PRESERVING PROBLEMS OF GEODESIC-AFFINE MAPS AND RELATED TOPICS ON POSITIVE DEFINITE MATRICES

PATRÍCIA SZOKOL, MING-CHENG TSAI, AND JUN ZHANG

Abstract. Based on affine maps in geometry, we study the geodesic-affine maps on Riemannian manifolds $\mathbb{P}_n$ of complex positive definite matrices that are induced by different so-called kernel functions. In this article, we are going to describe the structure of all continuous bijective geodesic-affine maps on these manifolds. We also prove that geodesic distance isometries are geodesic-affine maps. Moreover, the forms of all bijective maps which preserve norms of geodesic correspondence are characterized. Indeed, these maps are special examples of geodesic-affine maps.

Keywords: Positive definite matrices; Riemannian metric; geodesic-affine map; relative entropy; preserving problems
AMS classification: 47B49, 15B48, 53C22

1. Introduction

Let $\mathbb{P}_n$ be the set of all $n \times n$ complex positive definite matrices. An affine map ϕ on $\mathbb{P}_n$ is a map satisfying that

$$\phi((1-t)A + tB) = (1-t)\phi(A) + t\phi(B), \qquad t \in [0,1],\ A, B \in \mathbb{P}_n.$$

Namely, ϕ maps the segment joining A, B onto segment joining $\phi(A), \phi(B)$. It is known that the set $\mathbb{P}_n$ can be equipped with certain Riemannian structures via different Riemannian metrics and the geodesics in these manifolds play the same role as segments. Therefore we can define the geodesic-affine map ϕ as a map sending the points of geodesic $\gamma_{A,B}(t)$ joining A, B into geodesic $\gamma_{\phi(A),\phi(B)}(t)$ joining $\phi(A), \phi(B)$, i.e.

$$\phi(\gamma_{A,B}(t)) = \gamma_{\phi(A),\phi(B)}(t), \qquad t \in [0,1],\ A, B \in \mathbb{P}_n.$$

In addition, we say that ϕ preserves norms of geodesic correspondence if

$$\|\gamma_{A,B}(t)\| = \|\gamma_{\phi(A),\phi(B)}(t)\|, \qquad t \in [0,1],\ A, B \in \mathbb{P}_n.$$

In this paper, we will study the geodesic-affine maps in $\mathbb{P}_n$. The paper is organized as follows. In the present section we introduce the necessary notion and notation which will be

Date: March 31, 2015.

The first author was supported by the "Lendület" Program (LP2012-46/2012) of the Hungarian Academy of Sciences. The second author was supported by the Taiwan NSC Grant 102-2811-M-110-018.

used throughout the paper. In Section 2, we characterize the structural results of the geodesic-affine maps on these Riemannian manifolds. Section 3 is concerned with the relationship between geodesic-affine maps and geodesic distance isometries. We will show that every geodesic distance isometry is a geodesic-affine map, and the converse is not always true. In Section 4, the forms of maps which preserve the norm of all points of geodesics in Riemannian manifolds corresponding to different kinds of kernel functions will be studied. We will show that these maps are also geodesic distance isometries, and hence are geodesic-affine maps.

Throughout this paper we denote by $\mathbb{M}_n$ the set of all $n \times n$ complex matrices and $\mathbb{H}_n$ the set of all $n \times n$ Hermitian matrices. The elements of $\mathbb{P}_n$ form an open subset of $\mathbb{H}_n$ and it can be equipped with a Riemannian structure such that the tangent space at any foot point $D \in \mathbb{P}_n$ is identified with $\mathbb{H}_n$. A Riemannian metric $K_D \colon \mathbb{H}_n \times \mathbb{H}_n \to [0,\infty)$ is a family of inner products on $\mathbb{H}_n$ depending smoothly on the foot point D. If $\varphi\colon (0,\infty)\times(0,\infty) \to (0,\infty)$ is a so-called kernel function, i.e. it is a symmetric function ($\varphi(x,y) = \varphi(y,x)$, for every $x, y \in (0,\infty)$), and is smooth in its both variables, and D has the spectral decomposition $\sum_{i=1}^{n} \lambda_i P_i$, then a Riemannian metric can be defined by

$$K_D^{\varphi}(H,K) := \sum_{i,j=1}^{n} \varphi(\lambda_i,\lambda_j)^{-1} \operatorname{tr} P_i H P_j K, \quad D \in \mathbb{P}_n,\ H, K \in \mathbb{H}_n, \tag{1.1}$$

where tr is the usual trace functional on matrices and the Hermitian matrices H, K are tangent vectors. We denote by $(\mathbb{P}_n, K^{\varphi})$ the Riemannian manifold induced by Riemannian metric $K_D^{\varphi}(H,K)$.

These Riemannian metrics are induced by different kernel functions and the corresponding Riemannian manifolds have been studied by many mathematicians. Among others in [5, 6, 16, 18], the authors determined the geodesic curves and geodesic distances between any pairs of positive definite matrices in Riemannian manifolds. Suppose that $\rho : [0,1] \to \mathbb{P}_n$ is a continuously differentiable curve (or more generally, a continuous and piecewise continuously differentiable curve). Then the length of ρ with respect to the metric K^{φ} is given by

$$L_{\varphi}(\rho) := \int_0^1 \sqrt{K_{\rho(t)}^{\varphi}(\rho'(t),\rho'(t))}\, dt. \tag{1.2}$$

It is known, that the length $L_{\varphi}(\rho)$ is independent of the choice of the parametrization of ρ. The geodesic distance $\delta^{\varphi}(A,B)$ between $A, B \in \mathbb{P}_n$ with respect to the metric K^{φ} is defined by

$$\delta^{\varphi}(A,B) = \inf\{L_{\varphi}(\rho) |\ \rho \text{ is a continuously differentiable path from A to B}\}.$$

A geodesic (shortest) curve between $A, B \in \mathbb{P}_n$ is a continuously differentiable curve γ from A to B such that $L_{\varphi}(\gamma) = \delta^{\varphi}(A,B)$, and it will be denoted by $\gamma_{A,B}^{\varphi}(t)$ in this article.

In this article, the geodesic curves in certain Riemannian manifolds $\mathbb{P}_n$ will be studied. More precisely, we will consider several parametrized families of kernel functions and the geodesic curves in the corresponding Riemannian manifolds.

The first parametrized family of kernel functions is defined by

$$h_\alpha(x,y) = \left(\alpha \frac{x-y}{x^\alpha - y^\alpha}\right)^2, \text{ with } \alpha \neq 0. \tag{1.3}$$

In [5] Petz and Hiai proved that there exists a unique geodesic curve from A to B in the Riemannian structure that are induced by kernel function h_α and it is given by

$$\gamma_{A,B}^{h_\alpha}(t) = \big((1-t)A^\alpha + tB^\alpha\big)^{\frac{1}{\alpha}}, \quad 0 \leq t \leq 1,$$

and the geodesic distance between A and B is

$$\delta^{h_\alpha}(A,B) = \frac{1}{|\alpha|}\|A^\alpha - B^\alpha\|_{\mathrm{HS}}.$$

The trivial choice $\alpha = 1$ gives the most general kernel function $h_1(x,y) \equiv 1$. It leads to a flat space where the Riemannian metric is the Hilbert-Schmidt inner product $K^\varphi(H,K) = \langle H,K\rangle_{\mathrm{HS}}$ on $\mathbb{H}_n$. We recall that the Hilbert-Schmidt inner product $\langle .,.\rangle_{\mathrm{HS}} \colon \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{C}$ is defined by $\langle A,B\rangle_{\mathrm{HS}} = \operatorname{tr} A^*B$ for all $A, B \in \mathbb{M}_n$ and the Hilbert-Schmidt norm is $\|A\|_{\mathrm{HS}} = (\operatorname{tr} A^*A)^{\frac{1}{2}}$. In the corresponding manifold the geodesic curve joining $A, B \in \mathbb{P}_n$, is just the segment from A to B, i.e.

$$\gamma_{A,B}^{h_1}(t) = (1-t)A + tB, \quad 0 \leq t \leq 1.$$

For the second parametrized family, let κ be a positive number and g_κ be the kernel function defined by

$$g_\kappa(x,y) = \left(\kappa (xy)^{\frac{\kappa}{2}} \frac{x-y}{x^\kappa - y^\kappa}\right)^2. \tag{1.4}$$

In [5], Theorem 3.3 states that for every $A, B \in \mathbb{P}_n$ there exists a unique geodesic from A to B and it can be given by

$$\gamma_{A,B}^{g_\kappa}(t) = (A^\kappa \#_t B^\kappa)^{\frac{1}{\kappa}} := (A^{\frac{\kappa}{2}}(A^{-\frac{\kappa}{2}}B^\kappa A^{-\frac{\kappa}{2}})^t A^{\frac{\kappa}{2}})^{\frac{1}{\kappa}}, \quad 0 \leq t \leq 1.$$

The geodesic distance between A, B is

$$\delta^{g_\kappa}(A,B) = \|\log(A^{-\frac{\kappa}{2}}B^\kappa A^{-\frac{\kappa}{2}})^{\frac{1}{\kappa}}\|_{\mathrm{HS}}.$$

If we choose $\kappa = 1$ in (1.4), then we get the kernel function $g_1(x,y) = xy$. It leads to the so-called Fisher-Rao metric which is defined by $K_D^\varphi(H,K) = \operatorname{tr} D^{-1}HD^{-1}K$. We note, that this metric plays a significant role in the recent development of the geometric mean of matrices. By [3, 10, 12] it is known that the geodesic in this Riemannian manifold between $A, B \in \mathbb{P}_n$ is given by

$$\gamma_{A,B}^{g_1}(t) = A\#_t B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}, \quad 0 \leq t \leq 1;$$

and the geodesic distance between A and B is

$$\delta^{g_1}(A,B) = \|\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\|_{\mathrm{HS}}.$$

The midpoint of the geodesic between A and B is just the geometric mean of matrices A and B, which is defined by

$$A\#B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

Moreover, if α tends to 0 in (1.3) or κ tends to 0 in (1.4), then we get the kernel function defined by $\varphi(x,y) = \left(\frac{x-y}{\log x - \log y}\right)^2$. The Riemannian manifold induced by this kernel function will be denoted by $(\mathbb{P}_n, K^l)$. In [5], the authors have shown that the geodesic joining $A, B \in \mathbb{P}_n$ is given by

$$\gamma_{A,B}^{l}(t) = e^{(1-t)\log A + t\log B}, \qquad 0 \le t \le 1,$$

and the geodesic distance between $A, B \in \mathbb{P}_n$ is

$$\delta^{l}(A,B) = \|\log A - \log B\|_{\mathrm{HS}}.$$

2. Structure of Geodesic-affine maps on $(\mathbb{P}_n, K^{\varphi})$

In this section, we determine the structure of all geodesic-affine maps on $(\mathbb{P}_n, K^{h_\alpha})$ and $(\mathbb{P}_n, K^{g_\kappa})$.

Theorem 2.1. *Let $n \ge 3$. Suppose that $\phi : \mathbb{P}_n \to \mathbb{P}_n$ is a continuous bijective map on $(\mathbb{P}_n, K^{g_1})$. Then the following statements are equivalent:*

(1) *ϕ is a geodesic-affine map, i.e. $\phi(A\#_t B) = \phi(A)\#_t\phi(B)$ for all $t \in [0,1]$;*
(2) *ϕ preserves the geometric mean of A and B, i.e. $\phi(A\#B) = \phi(A)\#\phi(B)$;*
(3) *There is an invertible matrix S in $\mathbb{M}_n$ such that ϕ is of one of the forms*

$$\phi(A) = (\det A)^c SAS^*, \quad \phi(A) = (\det A)^c SA^TS^* \tag{2.1}$$

or of the forms

$$\phi(A) = (\det A)^c SA^{-1}S^*, \quad \phi(A) = (\det A)^c S(A^{-1})^TS^*, \tag{2.2}$$

where c is a real number for which $c \ne -\frac{1}{n}$ in (2.1) and $c \ne \frac{1}{n}$ in (2.2). Here A^T denotes the transpose of A.

Proof. Since (1) ⇒ (2) is trivial and (3) ⇒ (1) is a well-known result (see e.g. Chapter 4 in [2]), it is enough to verify that (2) ⇒ (3) holds. Consider the continuous map $\psi : \mathbb{P}_n \to \mathbb{P}_n$ defined by

$$\psi(A) = \phi(I)^{-\frac{1}{2}}\phi(A)\phi(I)^{-\frac{1}{2}}.$$

Using a similar argument as in [13], we can obtain that ψ is a so-called Jordan triple automorphism on $\mathbb{P}_n$. Using Molnár's structural result appearing in [15, Corollary 2] and noting that $\psi(A) = \phi(I)^{-\frac{1}{2}}\phi(A)\phi(I)^{-\frac{1}{2}}$, we deduce that ϕ is of one of the forms in (2.1) or (2.2). ☐

Corollary 2.2. *Let $n \ge 3$ and $\phi : \mathbb{P}_n \to \mathbb{P}_n$ be a continuous bijective map on $(\mathbb{P}_n, K^{g_\kappa})$. Then ϕ is a geodesic-affine map if and only if there is an invertible matrix $S \in \mathbb{M}_n$ such that for all $A \in \mathbb{P}_n$, ϕ is of one of the forms*

$$\phi(A) = (\det A)^c (SA^{\kappa}S^*)^{\frac{1}{\kappa}}, \quad \phi(A) = (\det A)^c (S(A^T)^{\kappa}S^*)^{\frac{1}{\kappa}} \tag{2.3}$$

or of the forms

$$\phi(A) = (\det A)^c (SA^{-\kappa}S^*)^{\frac{1}{\kappa}}, \quad \phi(A) = (\det A)^c (S(A^T)^{-\kappa}S^*)^{\frac{1}{\kappa}}. \tag{2.4}$$

Here c is a real number for which $c \neq -\frac{1}{n}$ in (2.3) and $c \neq \frac{1}{n}$ in (2.4).

Proof. Let $\psi\colon \mathbb{P}_n \to \mathbb{P}_n$ be a bijective map defined by

$$\psi(A) = \phi(A^{\frac{1}{\kappa}})^{\kappa}, \quad A \in \mathbb{P}_n.$$

One can see that

$$\psi(A\#_t B) = \psi(A)\#_t\psi(B), \quad t \in [0,1],\ A, B \in \mathbb{P}_n.$$

The rest of the proof is easy to obtain from the previous Theorem 2.1. □

Remark 2.3. *We note that the continuity assumption in Theorem 2.1 cannot be omitted. Indeed, let $f\colon (0,\infty) \to (0,\infty)$ be a multiplicative, non-continuous function and $\phi\colon \mathbb{P}_n \to \mathbb{P}_n$ be defined by*

$$\phi(A) = f(\det A)SAS^*.$$

One can check easily that ϕ preserves the geometric mean of any two points $A, B \in \mathbb{P}_n$, but it is not of the forms appearing in (2.1) or in (2.2).

Theorem 2.4. *Suppose that $\phi : \mathbb{P}_n \to \mathbb{P}_n$ is a geodesic-affine map on $(\mathbb{P}_n, K^l)$, i.e.*

$$\phi\left(e^{(1-t)\log A + t\log B}\right) = e^{(1-t)\log\phi(A) + t\log\phi(B)}, \tag{2.5}$$

if and only if there exist real numbers $\delta_1, \dots, \delta_k$, $n \times n$ matrices $M_1, \dots, M_k$ and a Hermitian matrix N such that

$$\phi(A) = e^{\sum_{i=1}^k \delta_i M_i(\log A){M_i}^* + N}, \quad A \in \mathbb{P}_n. \tag{2.6}$$

Proof. It is easy to verify that a map ϕ which is of the form (2.6) satisfies equation (2.5) for all $t \in [0,1]$.

On the other hand, we can define $\psi : \mathbb{H}_n \to \mathbb{H}_n$ by $\psi(T) = \log\phi(e^T)$ for $T \in \mathbb{H}_n$, then $\phi(A) = e^{\psi(\log A)}$ for $A \in \mathbb{P}_n$. From equation (2.5), we have

$$\psi((1-t)\log A + t\log B) = (1-t)\psi(\log A) + t\psi(\log B)$$

for all $A, B \in \mathbb{P}_n$.

Putting $Q_1 = \log A$ and $Q_2 = \log B$, we have

$$\psi((1-t)Q_1 + tQ_2) = (1-t)\psi(Q_1) + t\psi(Q_2)$$

for all $Q_1, Q_2 \in \mathbb{H}_n$. Hence ψ is an affine map from $\mathbb{H}_n$ to $\mathbb{H}_n$. From [7, Theorem 2] we know that, the linear map which maps Hermitian matrices into Hermitian matrices has the form

$\sum_{i=1}^{k} \delta_i M_i T M_i{}^*$ for all $T \in \mathbb{H}_n$, where $\delta_1, \dots, \delta_k$ are real numbers and $M_1, \dots, M_k$ are $n \times n$ matrices. Therefore,

$$\psi(T) = \sum_{i=1}^{k} \delta_i M_i T M_i{}^* + N,$$

where $N := \psi(O)$ is a Hermitian matrix. Hence for any $A \in \mathbb{P}_n$, $\phi(A)$ has the form

$$\phi(A) = e^{\psi(\log A)} = e^{\sum_{i=1}^{k} \delta_i M_i (\log A) M_i{}^* + N},$$

for some real numbers $\delta_1, \dots, \delta_k$, $n \times n$ matrices $M_1, \dots, M_k$ and a Hermitian matrix N. $\square$

Remark 2.5. *It is easy to see that if we set $N = O$, $M = U$ for some unitary matrix U and $\delta = 1$ or -1 in the above equation (2.6), then $\phi(A) = UAU^*$ or $\phi(A) = UA^{-1}U^*$ are the special cases of (2.6). Here O is the zero matrix. Furthermore, set $N = O$, $M = I$ and $\delta = \alpha \neq 0$, we can see that $\phi(A) = A^\alpha$ is a geodesic-affine map. But in general, equation (2.6) cannot be reduced to a single congruence.*

Remark 2.6. *Suppose that $\phi\colon \mathbb{P}_n \to \mathbb{P}_n$ is a geodesic-affine map on $(\mathbb{P}_n, K^{h_\alpha})$. That is to say,*

$$\phi\left(((1-t)A^\alpha + tB^\alpha)^{\frac{1}{\alpha}}\right) = ((1-t)\phi(A)^\alpha + t\phi(B)^\alpha)^{\frac{1}{\alpha}}, \tag{2.7}$$

for all $t \in [0, 1]$ and $A, B \in \mathbb{P}_n$. Then we can define $\psi\colon \mathbb{P}_n \to \mathbb{P}_n$ by $\psi(A) = \phi\left(A^{\frac{1}{\alpha}}\right)^\alpha$ for all $A \in \mathbb{P}_n$. Since $A \mapsto A^{\frac{1}{\alpha}}$ is a bijective transformation, it is clear that ψ satisfies the following

$$\psi((1-t)A + tB) = (1-t)\psi(A) + t\psi(B), \quad A, B \in \mathbb{P}_n. \tag{2.8}$$

As we know, there are many nonstandard affine maps of the form (2.8) on $\mathbb{P}_n$ if the metric is flat. However, the structures of geodesic-affine maps on these Riemannian manifolds are unknown. One can easily check that,

$$\psi(A) = \sum_{i=1}^{k} S_i A S_i^*, \quad \text{or} \quad \psi(A) = \sum_{i=1}^{k} S_i A^T S_i^*$$

with invertible matrices $S_1, \dots, S_k$ are special forms of affine maps on $(\mathbb{P}_n, K^{h_1})$.

3. Geodesic distance isometries and geodesic-affine maps

The famous Mazur-Ulam theorem states that every surjective isometry between two normed linear spaces is necessarily affine. In this section, we can see that the geodesic distance isometries are geodesic-affine maps. However, geodesic-affine maps are not necessarily isometric with respect to geodesic distance.

Theorem 3.1. *Let $\phi : \mathbb{P}_n \to \mathbb{P}_n$ be a surjective geodesic distance isometry in $(\mathbb{P}_n, K^{h_\alpha})$, then ϕ is a geodesic-affine map.*

Proof. Define $\psi\colon \mathbb{P}_n \to \mathbb{P}_n$ by the formula

$$\psi(A) = \phi(A^{\frac{1}{\alpha}})^{\alpha}, \quad A \in \mathbb{P}_n.$$

Then $\delta^{h_\alpha}(\phi(A), \phi(B)) = \delta^{h_\alpha}(A, B)$ yields that

$$\|\psi(A^\alpha) - \psi(B^\alpha)\|_{\mathrm{HS}} = \|A^\alpha - B^\alpha\|_{\mathrm{HS}}, \quad A, B \in \mathbb{P}_n.$$

As the map $A \mapsto A^{\frac{1}{\alpha}}$ is bijective, ψ is surjective and satisfies

$$\|\psi(A) - \psi(B)\|_{\mathrm{HS}} = \|A - B\|_{\mathrm{HS}}, \quad A, B \in \mathbb{P}_n.$$

Since $\mathbb{P}_n$ is a connected open subset of $\mathbb{H}_n$, Mankiewicz's result [11, Theorem 2] states that ψ can be uniquely extended to be an affine isometry from $\mathbb{H}_n$ to $\mathbb{H}_n$. Hence, we have $\psi((1-t)A + tB) = (1-t)\psi(A) + t\psi(B)$, $A, B \in \mathbb{P}_n$. Recalling that $\psi(A) = \phi\left(A^{\frac{1}{\alpha}}\right)^{\alpha}$, we obtain that

$$\phi\left(((1-t)A^\alpha + tB^\alpha)^{\frac{1}{\alpha}}\right) = ((1-t)\phi(A)^\alpha + t\phi(B)^\alpha)^{\frac{1}{\alpha}}.$$

This shows that ϕ is a geodesic-affine map on $(\mathbb{P}_n, K^{h_\alpha})$. □

Theorem 3.2. *Let $n \geq 3$ and $\phi : \mathbb{P}_n \to \mathbb{P}_n$ be a surjective geodesic distance isometry in $(\mathbb{P}_n, K^{g_\kappa})$, then there exists an invertible matrix $S \in \mathbb{M}_n$ such that ϕ is of one of the forms*

$$\phi(A) = (SA^\kappa S^*)^{\frac{1}{\kappa}}, (S(A^T)^\kappa S^*)^{\frac{1}{\kappa}}, (SA^{-\kappa}S^*)^{\frac{1}{\kappa}} \quad or, \quad (S(A^T)^{-\kappa}S^*)^{\frac{1}{\kappa}},$$

or of the forms

$$\phi(A) = (\det A)^{-\frac{2}{n}}(SA^\kappa S^*)^{\frac{1}{\kappa}}, (\det A)^{-\frac{2}{n}}(S(A^T)^\kappa S^*)^{\frac{1}{\kappa}},$$
$$(\det A)^{-\frac{2}{n}}(SA^{-\kappa}S^*)^{\frac{1}{\kappa}} \quad or, \quad (\det A)^{-\frac{2}{n}}(S(A^T)^{-\kappa}S^*)^{\frac{1}{\kappa}},$$

for all $A \in \mathbb{P}_n$. Furthermore, ϕ is a geodesic-affine map.

Proof. Since ϕ is a geodesic distance isometry, we have $\delta^{g_\kappa}(\phi(A), \phi(B)) = \delta^{g_\kappa}(A, B)$. Define $\psi\colon \mathbb{P}_n \to \mathbb{P}_n$ by the formula

$$\psi(A) = \phi\left(A^{\frac{1}{\kappa}}\right)^{\kappa}, \quad A \in \mathbb{P}_n.$$

This yields that

$$\|\log(\psi(A^\kappa)^{-\frac{1}{2}}\psi(B^\kappa)\psi(A^\kappa)^{-\frac{1}{2}})^{\frac{1}{\kappa}}\|_{\mathrm{HS}} = \|\log(A^{-\frac{\kappa}{2}}B^\kappa A^{-\frac{\kappa}{2}})^{\frac{1}{\kappa}}\|_{\mathrm{HS}}, \quad A \in \mathbb{P}_n.$$

Since the map $A \mapsto A^\kappa$ is bijective on $\mathbb{P}_n$ for $\kappa > 0$, one can obtain that

$$\|\log(\psi(A)^{-\frac{1}{2}}\psi(B)\psi(A)^{-\frac{1}{2}})\|_{\mathrm{HS}} = \|\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\|_{\mathrm{HS}}.$$

Using Theorem 3 in [15] we can obtain the structure of ψ and then by $\psi(A) = \phi\left(A^{\frac{1}{\kappa}}\right)^{\kappa}$ we also get the desired forms of ϕ. Furthermore, ϕ is a geodesic-affine map from Corollary 2.2, which ends the proof. □

In [4], Hatori and Molnár have determined the Thompson isometries of the spaces of invertible positive elements in unital C^*-algebras. They have shown that any Thompson isometry is necessarily a geodesic-affine map. Honma and Nogawa studied a more general case in [8]. We emphasize that in the just mentioned papers the set of $\mathbb{P}_n$ is equipped with a Finsler-type structure.

Next, we will consider the geodesic distance isometry on Riemannian manifold $(\mathbb{P}_n, K^l)$.

Theorem 3.3. *Let $\phi : \mathbb{P}_n \to \mathbb{P}_n$ be a surjective isometry on the Riemannian manifold $(\mathbb{P}_n, K^l)$. Then there exists a unitary matrix U and a Hermitian matrix N such that*

$$\phi(A) = e^{\pm U(\log A)U^* + N}, \quad \textit{or } \phi(A) = e^{\pm U(\log A)^T U^* + N}, \qquad A \in \mathbb{P}_n. \tag{3.1}$$

Moreover, ϕ is a geodesic-affine map. If ϕ is assumed to be unital, then we have that $N = O$ in (3.1).

Proof. Define $f : \mathbb{H}_n \to \mathbb{H}_n$ by $f(T) = \log \phi(e^T)$ for all $T \in \mathbb{H}_n$. Clearly, $\phi(A) = e^{f(\log A)}$, $A \in \mathbb{P}_n$. Hence, the geodesic distance isometry $\|\log \phi(A) - \log \phi(B)\|_{\mathrm{HS}} = \|\log A - \log B\|_{\mathrm{HS}}$ implies that

$$\|f(\log A) - f(\log B)\|_{\mathrm{HS}} = \|\log A - \log B\|_{\mathrm{HS}} \quad A, B \in \mathbb{P}_n.$$

Since $A \mapsto \log A$ is a bijective map from $\mathbb{P}_n$ to $\mathbb{H}_n$, we get a surjective f satisfying

$$\|f(A) - f(B)\|_{\mathrm{HS}} = \|A - B\|_{\mathrm{HS}} \quad A, B \in \mathbb{H}_n.$$

Set $g(A) = f(A) - f(O)$, which is real linear from Mazur-Ulam Theorem. Then we can extend $g : \mathbb{M}_n \to \mathbb{M}_n$ by $g(A) = g(P) + ig(Q)$ for $A = P + iQ$ with $P, Q \in \mathbb{H}_n$. Hence, g is a surjective linear isometry on $\mathbb{M}_n$. The well-known result of Kadison on the surjective linear isometries of C^*-algebras implies that there exist unitary matrices U, V such that

$$g(A) = UAV, \quad \textit{or } g(A) = UA^T V, \quad A \in \mathbb{M}_n.$$

Since f maps $\mathbb{H}_n$ to $\mathbb{H}_n$, we can see $V = \pm U^*$. Therefore, we have

$$\phi(A) = e^{\pm U(\log A)U^* + N}, \quad \textit{or } \phi(A) = e^{\pm U(\log A)^T U^* + N},$$

where $N := f(O)$ is a Hermitian matrix. Obviously, $N = O$ when ϕ is unital.

Together with Theorem 2.4, we can see immediately that ϕ is a geodesic-affine map. $\square$

As an application of Theorem 3.2, we consider the maps preserving the length of every differentiable curves in $(\mathbb{P}_n, K^{g_1})$. This problem has been introduced by many authors. In [3], the authors studied the map $\phi(A) = SAS^*$ for any invertible matrix S. They have proved that $L_{g_1}(\rho)$ is invariant under this map. In [5, Proposition 2.3], the authors have shown that $A \mapsto A^{-1}$ also preserves the length with respect to the metric K^{g_1}. Since being a length preserving map is equivalent to being a geodesic distance preserving map, we can see that ϕ is a length preserving map if and only if it is one of the forms in Theorem 3.2.

4. Maps preserving norms of geodesic correspondence on $(\mathbb{P}_n, K^{\varphi})$

Suppose that a geodesic-affine map ϕ on $\mathbb{P}_n$ is also an isometry with respect to certain norm $\|\cdot\|$, then we can obtain that ϕ preserves norm of geodesic correspondence since

$$\|\gamma_{\phi(A),\phi(B)}(t))\| = \|\phi(\gamma_{A,B}(t))\| = \|\gamma_{A,B}(t))\|.$$

In this section, we consider the converse problem. We will characterize the structure of maps preserving norm of geodesic correspondence, which shows that these maps are also geodesic distance isometries and geodesic-affine maps.

Recall that the Schatten p-norm of A in $\mathbb{P}_n$ is defined by $\|A\|_p = (\operatorname{tr} A^p)^{\frac{1}{p}}$. In particular case, $\|.\|_1$ is the so-called trace norm. We denote all $n \times n$ positive semi-definite matrices with unit trace by S_n and $\mathcal{M}_n$ denotes the set of all invertible elements of S_n, which is a submanifold of $\mathbb{P}_n$ from differential geometric point of view. For two self-adjoint matrices A and B we write $A \leq B$ if and only if $B - A$ is positive semi-definite, i.e., $0 \leq \langle (B-A)x, x\rangle$ holds for every vector x. We emphasize that we are going to use arguments similar to that appearing in [14].

Theorem 4.1. *Let $p \geq 1$ and $\phi : \mathbb{P}_n \to \mathbb{P}_n$ be a bijective map. Then the following statements are equivalent:*

(1) *ϕ preserves the Schatten p-norms of geodesic correspondence on $(\mathbb{P}_n, K^{h_\alpha})$, where α is a given positive number such that $\alpha \neq p$. That is to say,*
$$\left\|((1-t)A^\alpha + tB^\alpha)^{\frac{1}{\alpha}}\right\|_p = \left\|((1-t)\phi(A)^\alpha + t\phi(B)^\alpha)^{\frac{1}{\alpha}}\right\|_p, \tag{4.1}$$
for all $t \in [0,1]$ and $A, B \in \mathbb{P}_n$.

(2) *ϕ preserves the Schatten p-norms of geodesic correspondence on $(\mathbb{P}_n, K^{g_\kappa})$, where κ is a given positive number. That is to say,*
$$\|(A^\kappa \#_t B^\kappa)^{\frac{1}{\kappa}}\|_p = \|(\phi(A)^\kappa \#_t \phi(B)^\kappa)^{\frac{1}{\kappa}}\|_p, \tag{4.2}$$
for all $t \in [0,1]$ and $A, B \in \mathbb{P}_n$.

(3) *ϕ preserves the Schatten p-norms of geodesic correspondence on $(\mathbb{P}_n, K^{l})$. That is to say,*
$$\left\|e^{(1-t)\log A + t\log B}\right\|_p = \left\|e^{(1-t)\log\phi(A) + t\log\phi(B)}\right\|_p, \tag{4.3}$$
for all $t \in [0,1]$ and $A, B \in \mathbb{P}_n$.

(4) *There exists a unitary matrix U such that*
$$\phi(A) = UAU^*, \quad \text{or } \phi(A) = UA^TU^*, \tag{4.4}$$
for all $A \in \mathbb{P}_n$.

Moreover, the maps appearing in (1)–(3) are geodesic distance isometries with respect to their corresponding Riemannian metrics and by the assertions in Section 3 these maps are geodesic-affine maps.

Since the Schatten p-norm is invariant under unitarily equivalence, it is easy to check that (4) in Theorem 4.1 implies the other assertions (1)–(3). In the proofs of the converse implications ($(1) \Rightarrow (4), (2) \Rightarrow (4), (3) \Rightarrow (4)$) , we will consider the derivative of the Schatten p-norm of the corresponding geodesics with respect to t at $t = 0$. To do this we need the following proposition. Since it is a well-known result we present it without proof (for details see e.g. Theorem 6.6.30(1) in [9]).

Proposition 4.2. *Suppose that $X(t)$ is a positive definite matrix for each $t \in \mathbb{R}$ and $X(\cdot)$ is continuously differentiable then*

$$\frac{d}{dt} \operatorname{tr}[X(t)^p] = p \operatorname{tr}[X(t)^{p-1} \frac{d}{dt} X(t)]$$

for all $p > 0$.

Proof of Theorem 4.1$(1) \Rightarrow (4)$. Define the map $\psi\colon \mathbb{P}_n \to \mathbb{P}_n$ by the formula

$$\psi(A) = \phi(A^{\frac{1}{\alpha}})^{\alpha}, \quad A \in \mathbb{P}_n.$$

Since $A \mapsto A^{\frac{1}{\alpha}}$ is bijective, it is obvious that ψ is also a bijective transformation and it has the following preserver property

$$\left\| ((1-t)A + tB)^{\frac{1}{\alpha}} \right\|_p = \left\| ((1-t)\psi(A) + t\psi(B))^{\frac{1}{\alpha}} \right\|_p, \quad A, B \in \mathbb{P}_n. \tag{4.5}$$

Using Proposition 4.2 and differentiating both sides of this equation at $t = 0$, we have

$$\operatorname{tr}[A^{\frac{p-\alpha}{\alpha}}(B - A)] = \operatorname{tr}[\psi(A)^{\frac{p-\alpha}{\alpha}}(\psi(B) - \psi(A))], \quad A, B \in \mathbb{P}_n.$$

We also get easily that $\operatorname{tr} A^{\frac{p}{\alpha}} = \operatorname{tr} \psi(A)^{\frac{p}{\alpha}}$, hence it follows that

$$\operatorname{tr}(A^{\frac{p-\alpha}{\alpha}} B) = \operatorname{tr}(\psi(A)^{\frac{p-\alpha}{\alpha}} \psi(B)), \quad A, B \in \mathbb{P}_n. \tag{4.6}$$

Now, we assert that ψ preserves the order between positive definite matrices. To show this, we need to verify that

$$B \leq \tilde{B} \Longleftrightarrow \operatorname{tr} A^{\frac{p-\alpha}{\alpha}} B \leq \operatorname{tr} A^{\frac{p-\alpha}{\alpha}} \tilde{B}, \quad A \in \mathbb{P}_n. \tag{4.7}$$

The sufficiency is obvious. For the necessity assume that $\operatorname{tr} A^{\frac{p-\alpha}{\alpha}} B \leq \operatorname{tr} A^{\frac{p-\alpha}{\alpha}} \tilde{B}$. Then taking limits under the trace we see that the inequality holds for every positive semi-definite matrix A. Choosing $A^{\frac{p-\alpha}{\alpha}} = x \otimes x$ for an arbitrary unit vector x into that inequality, we obtain $\langle Bx, x\rangle \leq \langle \tilde{B}x, x\rangle$. Together with the bijectivity and preserving property (4.6), we have

$$B \leq \tilde{B} \Longleftrightarrow \psi(B) \leq \psi(\tilde{B}), \quad B \in \mathbb{P}_n.$$

The structure of all order automorphism on $\mathbb{P}_n$ is known and it is described by Molnár in [14, Theorem 1]. It states that there exists an invertible matrix S such that

$$\psi(A) = SAS^*, \quad \text{or} \quad \psi(A) = SA^T S^*, \quad A \in \mathbb{P}_n.$$

Substituting the case $\psi(A) = SAS^*$ into (4.6), we get

$$\operatorname{tr} A^{\frac{p-\alpha}{\alpha}} B = \operatorname{tr}(SAS^*)^{\frac{p-\alpha}{\alpha}} SBS^*. \tag{4.8}$$

Since the equality (4.8) holds for all $A \in \mathbb{P}_n$ and $B \geq 0$ by taking limits under the trace, we can choose $A = I$ and $B = x \otimes x$ for any unit vector x. Taking limit under the trace, the equation (4.8) holds for all $A \in \mathbb{P}_n$ and positive semi-definite matrix B. Therefore, we have

$$\langle x, x\rangle = \langle S^*(SS^*)^{\frac{p-\alpha}{\alpha}} Sx, x\rangle$$

for all unit vector $x \in \mathbb{R}^n$. Consequently, $S^*(SS^*)^{\frac{p-\alpha}{\alpha}} S = I$ and it follows that S is a unitary matrix. Similarly, one can easily check that the matrix S in the case $\psi(A) = SA^T S^*$ is also a unitary matrix. Therefore we obtain that ψ is of one of the following forms

$$\psi(A) = UAU^*, \quad \text{or} \quad \psi(A) = UA^T U^*.$$

for all $A \in \mathbb{P}_n$. Recalling that $\phi(A) = (\psi(A^\alpha))^{\frac{1}{\alpha}}$, the proof is then completed. □

Proof of Theorem 4.1 (2) ⇒ (4). Let $\psi\colon \mathbb{P}_n \to \mathbb{P}_n$ be a bijective map defined by $\psi(A) = \phi\left(A^{\frac{1}{\kappa}}\right)^{\kappa}$, $A \in \mathbb{P}_n$. Then, the map ψ has the following preserving property

$$\operatorname{tr}[(A\#_t B)^{\frac{p}{\kappa}}] = \operatorname{tr}[(\psi(A)\#_t\psi(B))^{\frac{p}{\kappa}}], \quad A, B \in \mathbb{P}_n. \tag{4.9}$$

Consequently, using Proposition 4.2 and differentiating both sides of (4.9) at $t = 0$, one can obtain that

$$\operatorname{tr}\left[A^{\frac{p}{\kappa}} \log(A^{\frac{1}{2}} B^{-1} A^{\frac{1}{2}})\right] = \operatorname{tr}\left[\psi(A)^{\frac{p}{\kappa}} \log(\psi(A)^{\frac{1}{2}} \psi(B)^{-1} \psi(A)^{\frac{1}{2}})\right], \tag{4.10}$$

for all $A, B \in \mathbb{P}_n$.

Using a similar proof to that in [14, Lemma 4], we have

$$B \leq \tilde{B} \iff \psi(B) \leq \psi(\tilde{B}), \quad B, \tilde{B} \in \mathbb{P}_n.$$

Hence, there exists a unitary matrix U such that ϕ is of the form

$$\phi(A) = UAU^*, \quad \text{or} \quad \phi(A) = UA^T U^*, \quad A \in \mathbb{P}_n,$$

from [14, Theorem 5] and $\phi(A) = (\psi(A^\kappa))^{\frac{1}{\kappa}}$. □

Remark 4.3. *Taking $\alpha = p$ in equation (4.1), we have*

$$\operatorname{tr}[(1-t)A^p + tB^p] = \operatorname{tr}[(1-t)\phi(A)^p + t\phi(B)^p], \quad A, B \in \mathbb{P}_n. \tag{4.11}$$

In this case, the map can be very complicate. For example, define

$$\phi(A) = U(s)AU(s)^*, \quad A \in \mathbb{P}_n,\ \|A\|_p = s,$$

where $U(s)$ is a unitary matrix depending on the Schatten p-norm of A. One can check easily that ϕ is bijective and satisfies (4.11), but it is not of one of the forms appearing in Theorem 4.1.

Proof of Theorem 4.1(3) ⇒ (4). From the definition of Schatten p-norm, one can get

$$\operatorname{tr}\left[e^{(1-t)\log A + t\log B}\right]^p = \operatorname{tr}\left[e^{(1-t)\log\phi(A) + t\log\phi(B)}\right]^p, \quad A, B \in \mathbb{P}_n. \tag{4.12}$$

In [17] Pedersen studied the differentiability properties of some operator valued functions of one operator variable. Using one of the results appearing in [17], we get that

$$\frac{d}{dt}e^{(1-t)\log A+t\log B}\Big|_{t=0}=\int_0^1 e^{s\log A}(\log B-\log A)e^{(1-s)\log A}\,ds.$$

By the above formula and Proposition 4.2 we can differentiate both sides of (4.12) at $t=0$ and then we obtain that

$$\mathrm{tr}[A^p(\log B-\log A)]=\mathrm{tr}[\phi(A)^p(\log\phi(B)-\log\phi(A))],\quad A,B\in\mathbb{P}_n. \tag{4.13}$$

In the following we apply an approach similar to that appearing in [14, Theorem 3]. We assert that for any $B,\tilde{B}\in\mathbb{P}_n$,

$$\log B\le\log\tilde{B}\Longleftrightarrow\mathrm{tr}[A^p(\log B-\log A)]\le\mathrm{tr}[A^p(\log\tilde{B}-\log A)],\quad A\in\mathbb{P}_n. \tag{4.14}$$

Using the above characterization of ordering (4.14) and the bijectivity and preserver property (4.13) of ϕ we deduced that

$$\log B\le\log\tilde{B}\Longleftrightarrow\log\phi(B)\le\log\phi(\tilde{B})$$

holds for all $B,\tilde{B}\in\mathbb{P}_n$. Now, we are in a position to apply Theorem 2 in [14]. Using that result and an approach similar to that appearing in the proof of Theorem 3 in [14] we can deduce that there exists a unitary matrix U such that

$$\phi(A)=UAU^*,\quad\text{or}\quad\phi(A)=UA^TU^*,\quad A\in\mathbb{P}_n.$$

□

Definition 4.4. *For any $A,B\in S_n$, the Belavkin-Staszewski relative entropy $S_{BS}(A||B)$ (see [1]) is defined by*

$$S_{BS}(A||B)=\begin{cases}\mathrm{tr}[A\log(A^{1/2}B^{-1}A^{1/2})], & \text{if suppA}\subset\text{suppB},\\ +\infty, & \text{otherwise}.\end{cases}$$

Here and throughout this paper supp denotes the support of operators, that is the orthogonal complement of the kernel of an element in S_n.

For any pair $A,B\in S_n$ of states, the Umegaki relative entropy $S_U(A||B)$ is defined by

$$S_U(A||B)=\begin{cases}\mathrm{tr}[A(\log A-\log B)], & \text{if suppA}\subset\text{suppB},\\ +\infty, & \text{otherwise}.\end{cases}$$

It can be seen easily that relative entropies are always finite for any $A,B\in\mathcal{M}_n$. From Theorem 4.1, we can get the following corollaries.

Corollary 4.5. *Let $\phi:\mathbb{P}_n\to\mathbb{P}_n$ be a bijective map preserving the trace norm of geodesic correspondence on $(\mathbb{P}_n,K^{g_1})$. Then ϕ preserves the Belavkin-Staszewski relative entropy on $\mathcal{M}_n$ and there exists a unitary matrix U such that*

$$\phi(A)=UAU^*,\quad\textit{or}\quad\phi(A)=UA^TU^*,$$

for all $A \in \mathbb{P}_n$.

Corollary 4.6. *Let* $\phi : \mathbb{P}_n \to \mathbb{P}_n$ *be a bijective map preserving trace norm of geodesic correspondence on* $(\mathbb{P}_n, K^l)$. *Then* ϕ *preserves the Umegaki relative entropy on* $\mathcal{M}_n$ *and there exists a unitary matrix* U *such that*

$$\phi(A) = UAU^*, \quad or \quad \phi(A) = UA^T U^*,$$

for all $A \in \mathbb{P}_n$.

5. Acknowledgement

The authors would like to thank Professor Lajos Molnár and Professor Ngai-Ching Wong for the motivation of the present paper. We are grateful for their important observations and suggestions which helped to improve this paper. We would also like to express our gratitude to the unknown referee for his valuable suggestions.

References

[1] V. P. Belavkin, P. Staszewski, *C^*-algebraic generalization of relative entropy and entropy,* Ann. Inst. H. Poincare Sect. A **37** (1982) 51–58.

[2] R. Bhatia, Positive definite matrices. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2007.

[3] R. Bhatia, J. Holbrook, *Riemannian geometry and matrix geometric means,* Linear Algebra Appl. **413** (2006) 594–618.

[4] O. Hatori, L. Molnár, *Isometries of the unitary groups and Thompson isometries of the spaces of invertible positive elements in C^*-algebras,* J. Math. Anal. Appl. **409** (2014) 158–167.

[5] F. Hiai, D. Petz, *Riemannian metrics on positive definite matrices related to means,* Linear Algebra Appl. **430** (2009) 3105–3130.

[6] F. Hiai, D. Petz, *Riemannian metrics on positive definite matrices related to means. II,* Linear Algebra Appl. **436** (2012), 2117–2136.

[7] R. D. Hill, *Linear transformations which preserve Hermitian matrices,* Linear Algebra Appl. **6** (1973) 257–262.

[8] S. Honma, T. Nogawa, *Isometries of the geodesic distances for the space of invertible positive operators and matrices,* Linear Algebra Appl. **444** (2014) 152–164.

[9] R. A. Horn, C. R. Johnson, *Topics in matrix analysis*, tenth ed., Cambridge, New York, 2008.

[10] J. D. Lawson, Y. Lim, *The geometric mean, matrices, metrics, and more,* Amer. Math. Monthly **108** (2001) 797–812.

[11] P. Mankiewicz, *On extension of isometries in normed linear spaces,* (Russian summary) Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 20 (1972), 367–371.

[12] M. Moakher, *A differential geometric approach to the geometric mean of symmetric positive definite matrices,* SIAM J. Matrix Anal. Appl. **26** (2005), 735–747.

[13] L. Molnár, *Maps preserving the geometric mean of positive operators,* Proc. Amer. Math. Soc. **137** (2009), 1763–1770.

[14] L. Molnár, *Order automorphisms on positive definite operators and a few applications,* Linear Algebra Appl. **434** (2011) 2158–2169.

[15] L. Molnár, *Jordan triple endomorphisms and isometries of spaces of positive definite matrices,* Linear Multilinear Algebra (2013), in press, published online: 30 Oct 2013.

[16] A. Ohara, N. Suda and S. Amari, *Dualistic differential geometry of positive definite matrices and its applications to related problems,* Linear Algebra Appl. **247** (1996), 31–53.

[17] G. K. Pedersen, *Operator differentiable functions,* Publ. RIMS Kyoto Univ. **36** (2000), 139–157.

[18] L. T. Skovgaard, *A Riemannian geometry of the multivariate normal model,* Scand. J. Statist. **11** (1984), 211–223.

(Szokol) MTA-DE "Lendület" Functional Analysis Research Group, Institute of Mathematics, University of Debrecen, P.O. Box 12, 4010 Debrecen, Hungary

E-mail address, Szokol: szokolp@science.unideb.hu

(Tsai) Department of Applied Mathematics, National Sun Yat-Sen University, Kaohsiung 80424, Taiwan.

E-mail address, Tsai: mctsai2@gmail.com

(Zhang) School of Mathematics and Statistics, Central China Normal University, Wuhan, Hubei, 430079, China.

E-mail address, Zhang: zhjun@mail.ccnu.edu.cn