Optimal pebbling of grids

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Abstract: A pebbling move on a graph removes two pebbles at a vertex and adds one pebble at an adjacent vertex. Rubbling is a version of pebbling where an additional move is allowed. In this new move, one pebble each is removed at vertices v and w adjacent to a vertex u, and an extra pebble is added at vertex u. A vertex is reachable from a pebble distribution if it is possible to move a pebble to that vertex using rubbling moves. The optimal pebbling (rubbling) number is the smallest number m needed to guarantee a pebble distribution of m pebbles from which any vertex is reachable using pebbling (rubbling) moves. We determine the optimal rubbling number of ladders ($P_n \square P_2$), prisms ($C_n \square P_2$) and Möblus-ladders. We also give upper and lower bounds for the optimal pebbling and rubbling numbers of large grids ($P_n \square P_n$).

Keywords: pebbling, rubbling, optimal pebbling

1 Introduction

Graph pebbling has its origin in number theory. It is a model for the transportation of resources. Starting with a pebble distribution on the vertices of a simple connected graph, a *pebbling move* removes two pebbles from a vertex and adds one pebble at an adjacent vertex. We can think of the pebbles as fuel containers. Then the loss of the pebble during a move is the cost of transportation. A vertex is called *reachable* if a pebble can be moved to that vertex using pebbling moves. There are several questions we can ask about pebbling. One of them is: How can we place the smallest number of pebbles such that every vertex is reachable (*optimal pebbling number*)? For a comprehensive list of references for the extensive literature see the survey papers [4, 5, 6]. Results on special grids can be found in [2] where

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the authors show that $\pi_{\text{opt}}(P_n \square P_2) = \pi_{\text{opt}}(C_n \square P_2) = n$ apart from a few smaller case, and in [11] the author gave upper bounds for the optimal pebbling number of various grids.

In the present paper we give better upper and lower bounds for the optimal pebbling numbers of large grids ($P_n \square P_n$).

Graph rubbling is an extension of graph pebbling. In this version, we also allow a move that removes a pebble each from the vertices v and w that are adjacent to a vertex u, and adds a pebble at vertex u. The basic theory of rubbling and optimal rubbling is developed in [1]. The rubbling number of complete m-ary trees are studied in [3], while the rubbling number of caterpillars are determined in [10]. In [7] the authors gives upper and lower bounds for the rubbling number of diameter 2 graphs.

In the present paper we determine the optimal rubbling number of ladders ($P_n \square P_2$), prisms ($C_n \square P_2$) and Möblus-ladders. We also give upper and lower bounds for the optimal rubbling numbers of large grids ($P_n \square P_n$).

2 Definitions

Throughout the paper, let G be a simple connected graph. We use the notation $V(G)$ for the vertex set and $E(G)$ for the edge set. A *pebble function* on a graph G is a function $p : V(G) \to \mathbb{Z}$ where $p(v)$ is the number of pebbles placed at v. A *pebble distribution* is a nonnegative pebble function. The *size* of a pebble distribution p is the total number of pebbles $\sum_{v \in V(G)} p(v)$. We say that a vertex v is *occupied* if $p(v) > 1$, else it is *unoccupied*.

Consider a pebble function p on the graph G. If $\{v, u\} \in E(G)$ then the *pebbling move* $(v, v \to u)$ removes two pebbles at vertex v, and adds one pebble at vertex u to create a new pebble function p', so $p'(v) = p(v) - 2$ and $p'(u) = p(u) + 1$. If $\{w, u\} \in E(G)$ and $v \neq w$, then the *strict rubbling move* $(v, w \to u)$ removes one pebble each at vertices v and w, and adds one pebble at vertex u to create a new pebble function p', so $p'(v) = p(v) - 1$, $p'(w) = p(w) - 1$ and $p'(u) = p(u) + 1$.

A *rubbling move is* either a pebbling move or a strict rubbling move. A *rubbling sequence* is a finite sequence $T = (t_1, \dots, t_k)$ of rubbling moves. The pebble function obtained from the pebble function p after applying the moves in T is denoted by p_T. The concatenation of the rubbling sequences $R = (r_1, \dots, r_k)$ and $S = (s_1, \dots, s_l)$ is denoted by $RS = (r_1, \dots, r_k, s_1, \dots, s_l)$.

A rubbling sequence T is *executable* from the pebble distribution p if $p_{(t_1,\dots,t_i)}$ is nonnegative for all i. A vertex v of G is *reachable* from the pebble distribution p if there is an executable rubbling sequence T such that $p_T(v) \geq 1$. p is a solvable distribution when each vertex is reachable. All the above notions are defined for pebbling as well, just we restrict ourselves to pebbling moves.

The *optimal pebbling* $\pi_{\text{opt}}(G)$ *and rubbling number* $\varrho_{\text{opt}}(G)$ of a graph G is the size of a distribution with the least number of pebbles from which every vertex is reachable using pebbling/rubbling moves. For large graphs it is better to consider the ratio of the optimal pebbling or rubbling number and the number of the vertices of the graph. So the *Optimal Pebbling Density* is $\text{OPD}(G) = \pi_{\text{opt}}(G)/|V(G)|$ and the *Optimal Rubbling Density* is $\text{ORD}(G) = \varrho_{\text{opt}}(G)/|V(G)|$.

Let G and H be simple graphs. Then the *Cartesian product* of graphs G and H is the graph whose vertex set is $V(G) \times V(H)$ and (g, h) is adjacent to (g', h') if and only if $g = g'$ and $(h, h') \in E(H)$ or if $h = h'$ and $(g, g') \in E(G)$. This graph is denoted by $G \square H$.

P_n and C_n denotes the path and the cycle containing n distinct vertices, respectively. We call $P_n \square P_2$ a *ladder* and $C_n \square P_2$ a *prism*, and $P_{n_1} \square P_{n_2}$ in general a *grid*. It is clear that the prism can be obtained from the ladder by joining the 4 endvertices by two edges to form two vertex disjoint C_n subgraphs. If the four endvertices are joined by two new edges in a switched way to get a C_{2n} subgraph, then a *Möbius-ladder* is obtained.

3 Optimal rubbling number of the ladder, the n-prism and Möbius-ladder

Our main result is the following formula for the optimal rubbling number of ladders:

Theorem 1 *Let $n = 3k + r$ such that $0 \leq r < 3$ and $n, r \in \mathbb{N}$, so $k = \lfloor \frac{n}{3} \rfloor$.*

$$\varrho_{opt}(P_n \Box P_2) = \begin{cases} 1 + 2k & if\ r = 0, \\ 2 + 2k & if\ r = 1, \\ 2 + 2k & if\ r = 2. \end{cases}$$

So $\mathrm{ORD}(P_n \Box P_2) \approx \frac{1}{3}$.

To show that the above values are upper bounds for $\varrho_{opt}(P_n \Box P_2)$ it is enough to give a solvable distribution. It is not too hard to show that these are really solvable. Such distributions are shown on Figure 1.

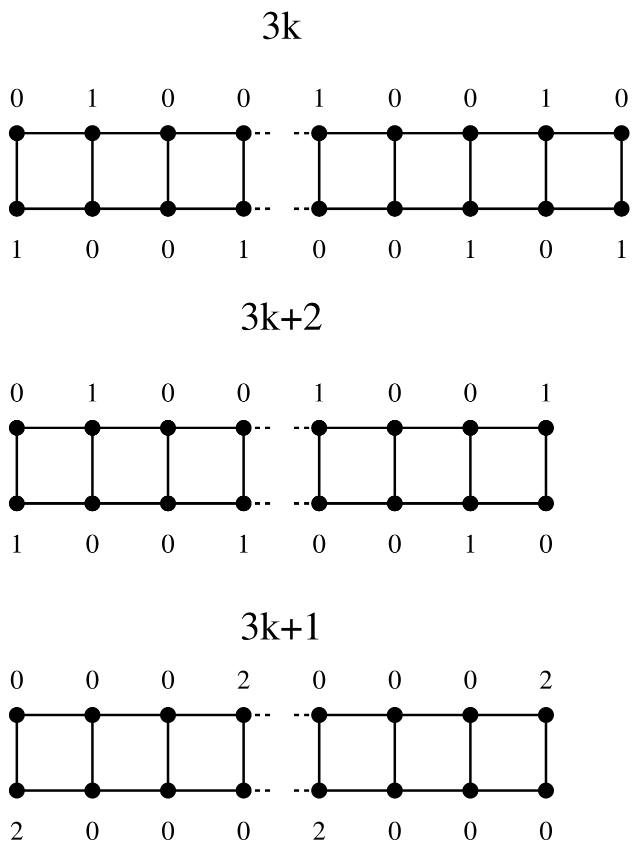

Figure 1: Optimal distributions.

We also need to prove that it is a lower bound. This is done by induction on n, we give a summery of the proof.

Consider a p optimal distribution on $P_n \Box P_2$. Choose an appropriate $R = P_3 \Box P_2$ subgraph, delete the vertices of R and reconnect the remaining two parts to obtain $G^R = P_{n-3} \Box P_2$, called the *reduced graph*, see Fig 2.

Now construct a solvable p' distribution for the new $P_{n-3} \Box P_2$ graph in the following way: p induces a distribution on the vertices which we have not deleted. Place $p(v)$ pebbles to all $v \in V(G) \backslash V(R)$. (In

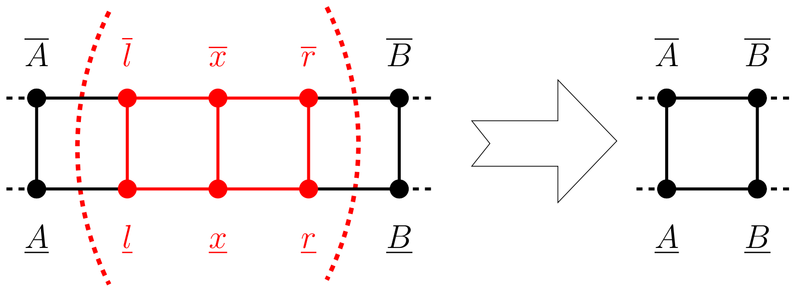

Figure 2: Deleting a $P_3 \Box P_2$ subgraph.

some cases we need to swap the pebbles on the rungs.) Finally, distribute and place $R_p - 2$ pebbles at vertices $\overline{A}$, $\underline{A}$, $\overline{B}$ and $\underline{B}$ in an appropriate way so that the new distribution on $P_{n-3} \Box P_2$ is solvable. Our aim is to show that it is always possible to find such a new distribution. This is proved in several lemmas and case analysis. These will imply

$$\varrho_{opt}(P_n \Box P_2) \geq \varrho_{opt}(P_{n-3} \Box P_2) + 2.$$

It is easy to see that this implies the theorem if we show that the theorem holds for $n = 1, 2, 3$.

Lemma 2

$$\varrho_{opt}(P_2) = 2, \varrho_{opt}(P_2 \Box P_2) = 2, \varrho_{opt}(P_3 \Box P_2) = 3.$$

PROOF: The optimal distributions are shown in Fig. 3. It is an easy exercise to check that these distributions are optimal.

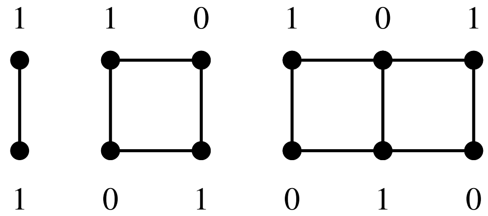

Figure 3: Optimal distributions of P_2, $P_2 \Box P_2$ and $P_3 \Box P_2$.

□

In the rest of the paper we also determine the optimal pebbling number for the n-prism (and the Möbius-ladder).

Theorem 3 $\varrho_{opt}(C_{3k-1} \Box P_2) = \varrho_{opt}(P_{3k-2} \Box P_2) = 2k$, $\varrho_{opt}(C_{3k} \Box P_2) = \varrho_{opt}(P_{3k-1} \Box P_2) = 2k$, $\varrho_{opt}(C_{3k+1} \Box P_2) = \varrho_{opt}(P_{3k} \Box P_2) = 2k + 1$. *Except:* $\varrho_{opt}(C_3 \Box P_2) = 3$, $\varrho_{opt}(C_4 \Box P_2) = 4$.

4 Optimal pebbling and rubbling numbers of large grids

We turn our attention to larger grids now, in the following we assume that n is large enough (say ≥ 100). Shiue [11] proved that the analogue of Graham's conjecture for optimal pebbling is true: $\pi_{opt}(G_1 \Box G_2) \leq \pi_{opt}(G_1)\pi_{opt}(G_2)$. Since in [9] it was proved that $\pi_{opt}(P_n) = \lceil 2n/3 \rceil$, this implies that $OPD(P_n \Box P_n) \leq \frac{4}{9} + o(1)$. In [12] the authors gave a construction showing that $OPD(P_n \Box P_n) \leq \frac{4}{13} + o(1)$. Our first result is better construction.

Theorem 4
$$\pi_{opt}(P_n \square P_n) \le \frac{2}{7}n^2 + O(n),$$
so $OPD(P_n \square P_n) \le \frac{2}{7} + o(1)$.

We conjecture that this is a sharp bound. Applying the well known weight argument, it is fairly easy to obtain that $OPD(P_n \square P_n) \ge \frac{1}{9}$. The authors in [12] claim $OPD(P_n \square P_n) \ge \frac{1}{6}$. Unfortunately, we believe that their proof contains an error, may be it can be corrected easily, but we do not see how. However, they introduced an interesting notion: excess weight. Using this notion, but following a different approach we proved the following lower bound.

Theorem 5 $OPD(P_n \square P_n) \ge \frac{2}{13}$.

For the optimal rubbling number of large grids we do not know any previous results. We give a construction to prove:

Theorem 6
$$\varrho_{opt}(P_n \square P_n) \le \frac{1}{5}n^2 + O(n),$$
so $ORD(P_n \square P_n) \le \frac{1}{5} + o(1)$.

We conjecture that this is a sharp bound. A similar argument to the one we used for pebbling also gives a nontrivial lower bound for the optimal rubbling number.

Theorem 7 $ORD(P_n \square P_n) \ge \frac{5}{37}$.

References

[1] Ch. Belford and N. Sieben, *Rubbling and optimal rubbling of graphs*, Discrete Math. **309** no. 10,(2009) pp. 3436–3446.

[2] D.P. Bunde, E. W. Chambers, D. Cranston, K. Milans, D. B. West, *Pebbling and optimal pebbling in graphs* J. Graph Theory 57 no. 3. (2008) pp. 215–238.

[3] L. Danz, *Optimal t-rubbling of complete m-ary trees*, REU project report, University of Minnesota Duluth, Department of Mathematics and Statistics, (2010).

[4] G. Hurlbert, *A survey of graph pebbling*, Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1999), vol. 139, (1999), pp. 41–64.

[5] G. Hurlbert, *Recent progress in graph pebbling*, Graph Theory Notes N. Y. **49** (2005), pp. 25–37.

[6] G. Hurlbert, *Graph pebbling*, In: Handbook of Graph Theory, Ed: J. L. Gross, J. Yellen, P. Zhang, Chapman and Hall/CRC, Kalamazoo (2013), pp. 1428– 1449.

[7] G. Y. Katona, N. Sieben, *Bounds on the Rubbling and Optimal Rubbling Numbers of Graphs* Electronic Notes in Discrete Mathematics, Volume 38, (2011), pp. 487-492.

[8] D. Moews, *Pebbling graphs*, J. Combin. Theory (B), 55 (1992) pp. 244–252.

[9] L. Pachter, S.S. Hunter, B. Voxman, *On pebbling graphs*, Proceedings of the Twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995), Congr. Numer.,**107**, (1995)

[10] L. Papp, *Optimal rubbling numbers of graphs (in Hungarian)*, thesis, Budapest University of Technology and Economics, Department of Computer Science and Information Theory, (2010).

[11] C. L. Shiue, *Optimally pebbling graphs*, Ph. D. Dissertation, Department of Applied Mathematics, National Chiao Tung University, (1999) Hsin chu, Taiwan.

[12] C. Xue, C. Yerger, *Optimal Pebbling on Grids*, manuscript (2014)