TWO CHARACTERIZATIONS OF UNITARY-ANTIUNITARY SIMILARITY TRANSFORMATIONS OF POSITIVE DEFINITE OPERATORS ON A FINITE DIMENSIONAL HILBERT SPACE

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Dedicated to Professor Zoltán Sebetyén on the occasion of his seventieth birthday

ABSTRACT. In this paper we characterize the unitary-antiunitary similarity transformations on the set of all positive definite operators on a finite dimensional Hilbert space by means of two particular invariance properties. The first one concerns the preservation of a general relative entropy like quantity while the second one is related to the preservation of a measure of difference between the arithmetic mean as the maximal symmetric operator mean and any other given symmetric operator mean.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let us begin with a general remark. As the title and the abstract suggest the results of this paper are about transformations on certain structures of linear operators acting on a finite dimensional Hilbert space. Of course, the problems and results could be formulated in the context of matrices but, on the one hand we prefer treating operators to matrices, on the other hand we believe our present approach may help someone to extend these investigations to other settings, e.g., to that of certain classes of von Neumann algebras.

So, in what follows let H be any finite dimensional complex Hilbert space with $\dim H > 1$. Denote by $B(H)$ the algebra of all linear operators on H. An element A of $B(H)$ is called positive semidefinite if $\langle Ax, x\rangle \geq 0$ holds for all $x \in H$. The set of all positive semidefinite operators in $B(H)$ is denoted by $B(H)_+$. The invertible elements of $B(H)_+$ are called positive definite and $B(H)_+^{-1}$ stands for their collection. Denote by Tr the usual trace functional on $B(H)$.

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Our aim in this paper is to give two characterizations of the unitary-antiunitary similarity transformations on $B(H)_+^{-1}$. These maps are of particular importance, they originate from the algebra *-automorphisms and the algebra *-antiautomorphisms of $B(H)$ (which, anyway, are just the so-called Jordan *-automorphisms of that algebra). Those transformations appear and play fundamental roles in very many applications of the theory of operator algebras.

Our first characterization is in a way connected with quantum information theory. One of the most important concepts there is that of the relative entropy. In fact, there is not just one but several notions of quantum relative entropy. Here we recall the one named after Belavkin and Staszewski which is defined by the formula

$$S_{BS}(A||B) = -\operatorname{Tr} A\log(A^{-1/2}BA^{-1/2}), \quad A,B \in B(H)_+^{-1}. \tag{1}$$

Motivated by Wigner's famous theorem on the structure of quantum mechanical symmetry transformations (i.e., bijections of the space of all pure states preserving transition probability), in Theorem 5 in the paper [5] we determined the structure of all bijective maps ϕ on $B(H)_+^{-1}$ which leave the Belavkin-Staszewski relative entropy invariant, i.e., which satisfy

$$S_{BS}(\phi(A)||\phi(B)) = S_{BS}(A||B), \quad A,B \in B(H)_+^{-1}. \tag{2}$$

It turned out that in spite of the high nonlinearity reflected in the definition (1) of the quantity $S_{BS}(\cdot,\cdot)$, the transformations ϕ which leave it invariant are unitary-antiunitary similarity transformations. To be honest in [5, Theorem 5] the map ϕ was originally defined on the class of all nonsingular density operators (i.e., elements of $B(H)_+^{-1}$ with unit trace which represent mixed quantum states), but in the very first step of the proof we extended it to the whole space $B(H)_+^{-1}$ keeping its bijectivity and preserver property (2). In Theorem 1 below we extend this result to a much more general setting, namely, we consider bijective maps on $B(H)_+^{-1}$ which preserve a quantity similar to (1) but the function $-\log$ is replaced by any nonconstant real valued operator monotone decreasing function f on $]0,\infty[$ having the property $\lim_{t\to\infty} f(t)/t = 0$. The precise formulation of the result reads as follows.

Theorem 1. *Let* $f :]0,\infty[\to \mathbb{R}$ *be a nonconstant operator monotone decreasing function such that* $\lim_{t\to\infty} f(t)/t = 0$. *The bijective map* $\phi : B(H)_+^{-1} \to B(H)_+^{-1}$ *satisfies*

$$\operatorname{Tr}\phi(A)f(\phi(A)^{-1/2}\phi(B)\phi(A)^{-1/2}) = \operatorname{Tr} Af(A^{-1/2}BA^{-1/2}), \quad A,B \in B(H)_+^{-1}$$

if and only if there is a unitary or antiunitary operator U on H such that we have

$$\phi(A) = UAU^*, \quad A \in B(H)_+^{-1}.$$

We now turn to our second characterization of unitary-antiunitary similarity transformations on $B(H)_+^{-1}$.

In the recent paper [6] we have discussed transformations on structures of positive semidefinite operators which preserve a given norm of a given operator mean. Our second theorem in this paper is also connected to means, hence we briefly collect the necessary preliminaries on the Kubo-Ando theory of operator means. Following the fundamental paper [2], a binary operation σ on $B(H)_+$ is called a connection if it satisfies the following conditions. For any operators $A, B, C, D \in B(H)_+$ and sequences $(A_n), (B_n)$ in $B(H)_+$ we have

(O1) if $A \leq C$ and $B \leq D$ then $A\sigma B \leq C\sigma D$;
(O2) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$;
(O3) if $A_n \downarrow A$ and $B_n \downarrow B$ then $A_n \sigma B_n \downarrow A\sigma B$,

where the arrow $\downarrow$ refers to monotone decreasing convergence in the strong operator topology. (We remark that in this definition H should be an infinite dimensional Hilbert space in order to finally obtain the Kubo-Ando theory of operator means which is independent of the underlying Hilbert space.) A connection σ is called a mean if it is normalized in the sense that for the identity operator I on H we have $I\sigma I = I$. Clearly, for any connection (or mean) σ on $B(H)_+$, its so-called transpose σ' defined by

$$A\sigma' B = B\sigma A, \quad A, B \in B(H)_+$$

is a connection (or mean) again and σ is called symmetric if $\sigma = \sigma'$. The most simple means are the weighted arithmetic means (which are just the fixed convex combinations); $A\sigma B = \lambda A + (1-\lambda)B$ with some given $\lambda \in [0,1]$.

One of the most important results in the Kubo-Ando theory says that there is an affine order-isomorphism between the class of all connections σ on $B(H)_+$ and the class of all nonnegative real valued operator monotone increasing functions f on $]0,\infty[$, see Theorem 3.2 in [2]. In fact, as seen in the proof of that theorem, if σ is a connection then the operator monotone function f associated with it is $f(t) = I\sigma(tI)$, $t > 0$. Conversely, if f is a nonnegative real valued operator monotone increasing function on $]0,\infty[$ then the connection with which it is associated satisfies

$$A\sigma B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2} \tag{3}$$

for all invertible elements A, B of $B(H)_+$. We remark that the property (O3) implies that the above formula extends also to the case where $A \in B(H)_+$ is invertible but $B \in B(H)_+$ is arbitrary (to be correct, in that case the quantity $f(0)$ should be defined; we set $f(0) = \lim_{t\to 0} f(t)$). The case where $f(t) = \sqrt{t}$, $t > 0$ is especially important. The corresponding mean is called the geometric mean of positive semidefinite operators which has very many applications in different areas of science. By Corollary 4.2 in [2], if σ is a connection with associated operator monotone function f, then $t \mapsto tf(1/t)$, $t > 0$ is the operator monotone function on $]0,\infty[$ associated with the transpose σ' of σ. Therefore, for symmetric means we have that the corresponding operator monotone function f satisfies $f(t) = tf(1/t)$ for all $t > 0$.

By Theorem 4.5 in [2] the arithmetic mean $A\nabla B = (A+B)/2$, $A, B \in B(H)_+$ is the maximal symmetric mean. That is, for any symmetric mean σ we have

$$A\sigma B \leq A\nabla B \tag{4}$$

for all $A, B \in B(H)_+$. One can use the trace-norm (which, for positive semidefinite operators, simply equals the trace) to measure the gap between the two sides of the inequality in (4). In our second result we show the hopefully interesting fact that the bijective transformations on $B(H)_+$ which preserve this quantity for any given symmetric mean (satisfying some mild assumptions) are exactly the unitary-antiunitary similarity transformations on $B(H)_+$.

Theorem 2. *Let σ be a symmetric operator mean and $f :]0,\infty[\to [0,\infty[$ the nonnegative scalar valued operator monotone function associated with σ. Assume $f(0) := \lim_{t\to 0} f(t) = 0$ and $\lim_{t\to\infty} f(t) = \infty$. A bijective map $\phi : B(H)_+ \to B(H)_+$ satisfies*

$$\mathrm{Tr}(\phi(A)\nabla\phi(B) - \phi(A)\sigma\phi(B)) = \mathrm{Tr}(A\nabla B - A\sigma B), \quad A, B \in B(H)_+ \tag{5}$$

if and only if there is a unitary or antiunitary operator U on H such that

$$\phi(A) = UAU^*, \quad A \in B(H)_+.$$

Observe that many means, among them the geometric mean, satisfy the requirements of the theorem. Furthermore, we remark that applying very simple modifications in the proof one check easily that the same result holds true also in the case where ϕ acts on $B(H)_+^{-1}$, not on $B(H)_+$.

2. PROOFS

In this section we present the proofs of our results. Our strategy is the following. We present characterizations of the usual order $\leq$ between self-adjoint operators in terms of the quantities what our transformations preserve. This will imply that they are order automorphisms. Then we can apply our former results on the structures of those automorphisms and then finish the proof rather easily.

The proof of Theorem 1 is based on the following characterization of the order $\leq$ on $B(H)_+^{-1}$. Recall that for any two self-adjoint operators A, B in $B(H)$ we write $A \leq B$ if and only if $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ holds for all $x \in H$.

Lemma 3. *Let f be as in Theorem 1. For any $A, B \in B(H)_+^{-1}$ we have $A \leq B$ if and only if*

$$\operatorname{Tr} X f(X^{-1/2}AX^{-1/2}) \geq \operatorname{Tr} X f(X^{-1/2}BX^{-1/2})$$

holds for all $X \in B(H)_+^{-1}$.

Proof. First, let g be any real valued continuous function on the interval $]0,\infty[$. We claim that for any invertible operator $A \in B(H)$ we have

$$Ag(A^*A) = g(AA^*)A. \tag{6}$$

Indeed, since $|A|$ clearly commutes with $g(|A|^2)$, we infer $|A|g(|A|^2) = g(|A|^2)|A|$. Let U be the unitary operator in the polar decomposition of A, i.e., $A = U|A|$. We have $g(UXU^*) = Ug(X)U^*$ for any self-adjoint operator $X \in B(H)$ (in fact, this follows from the fact that g can be uniformly approximated by polynomials on every compact subinterval of $]0,\infty[$). Then we deduce

$$Ag(A^*A) = U|A|g(|A|^2) = Ug(|A|^2)|A| = g(U|A|^2U^*)U|A| = g(AA^*)A.$$

Now, let g be defined by $g(t) = f(1/t)$, $t > 0$. Using (6) we compute

$$\begin{aligned}
\operatorname{Tr} Af(A^{-1/2}BA^{-1/2}) &= \operatorname{Tr} Ag(A^{1/2}B^{-1}A^{1/2}) = \operatorname{Tr} Ag((A^{1/2}B^{-1/2})(A^{1/2}B^{-1/2})^*) \\
&= \operatorname{Tr} B^{1/2}A^{1/2}g((A^{1/2}B^{-1/2})(A^{1/2}B^{-1/2})^*)A^{1/2}B^{-1/2} \\
&= \operatorname{Tr} B^{1/2}A^{1/2}A^{1/2}B^{-1/2}g((A^{1/2}B^{-1/2})^*(A^{1/2}B^{-1/2})) \\
&= \operatorname{Tr} BB^{-1/2}AB^{-1/2}g(B^{-1/2}AB^{-1/2}).
\end{aligned} \tag{7}$$

Denote h the function defined by $h(t) = tg(t)$, $t > 0$ and let $h(0) = 0$. Since $\lim_{t\to\infty} f(t)/t = 0$, we have $\lim_{t\to 0} h(t) = 0$ and hence h is a continuous function on $[0,\infty[$ implying that $X \longmapsto h(X)$ is continuous on $B(H)_+$.

Now, select any $A, B \in B(H)_+^{-1}$. If $A \leq B$, then by the operator monotone decreasing property of f it follows that

$$\operatorname{Tr} X f(X^{-1/2}AX^{-1/2}) \geq \operatorname{Tr} X f(X^{-1/2}BX^{-1/2}) \tag{8}$$

holds for all $X \in B(H)_+^{-1}$. Conversely, assume that the equality in (8) is valid for all $X \in B(H)_+^{-1}$. By the definition of h, (7) and (8) we obtain that

$$\begin{aligned} &\operatorname{Tr} Ah(A^{-1/2}XA^{-1/2}) = \operatorname{Tr} AA^{-1/2}XA^{-1/2}g(A^{-1/2}XA^{-1/2}) \\ &\geq \operatorname{Tr} BB^{-1/2}XB^{-1/2}g(B^{-1/2}XB^{-1/2}) = \operatorname{Tr} Bh(B^{-1/2}XB^{-1/2}) \end{aligned}$$

holds for all $X \in B(H)_+^{-1}$. Let X converge to an arbitrary rank-one projection P on H. Then we have

$$\operatorname{Tr} Ah(A^{-1/2}PA^{-1/2}) \geq \operatorname{Tr} Bh(B^{-1/2}PB^{-1/2}). \tag{9}$$

Pick a unit vector $x \in H$ from the range of P. Obviously, $P = x \otimes x$ holds where the operator $x \otimes x$ is defined by $(x \otimes x)z = \langle z, x\rangle x$, $z \in H$. We compute

$$A^{-1/2}PA^{-1/2} = \|A^{-1/2}x\|^2(A^{-1/2}x/\|A^{-1/2}x\|) \otimes (A^{-1/2}x/\|A^{-1/2}x\|)$$

and hence

$$\begin{aligned} &A^{1/2}h(A^{-1/2}PA^{-1/2})A^{1/2} \\ &= A^{1/2}h(\|A^{-1/2}x\|^2)(A^{-1/2}x/\|A^{-1/2}x\|) \otimes (A^{-1/2}x/\|A^{-1/2}x\|)A^{1/2} \\ &= h(\|A^{-1/2}x\|^2)(x/\|A^{-1/2}x\|) \otimes (x/\|A^{-1/2}x\|) = g(\|A^{-1/2}x\|^2)x \otimes x. \end{aligned}$$

Consequently, by (9) we have

$$g(\|A^{-1/2}x\|^2) \geq g(\|B^{-1/2}x\|^2).$$

Since f is a nonconstant operator monotone function, it is strictly monotone (according to a famous theorem of Löwner, operator monotone functions on any open interval have analytic continuations onto the upper half plane). We deduce that g is strictly increasing which implies that $\|A^{-1/2}x\|^2 \geq \|B^{-1/2}x\|^2$, i.e., $\langle A^{-1}x, x\rangle \geq \langle B^{-1}x, x\rangle$. The rank-one projection P was arbitrary which means that this inequality holds for every unit vector $x \in H$. Therefore, $A^{-1} \geq B^{-1}$ which is equivalent to $A \leq B$. This completes the proof of the lemma. □

Now we are in a position to prove our first theorem.

Proof of Theorem 1. The necessity part of the statement is easy. As for sufficiency, it follows from the characterization of the order $\leq$ given in the above lemma that ϕ is an order automorphism of $B(H)_+^{-1}$. The structure of such maps is known. By Theorem 1 in [5] there is an invertible either linear or conjugate linear operator T on H such that $\phi(A) = TAT^*$, $A \in B(H)_+^{-1}$. In addition to that we have the preserver property of ϕ, i.e.,

$$\operatorname{Tr} TAT^* f((TAT^*)^{-1/2}(TBT^*)(TAT^*)^{-1/2}) = \operatorname{Tr} Af(A^{-1/2}BA^{-1/2})$$

holds for all $A, B \in B(H)_+^{-1}$. Let $t > 0$ be such that $f(t) \neq 0$. Plugging $B = tA$, we have $\operatorname{Tr} TAT^* f(tI) = \operatorname{Tr} Af(tI)$. Since $f(tI) = f(t)I$, we infer that $\operatorname{Tr} TAT^* = \operatorname{Tr} A$ and hence $\operatorname{Tr} AT^*T = \operatorname{Tr} A$ holds for all $A \in B(H)_+^{-1}$. This clearly implies $T^*T = I$, consequently T is a unitary or antiunitary operator on H. The proof is complete. $\square$

Before turning to the proof of our second theorem we present some more preliminaries which will be needed.

From the famous Löwner theory of operator monotone functions it is well-known that all such functions have a certain integral representation. In fact, the formula

$$f(t) = \int_{[0,\infty]} \frac{t(1+s)}{t+s}\, dm(s), \quad t > 0 \tag{10}$$

gives an affine order-isomorphism from the class of all positive Radon measures m on the extended interval $[0,\infty]$ onto the set of all nonnegative scalar valued operator monotone increasing functions f on $]0,\infty[$, see Lemma 3.1 in [2]. We note that here $f(0) := \lim_{t\to 0} f(t) = m(\{0\})$ and $\lim_{t\to\infty} f(t)/t = m(\{\infty\})$.

Next, let us recall the concept of the strength of a positive semidefinite operator along any rank-one projection. Following [1], p.329, for any $A \in B(H)_+$ and rank-one projection P on H we define the numerical quantity $\lambda(A,P)$ by

$$\lambda(A,P) = \sup\{t \geq 0 : tP \leq A\}$$

and call it the strength of A along P. The function $P \longmapsto \lambda(A,P)$ is said to be the strength function of A. By Theorem 1 in [1] we know that for any $A, B \in B(H)_+$ we have $A \leq B$ if and only if $\lambda(A,P) \leq \lambda(B,P)$ holds for all rank-one projections P on H.

The proof of Theorem 2 is again based on a characterization of the usual order $\leq$ on $B(H)_+$. This is given in the next lemma. Denote $d(A,B) = \operatorname{Tr}(A \nabla B - A\sigma B)$, $A, B \in B(H)_+$.

Lemma 4. *Let σ, f be as in Theorem 2. For any $A, B \in B(H)_+$ we have $A \leq B$ if and only if the set $\{d(A,X) - d(B,X) : X \in B(H)_+\}$ is bounded from below.*

Proof. Assume $A \leq B$, then by the monotonicity of operator means (O1) we obtain

$$d(A,X) - d(B,X) = \operatorname{Tr}((A-B)/2) - \operatorname{Tr}(A\sigma X - B\sigma X) \geq \operatorname{Tr}((A-B)/2)$$

for any $X \in B(H)_+$. This gives us the necessity part of the statement. As for the converse, assume that the set $\{d(A,X) - d(B,X) : X \in B(H)_+\}$ is bounded from below. This means that

$$\operatorname{Tr}(B\sigma X - A\sigma X) \geq c$$

holds with a given constant c for arbitrary $X \in B(H)_+$. By Lemma 2.6 in [4] we have the formula

$$A\sigma P = f(\lambda(A,P))P \tag{11}$$

for every $A \in B(H)_+$ and rank-one projection P on H. Since the operator means are clearly homogeneous (see (3) and then use the continuity property (O3)), we obtain

$$A\sigma(tP) = t((1/t)A)\sigma P) = tf((1/t)\lambda(A,P))P$$

for all $t > 0$. Therefore, letting t range over the set of all positive real numbers and P range over the set of all rank-one projections on H, it follows that the set of all quantities

$$tf((1/t)\lambda(B,P)) - tf((1/t)\lambda(A,P))$$

is bounded from below by the number c given above. We fix P and claim that $\lambda(A,P) \leq \lambda(B,P)$. Obviously, we need to check this only in the case where $\lambda(A,P)$ is positive. Assuming that, $\lambda(B,P)$ is also positive. Indeed, by the symmetricity of σ we have $f(t) = tf(1/t)$ for all $t > 0$ and hence, if $\lambda(B,P)$ were zero, we would get that

$$-tf((1/t)\lambda(A,P)) = -\lambda(A,P)f(t/\lambda(A,P)), \quad t > 0$$

is bounded from below meaning that f is bounded from above, a contradiction. Hence, we have $\lambda(B,P) > 0$. Let us introduce the new variable $s = t/\lambda(A,P)$. Then denoting $\gamma = \lambda(A,P)/\lambda(B,P)$, simple calculation using the symmetricity of f again yields

$$tf((1/t)\lambda(B,P)) - tf((1/t)\lambda(A,P)) = \lambda(A,P)(f(\gamma s)/\gamma - f(s))$$

and hence we deduce that the function

$$t \longmapsto f(\gamma t)/\gamma - f(t), \quad t > 0$$

is bounded from below. We show that this implies $\gamma \leq 1$. To see this, we use the integral representation (10) of f (observe that in our present case $m(\{0\}) = m(\{\infty\}) = 0$) and compute

$$\begin{aligned} &f(\gamma t) - \gamma f(t) \\ &= \int_{]0,\infty[} \frac{\gamma t(1+s)}{\gamma t + s} - \frac{\gamma t(1+s)}{t+s} dm(s) \\ &= \gamma(1-\gamma) \int_{]0,\infty[} \frac{t^2(1+s)}{(\gamma t + s)(t+s)} dm(s). \end{aligned}$$

Assuming $\gamma > 1$, the quantity $\gamma(1-\gamma)$ is negative and it follows that

$$\int_{]0,\infty[} \frac{t^2(1+s)}{(\gamma t+s)(t+s)} dm(s) \\ = \int_{]0,\infty[} \frac{1+s}{(\gamma+s/t)(1+s/t)} dm(s)$$

as a function of $t > 0$ is bounded from above. However, for any fixed s, the values under the sign of integral increase as t increases to ∞. Therefore, applying Beppo Levi theorem we deduce that

$$\begin{aligned}
&\lim_{t\to\infty} \int_{]0,\infty[} \frac{1+s}{(\gamma+s/t)(1+s/t)} dm(s) \\
&= \int_{]0,\infty[} \lim_{t\to\infty} \frac{1+s}{(\gamma+s/t)(1+s/t)} dm(s) \\
&= (1/\gamma) \int_{]0,\infty[} 1+s dm(s) \\
&= (1/\gamma) \int_{]0,\infty[} \lim_{t\to\infty} \frac{t(1+s)}{t+s} dm(s) \\
&= (1/\gamma) \lim_{t\to\infty} \int_{]0,\infty[} \frac{t(1+s)}{t+s} dm(s) \\
&= (1/\gamma) \lim_{t\to\infty} f(t).
\end{aligned}$$

Hence we obtain that $\lim_{t\to\infty} f(t)$ is finite which is a contradiction. It follows that we necessarily have $\gamma \le 1$ implying that $\lambda(A,P) \le \lambda(B,P)$. Since this holds for every rank-one projection P on H, we obtain that $A \le B$. □

Having proven the above characterization, the proof of our second theorem is simple. We shall use the fact, usually referred to as the transfer property, that for any invertible linear or conjugate linear operator $T \in B(H)$ and $A, B \in B(H)_+$ we have $T(A\sigma B)T^* = (TAT^*)\sigma(TBT^*)$. This follows quite easily from the inequality (O2).

Proof of Theorem 2. The necessity part of the statement is apparent. As for the sufficiency, it follows from Lemma 4 that ϕ is an order isomorphism of $B(H)_+$. The structure of those transformations is also known. By Theorem 1 in [3], it follows that we have an invertible either linear or conjugate linear operator T on H such that

$$\phi(A) = TAT^*, \quad A \in B(H)_+.$$

Plugging $A = I$ into (5) and using the transfer property mentioned above we deduce

$$\begin{aligned}\operatorname{Tr} T((I+B)/2 - f(B))T^* &= \operatorname{Tr} T(I\nabla B - I\sigma B)T^* \\ &= \operatorname{Tr}(\phi(I)\nabla\phi(B) - \phi(I)\sigma\phi(B)) \\ &= \operatorname{Tr}(I\nabla B - I\sigma B) = \operatorname{Tr}((I+B)/2 - f(B)).\end{aligned}$$

Substituting any projection P on H into the place of B, by $f(P) = P$ we obtain from the last equalities that

$$\operatorname{Tr}(I-P)T^*T = \operatorname{Tr} T(I-P)T^* = \operatorname{Tr}(I-P).$$

Since $I - P$ ranges over the set of all projections on H we infer from this that $T^*T = I$, i.e., T is either a unitary or an antiunitary operator on H. The proof is complete. □

To conclude the paper we make a few remarks. Throughout we have assumed that H is at least two-dimensional. If $\dim H = 1$, then the treated problems reduce to certain functional equations that can possibly be solved with related methods. As for the first theorem, we mention that in the case where $f(1) \neq 0$ (which is not satisfied by the motivating example $f = -\log$) we immediately obtain that the only solution of the corresponding functional equation is the identity function. Otherwise, we need to consider an implicit functional equation and the situation is similar regarding the second theorem, too. Concerning Theorem 2 we further note that dual to the case of the arithmetic mean, the harmonic mean is known to be the minimal symmetric operator mean. So, it would be a natural question to investigate a similar problem for the harmonic mean in the place of the arithmetic mean. That question seems to be challenging. Finally, as already referred to it in the introduction, one might consider similar problems in more general operator algebras, e.g., in von Neumann algebras carrying scalar valued traces.

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