Upper Bound on the Optimal Rubbling Number in graphs with given minimum degree

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Abstract

A pebbling move on a graph removes two pebbles at a vertex and adds one pebble at an adjacent vertex. A vertex is reachable from a pebble distribution if it is possible to move a pebble to that vertex using pebbling moves. The optimal pebbling number is the smallest number m needed to guarantee a pebble distribution of m pebbles from which any vertex is reachable. Czygrinow proved that the optimal pebbling number of a graph is at most $\frac{4n}{\delta+1}$, where n is the number of the vertices and δ is the minimum degree of the graph. We improve this result and show that the optimal pebbling number is at most $\frac{3.75n}{\delta+1}$.

1 Introduction

Graph pebbling has its origin in number theory[8]. It is a model for the transportation of resources. Starting with a pebble distribution on the vertices of a simple connected graph, a *pebbling move* removes two pebbles from a vertex and adds one pebble at an adjacent vertex. We can think of the pebbles as fuel containers. Then the loss of the pebble during a move is the cost of transportation. A vertex is called *reachable* if a pebble can be moved to that vertex using pebbling moves. There are several questions we can ask about pebbling. For a comprehensive list of references for the extensive literature see the survey papers [5, 6, 7].

An interesting optimization question: How can we place the smallest number of pebbles such that every vertex is reachable? The amount of required pebbles is called *optimal pebbling number*. Even the decision version of the optimal pebbling number is NP-hard[9]. Optimal pebbling number has been studied for some time past and it has been determined for paths, cycles [13], ladders[1], complete m-ary trees[4] and for hypercubes[11].

There are several known lower and upper bounds on the optimal pebbling number π_{opt}, i.e. $\pi_{opt}(G) \geq \frac{2}{3}(\operatorname{diam}(G)+1)$, $\pi_{opt}(G) \leq 2^{\operatorname{diam}(G)}$ [12] or $\pi_{opt}(G) \leq \lceil \frac{2n}{3} \rceil$ [1]. Most of these bounds are sharp.

Czygrinow observed and proved that the optimal pebbling number is at most $\frac{4n}{\delta+1}$ where δ is the minimum degree of the graph. On the one hand, it gives a really nice bound when the minimum degree is high, but on the other hand, we do not know a graph where equality holds. For the best

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known construction the optimal pebbling number is $\frac{8n}{3(\delta+1)}$ [3]. In the present paper we show that Czygrinow's Theorem is not sharp, and we give a better upper bound which is $\frac{3.75n}{\delta+1}$.

2 Definitions

Throughout the paper, let G be a simple connected graph. We use the notation $V(G)$ for the vertex set and $E(G)$ for the edge set. A *pebble function* on a graph G is a function $p : V(G) \to \mathbb{Z}$ where $p(v)$ is the number of pebbles placed at v. A *pebble distribution* is a nonnegative pebble function. The *size* of a pebble distribution p is the total number of pebbles $\sum_{v \in V(G)} p(v)$.

Consider a pebble function p on the graph G. If $\{v, u\} \in E(G)$ then the *pebbling move* $(v, v \to u)$ removes two pebbles at vertex v, and adds one pebble at vertex u to create a new pebble function p', so $p'(v) = p(v) - 2$ and $p'(u) = p(u) + 1$.

A *pebbling sequence* is a finite sequence $T = (t_1, \ldots, t_k)$ of pebbling moves. The pebble function obtained from the pebble function p after applying the moves in T is denoted by p_T. The concatenation of the pebbling sequences $R = (r_1, \ldots, r_k)$ and $S = (s_1, \ldots, s_l)$ is denoted by $RS = (r_1, \ldots, r_k, s_1, \ldots, s_l)$.

A pebbling sequence T is *executable* from the pebble distribution p if $p_{(t_1,\ldots,t_i)}$ is nonnegative for all i. A vertex v of G is *reachable* from the pebble distribution p if there is an executable pebbling sequence T such that $p_T(v) \geq 1$. p is a solvable distribution when each vertex is reachable. Correspondingly, v is k*-reachable* under p if there is an executable T, that $p_T(v) \geq k$, and p is k*-solvable* when every vertex is k-reachable. We say that vertices u and v are *independently reachable* if there is an executable pebbling sequence T such that $p_T(u) = 1$ and $p_T(v) = 1$.

The *optimal pebbling number* $\pi_{\text{opt}}(G)$ of a graph G is the size of a distribution with the least number of pebbles from which every vertex is reachable.

We say that a pebble distribution q is an expansion of p, denoted by $p \leq q$, when they are on the same graph and $p(v) \leq q(v)$ for each vertex v. $p_1 \leq p_2 \leq \cdots \leq p_k$ is an expansion chain if $p_i \leq p_{i+1}$ for each i.

The distance $d(u, v)$ between vertices u and v is the length of the shortest path between them. The eccentricity of a vertex v is the maximum distance between v and any other vertex of the graph. The radius of G is the minimum eccentricity of any vertex of the graph, in contrast diameter of G is the maximum eccentricity of any vertex of the graph.

We say that u is a neighbor of v if $(u, v) \in E(G)$. The open neighborhood $N(v)$ of v contains the neighbors of v. The closed neighborhood $N[v]$ of v contains the open neigbourhood and vertex v itself.

A dominating set D is a subset of $V(G)$ such that each vertex of G is either in D or a neighbor of D. A k-dominating set D is such a subset of $V(G)$ that for each vertex v exists a vertex u in D such that $d(u, v) \leq k$.

3 Improving Czygrinow's theorem

Our main result is the following formula for the optimal pebbling number:

Theorem 3.1

$$\pi_{opt}(G) \leq \frac{15n}{4(\delta + 1)}$$

Before the presentation of our results, we sketch the proof of the original Czygrinow's theorem, which states that $\pi_{opt}(G) \leq 4n/(\delta + 1)$. We do this for two reasons: Firstly because it is simple and elegant. Secondly to show why it can not be improved.

To do this we use the following statement which is very useful in graph theory:

Statement 3.2 *If δ is the minimum degree of G, then G has a 2-dominating set D of size at most $\frac{n}{\delta+1}$.*

It is clear that all vertices are reachable when we put four pebbles at each vertex of a 2-dominating set. This and the statement prove the original theorem.

Such a distribution has the property that each vertex has a 4-reachable vertex in its distance-2 neighborhood. Hence, our first attempt to improve this was based on the following idea: If G is connected and we put three pebbles on a vertex and two pebbles on the other vertices of G, then all vertices are 4-reachable. This means that a connected 2-dominating set requires less number of pebbles in terms of its size to assure the 4-reachability of its vertices.

Therefore, if we can show that exist a connected 2-dominating set for each graph whose size is substantially less than $\frac{2n}{\delta+1}$ then we get an improvement for the original theorem.

Unfortunately, such a set does not exist for an arbitrary graph. A counterexample can be seen on Fig. 1.

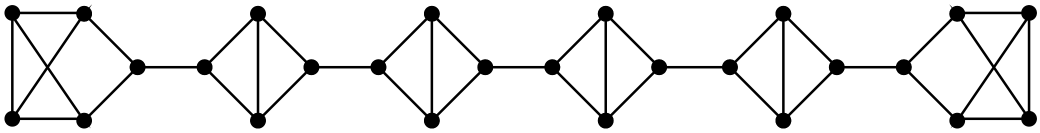

Figure 1: $n = 26$, $\delta = 3$, but a 2-dominating set has to contain at least 14 vertices.

Our investigations led to the recognition that domination theory alone may be not sufficient to make any improvement. We need to utilize the properties of pebbling also.

Our new tool is strengthening, which needs some definitions:

A vertex is strongly reachable under p if it and all of its neighbors are reachable. The strengthening ratio $\mathcal{E}(p_1, p_2)$ of a distribution expansion $p_1 \leq p_2$ is the number of the vertices newly become strongly reachable divided by the number of pebbles newly added during the expansion. The strengthening ratio of a distribution p is the the strengthening ratio of $0 \leq p$.

If p is solvable, then we have that:

$$\frac{n}{\mathcal{E}(0,p)} = |p|$$

For this reason, if we can show for each graph a solvable distribution whose strengthening ratio is greater than $\frac{\delta+1}{4}$ then we obtain an improved theorem.

The strengthening ratio has a very useful property:

Lemma 3.3 *Let p_1, p_2, p_3 be pebbling distributions such that $p_1 \leq p_2$ and $p_2 \leq p_3$. Then:*

$$\mathcal{E}(p_1, p_3) \geq \min(\mathcal{E}(p_1, p_2), \mathcal{E}(p_2, p_3))$$

Conversely, if p is a pebbling distribution and we have an expansion chain $0 = p_0 \leq p_1 \leq p_2 \leq \cdots \leq p_k = p$ then $\mathcal{E}(0,p) \geq \min_{0 \leq i < k}(p_i, p_i + 1)$.

Now we have everything to prove the improvement. The proof is by contradiction. Hence assume that there exists a graph G such that none of the solvable distributions of G has size at most $\frac{15n}{4(\delta+1)}$. This is equivalent with the statement that the strengthening ratio of these distributions are lower than $\frac{4}{15}(\delta+1)$. Let p be a pebbling distribution, whose strengthening ratio is greater than this and let p be maximal with respect to its size. Such a distribution exists according to our assumption. Thus there is no expansion of p with strengthening ratio at least $\frac{4}{15}(\delta+1)$.

To obtain a contradiction, we show that such an expansion is always possible.

Classify the vertex set of G in the following way: Let $\mathcal{T}(p)$ be the set of vertices which are strongly reachable under p, $\mathcal{H}(p)$ be the set which contains reachable but not strongly reachable vertices, and finally let $\mathcal{U}(p)$ be the set containing non reachable vertices.

In the full proof we have to deal with 8 different cases, but now in this abstract we are only showing two of them. One is the easiest case, the other is the hardest one.

Let S be a connected component of the subgraph induced by $\mathcal{U}(p)$.

Case 1: The radius of S is one.

Let v be the vertex, whose eccentricity in S is one. Put two pebbles at v. All vertices of S become reachable. Consider u which is a neighbor of v. If u is in S then $N(u) \subseteq N[S]$, therefore the vertices of $N[u]$ are reachable, which means that u is strongly reachable. Otherwise, u is reachable under p, and with the help of the two pebbles placed at v it is 2-reachable now. Thus it is strongly reachable again. So all neighbors and v itself are strongly reachable, hence the strengthening ratio of this expansion is at least $\frac{\delta+1}{2}$, which is greater than the required one.

Case 2: The diameter of S is at least four, and there is a path of length four in S whose length is also four in G. Let v_1, v_2, v_3, v_4, v_5 be this path. If $N[v_2] \cap N[v_4] \leq \frac{2}{15}(\delta+1)$, then put 3 pebbles at v_3 and 2-2 at vertices v_2 and v_4. It is clear that these three vertices are 4-reachable, hence their neighbors became strongly reachable. $|N[v_2, v_3, v_4]| \geq |N[v_2]| + |N[v_4]| - |N[v_2] \cap N[v_4]| \geq \frac{28}{15}(\delta+1)$. We have placed seven additional pebbles, thus the strengthening ratio is at least $\frac{4}{15}(\delta+1)$.

Now we handle the other case. So $N[v_2] \cap N[v_4] > \frac{2}{15}(\delta+1)$. Put 4-4 pebbles at v_1 and v_5. The neighbors of v_1 and v_5 become strongly reachable. Furthermore, the vertices of $N[v_2] \cap N[v_4]$ can get a pebble from both v_1 and v_5, hence they also became strongly reachable. The total number of vertices which become strongly reachable is $\frac{32}{15}(\delta+1)$ and we have placed eight more pebbles, so the strengthening ratio is at least $\frac{4}{15}(\delta+1)$.

The proofs of of the other cases will be given in the full version of the paper.

To bound the number of possible cases, and simplify the proof that no more cases are possible, we assume that p is an expansion of p^* whose properties are classified in the next lemma:

Lemma 3.4 *Let G be a graph, then there is a pebbling distribution p^* on G which fulfills the following two conditions:*

- *If (u, v) is an edge of G and $N[u, v] \geq \frac{29}{15}(\delta+1)$ then u and v are reachable under p^*.*
- *The strengthening ratio of p^* is at least $\frac{4}{15}(\delta+1)$.*

Here we present a construction for p^*, which is a greedy algorithm followed by an improvement. We say that an edge is interesting if it satisfies the first condition. The main idea is that if an edge has a large neighborhood, then if we put 3 pebbles at both ends of this edge, then its whole neighborhood, a lot of vertices, becomes strongly reachable. This can be contunued until the neighborhoods of the interesting edges are disjoint. When there is an intersection, we can choose to either increase the number of vertices that are reachable from an interesting edge by putting more pebbles on it, to make several interesting edges reachable, or put less pebbles only at one vertex of the edge. If there is no interesting edge then the trivial distribution is good to be p^*.

Create a list for each interesting edge and also for each vertex. These lists are empty in the beginning. So consider an interesting edge and put 3 pebbles at both of its vertices. Then consider an other interesting edge whose vertices are not in the distance-2 neighborhood of any vertex which has a pebble. Repeat this step until it can be repeated.

Each remaining interesting edge has the following property: Exactly one vertex of the edge is not reachable and the other vertex v is at distance two from an other interesting edge whose vertices has 3 pebbles. Maybe there is more than one such edge. Add v to the lists of these edges, and similarly add these edges to the list of v. Put 4 pebbles at vertex v. Do this step again until no interesting edge remains which has a non reachable vertex. Finally, consider the lists of the edges. If a list of an edge has length at least five, then remove a pebble from each element of the list and add five more pebbles at the edge in the following way: Put two pebbles at one vertex and three at the other. Therefore the distance-3 neighborhood of the edge becomes reachable. After that, delete the elements of the list from any other lists and also delete the list itself. Choose an other list of an edge whose size is at least five and do the same until no more such a list remains. Let p^* be the distribution what we have obtained.

It is clear that each vertex of every edge is reachable under p^*. We can think about p^* as an expansion chain. Let an element of the chain be one of that:

- We put 6 pebbles at an edge.
- We put 4 pebbles at a vertex.
- We put 11 pebbles at an edge and 3 pebbles at each vertex which has appeared on the list of it.

A lower bound for the strengthening ratio of each distribution expansion can be given by elementary analysis. Therefore by Lemma 3.3 we have a lower bound on the strengthening ratio of p^*, and it gives the desired bound.

4 Conclusion

We improved the upper bound on optimal pebbling given by Czygrinow. We introduced strengthening; a new tool which could be useful to give another upper bounds on optimal pebbling number.

Despite our improvement, it is still an open question that which is the exact bound in the sense of the minimum degree. We conjecture that the answer is $\frac{cn}{\delta+1}$ where c is closer to 3 than 4.

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