tag{12}$$

where $\tilde{\mathbf{B}}_i, i=m,b,s$ are smoothed strain gradient matrices that defined by

$$\tilde{\mathbf{B}}_i=\frac{A^{\Delta_1}\mathbf{B}_i^{\Delta_1}+A^{\Delta_2}\mathbf{B}_i^{\Delta_2}+A^{\Delta_3}\mathbf{B}_i^{\Delta_3}}{A^e}\tag{13}$$

where $A^{\Delta_i}\ (i=1,2,3)$ is the area of the $i^{\text{th}}$ sub-triangle and $A^e$ is the area of the whole triangular element.

Substituting Equation (12) into Equation (11), the equilibrium equations for static, free vibration and dynamic analysis are, respectively

$$\mathbf{Ku}=\mathbf{F};\ \mathbf{M\ddot{u}}+\mathbf{Ku}=0;\ \mathbf{M\ddot{u}}+\mathbf{C\dot{u}}+\mathbf{Ku}=\mathbf{F}(t) \quad (14)$$

in which **K**, **M**, **C** are defined as

$$\begin{aligned}\mathbf{K} &= \int_\Omega \tilde{\mathbf{B}}_m^T \mathbf{D}_m \tilde{\mathbf{B}}_m \mathrm{d}\Omega + \int_\Omega \tilde{\mathbf{B}}_b^T \mathbf{D}_b \tilde{\mathbf{B}}_b \mathrm{d}\Omega + \int_\Omega \tilde{\mathbf{B}}_s^T \mathbf{D}_s \tilde{\mathbf{B}}_s \mathrm{d}\Omega + \sum_{N_{st}} \int_{\Omega_{st}} \left(\mathbf{L}^{st}\right)^T \mathbf{D}^{st} \mathbf{L}^{st} \mathrm{d}\Omega_{st} \\ &+ \int_\Omega \mathbf{N}^T \left\{0,0,k_f,0,0\right\} \mathbf{N} \mathrm{d}\Omega \\ \mathbf{M} &= \int_\Omega \mathbf{N}^T \mathbf{m}_p \mathbf{N} \mathrm{d}\Omega + \sum_{N_{st}} \int_{\Omega_{st}} \mathbf{N}^T \mathbf{T}^T \mathbf{m}_p \mathbf{T} \mathbf{N} \mathrm{d}\Omega_{st} \\ \mathbf{C} &= \int_\Omega \mathbf{N}^T \left\{0,0,c_f,0,0\right\} \mathbf{N} \mathrm{d}\Omega \end{aligned} \quad (15)$$

where **N**, a diagonal matrix, contains three node linear shape functions. The dynamic behavior of stiffened plate resting on viscoelastic foundation is deduced by using Newmark's constant acceleration method **Error! Reference source not found.**].

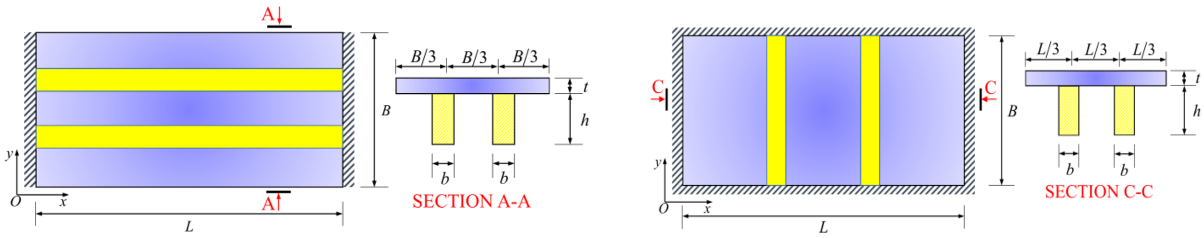
**4. Numerical results**

In this section, various numerical examples including static, free vibration, and dynamic response analyses of un-stiffened and stiffened plates resting on homogeneous Winkler viscoelastic foundation are performed to verify the accuracy and reliability of the CS-FEM-DSG3 element in comparison to others existing numerical solutions.

The non-dimensional coefficient of homogeneous Winkler elastic foundation is given by

$$K = k_f\, B^4 / D \quad (16)$$

where $D = Et^3/12(1-\nu)$ is the bending stiffness of the plate.

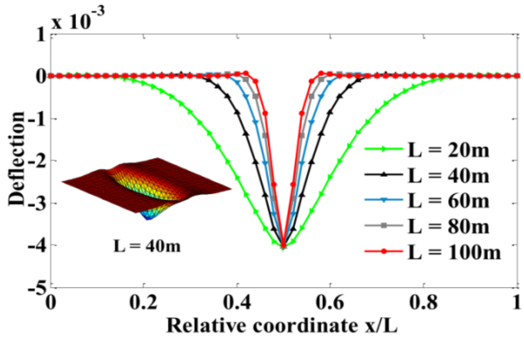



(a) (b)

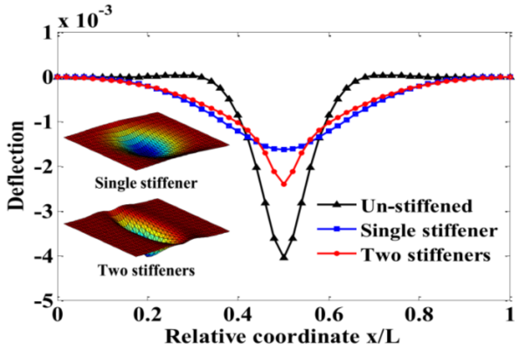
**Fig 3. A plate stiffened by**
**(a) double stiffener along x-axis; (b) double stiffener along y-axis**

***4.1. Static analysis of a stiffened plate resting on the elastic foundation***

Firstly, a rectangular plate resting on Winkler elastic foundation with the non-dimensional elastic foundation coefficient $K$ of 1000 is considered. The plate is subjected to a concentrated load $P$ = 1000N at its center. The width and the thickness are $B$ = 10m, $t$ = 0.5m respectively as shown in Fig 3 (a). The plate is free along two longer edges and is simply supported along two remaining edges. The material parameters of the plate are Young's modulus $E = 3.1\times10^{10}$ N/m$^2$ and Poisson's ratio $\nu = 0.2$. The non-dimensional deflection at middle line along the longitudinal direction $x$ of the plate with various length $L$ is shown in *Fig* 4 (a).




(a)




(b)

**Fig 4. The non-dimensional deflection at middle line** $\bar{w} = w\,D/(PB^2)$

This example was previously studied by Huang and Thambiratnam **Error! Reference source not found.**] using finite strip method. From the Figure 9 of the reference **Error! Reference source not found.**] on page 2556, it can be seen that the solution of the CS-FEM-DSG3 has good agreement to the reference solution. When the length of the plate is changed whereas the width is remained, the deflection of plate at load point is almost unchanged.

Next, the effect of stiffener to plate resting on elastic foundation is considered. The plate is stiffened by single and double stiffeners, respectively, parallel to *x*-axis as shown in *Fig 3* (a). The geometrical parameters of the stiffener are $h$ = 2m, $b$ = 0.5m and the material properties of the stiffener are the same as those of the plate. As shown in the *Fig 4* (b), the deflections of the plate are presented for three cases: i) the unstiffened plate, ii) the plate is stiffened by a single stiffener, and iii) the plate is stiffened by double stiffeners. It is obvious that the rigidity of the plate is significantly improved when the stiffeners are attached along $x$ direction of the plate. However, the central deflection of the plate with single stiffener is smaller than that of the plate with double stiffeners. It can be explained that the stiffest positions of stiffened plate is at the contact ones between plate and stiffeners.

### *4.2. Free vibration of a stiffened plate resting on the elastic foundation*

Free vibration analysis of the above structure is considered in this example. The plate is simply supported along all of edges. The dimensions of the plate are $L$ = 30m, $B$ = 10m and $t$ = 0.5m. The material parameters of the plate are given: Young's modulus $E = 3.1\times10^{10}$ N/m$^2$, Poisson's ratio $\nu = 0.3$ and the mass density of the plate $\rho = 2500$ kg/m$^3$. The non-dimensional foundation coefficient is $K$ = 100.

Non-dimensional frequencies of the first five modes are shown in Table 1. It is seen that the results by present method agree well with those by Huang and Thambiratnam [13]. The results also illustrate that the fundamental frequencies of the plate resting on foundation depend on the stiffness of the foundation.

**Table 1. Convergence of first five non-dimensional frequencies of the plate on the elastic foundation** $\bar{\omega} = \omega\sqrt{\rho h B^4 / D}$

| Stiffness factor *K* | Mode | Meshing 24 x 8 | Meshing 48 x 16 | Meshing 72 x 24 | Reference solution |
|---|---|---|---|---|---|
| 0 | 1 | 11.0927 | 10.9563 | **10.9313** | 10.9658 |
| | 2 | 14.4714 | 14.2397 | **14.1973** | 14.2558 |
| | 3 | 20.1532 | 19.7082 | **19.6271** | 19.7390 |
| | 4 | 28.2106 | 27.3562 | **27.2007** | 27.4157 |
| | 5 | 38.7398 | 37.1758 | **36.8907** | 37.2861 |
| 100 | 1 | 14.9272 | 14.8262 | **14.8077** | 14.4279 |
| | 2 | 17.5820 | 17.3919 | **17.3572** | 16.8449 |
| | 3 | 22.4888 | 22.0911 | **22.0187** | 19.7390 |
| | 4 | 29.9212 | 29.1173 | **28.9712** | 28.4173 |
| | 5 | 40.0003 | 38.4877 | **38.2125** | 38.6348 |

Next, the effect of the stiffeners to free vibration frequencies of the plate resting on foundation is considered. *Fig 3* (b) shows the plate stiffened by the stiffeners parallel to the *y*-axis. The material parameters of the stiffeners are the same as those of the plate in the example 4.1. *Table 2* compares the non-dimensional first five frequencies of the un-stiffened plate and stiffened ones. It is observed that frequencies of the plate significantly increase when the plate is stiffened, as expected.

**Table 2. First five non-dimension frequencies of the un-stiffened and stiffened plates**

| Number of stiffeners | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 |
|---|---|---|---|---|---|
| 0 | 14.8077 | 17.3572 | 22.0187 | 28.9712 | 38.2125 |
| 1 | 17.4462 | 18.0048 | 28.7950 | 28.9987 | 40.2121 |
| 2 | 22.1467 | 22.6871 | 23.8013 | 38.9191 | 39.0821 |
| 3 | 28.4480 | 28.6639 | 28.8977 | 29.1359 | 38.0381 |

First three mode shapes of un-stiffened and stiffened plate are shown in *Figure 5*. From these figures, it can be observed that the mode shapes of plate are changed considerably because of the effect of the stiffeners.

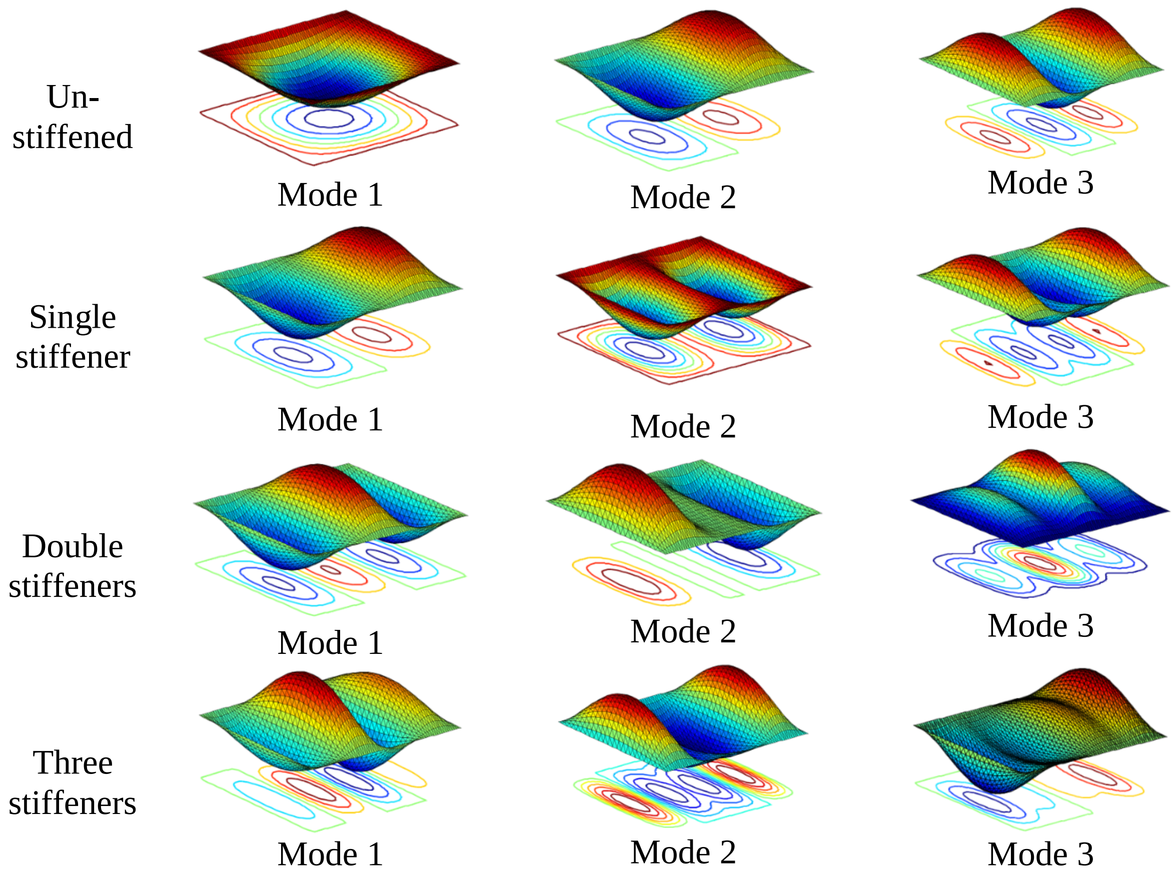



**Figure 5. Comparison of the first three mode shapes of the un-stiffened and stiffened plates rested on elastic foundation**

***4.3. Dynamic analysis of a stiffened plate resting on viscoelastic foundation subjected to a moving vehicle***

Finally, a rectangular plate simply supported along two shorter edges subjected to a moving vehicle at velocity $v$ = 40m/s in $T$ = 0.5s on the middle line along the longitudinal direction $x$ as shown in *Fig 3* (a) is investigated. The mass of the vehicle is $M$ = 1000kg. The dimension of the plate are $L$ = 20m, $B$ = 10m and $t$ = 0.3m. The material parameters of the plate are Young's modulus $E = 3.1\times10^{10}\,\mathrm{N/m^2}$, Poisson's ratio $\nu = 0.2$ and the density mass $\rho = 1000\,\mathrm{kg/m^3}$. The non-dimensional foundation stiffness coefficient is $K$ = 1000 and the damping coefficient is $c_f = 5\times10^5\,\mathrm{Ns/m^2}$ .

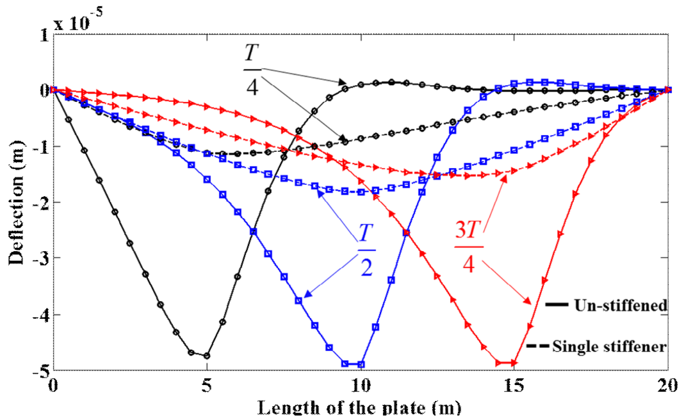



**Figure 6. Deflections of the un-stiffened and stiffened plates resting on viscoelastic foundation**

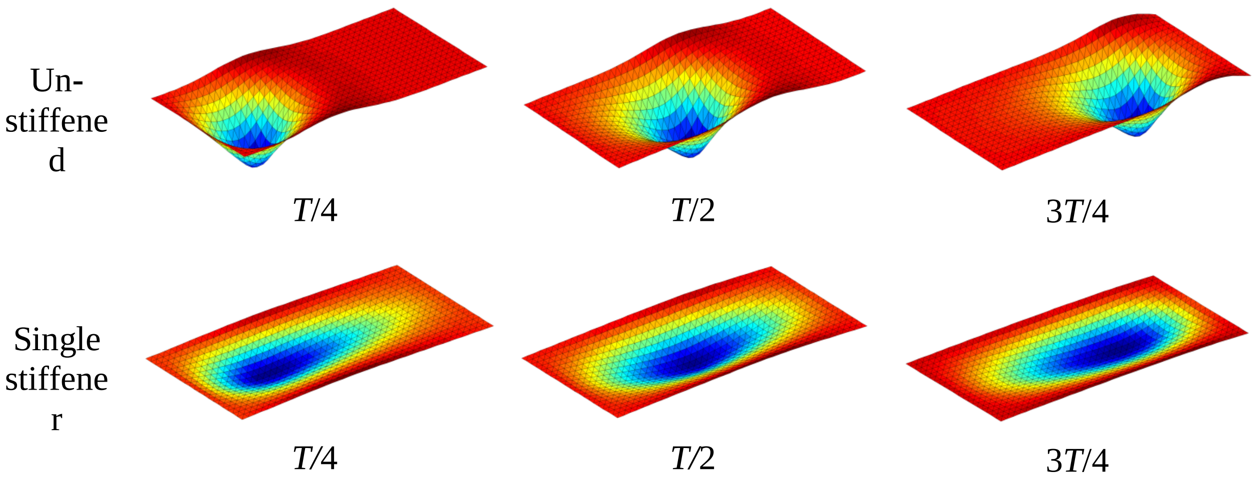



**Figure 7. Deformation corresponding to un-stiffened and stiffened plates**

It is assumed that the vehicle weight is transformed into only one concentrated load at the central point of the vehicle. *Figure 6* and *Figure 7* presents the deflection results of the un-stiffened and stiffened plates resting on viscoelastic foundation at $T/4$, $T/2$, $3T/4$. The results show that the deflection of the stiffened plate is much smaller than those of the un-stiffened plate, as expected.

**5. Conclusion**

The paper presents an extension of the CS-FEM-DSG3 using triangular element for static, free vibration, and dynamic response of the stiffened plate resting on viscoelastic foundation subjected to a moving vehicle. The viscoelastic foundation is modeled by discrete springs and dampers whereas the stiffened plate can be considered as the combination between the Mindlin plate and the Timoshenko beam elements. The moving vehicle is transformed into one concentrated load at its central point. Some numerical examples are investigated and numerical results show that the CS-FEM-DSG3 overcomes shear-locking phenomena and has a fast convergence to the solution. In the case of un-stiffened plates, the numerical results by the CS-FEM-DSG3 for static and free vibration analyses agree well with the previous published results. While to stiffened plates, the numerical results by the CS-FEM-DSG3 illustrated logically the expected property in which the deflection of the stiffened plate is much smaller than those of the un-stiffened plate.

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