Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **26**, 1–17; http://www.math.u-szeged.hu/ejqtde/

Fractional-order multivalued problems with non-separated integral-flux boundary conditions

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Received 20 January 2015, appeared 20 May 2015
Communicated by Gennaro Infante

Abstract. In this paper, we study the existence of solutions for a new kind of boundary value problem of Caputo type fractional differential inclusions with non-separated local and nonlocal integral-flux boundary conditions. We apply appropriate fixed point theorems for multivalued maps to obtain the existence results for the given problems covering convex as well as non-convex cases for multivalued maps. We also include Riemann–Stieltjes integral conditions in our discussion. Some illustrative examples are also presented. The paper concludes with some interesting observations.

Keywords: fractional differential inclusions, integral, flux, boundary conditions, fixed point.

2010 Mathematics Subject Classification: 34A60, 34A08.

1 Introduction

We investigate existence of solutions for the following fractional differential inclusion:

$$ {}^cD^{\alpha}x(t) \in F(t, x(t)), \qquad t \in [0,1], \qquad 1 < \alpha \le 2, \tag{1.1}$$

supplemented with non-separated local and nonlocal integral-flux boundary conditions respectively given by

$$ x(0) + x(1) = a\int_0^1 x(s)\,ds, \qquad x'(0) = b\,{}^cD^{\beta}x(1), \qquad 0 < \beta \le 1, \tag{1.2}$$

and

$$ x(0) + x(1) = aI^{\gamma}x(\eta), \qquad x'(0) = b\,{}^cD^{\beta}x(1), \qquad 0 < \beta, \gamma \le 1, \qquad 0 < \eta < 1, \tag{1.3}$$

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where ${}^cD^\alpha$, ${}^cD^\beta$ denotes the Caputo fractional derivatives of orders α, β, $F\colon [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, I^γ is the Riemann–Liouville fractional integral of order γ (see Definition 2.1) and a, b are appropriate real constants.

Fractional-order boundary problems involving a variety of boundary conditions have been extensively studied in the recent years. In view of the extensive development of single-valued nonlinear boundary value problems of fractional-order differential equations [1–4,17,19,26,27,29,31], it is natural to extend this work to the case of fractional-order multi-valued problems. For some recent results on fractional-order inclusions problems, we refer the reader to a series of papers [6–8, 11, 12, 14, 20, 35, 36] and the references cited therein. It is worthwhile to mention that fractional-order differential equations have attracted a great attention due to their widespread applications in applied and technical sciences such as physics, mechanics, chemistry, engineering, biomedical sciences, control theory, etc. One of the reasons for popularity of fractional calculus is that fractional-order operators can describe the hereditary properties of many important materials and processes. Further details can be found in the texts [9,23,28].

The purpose of this paper is to establish some existence results for the problems (1.1)–(1.2) and (1.1)–(1.3) for convex and non-convex values of the multivalued maps involved in the problems. Our main results rely on the well known tools of fixed point theory of multivalued maps such as the nonlinear alternative of Leray–Schauder type and a fixed point theorem for contraction multivalued maps due to Covitz and Nadler. We also discuss the case when the multivalued map is not necessarily convex valued. In this case, we make use of the nonlinear alternative of Leray–Schauder type for single-valued maps and a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values. We emphasize that the tools employed in the present work are well known, however their application in the present framework facilitates to obtain the existence results for the problems (1.1)–(1.2) and (1.1)–(1.3), which is indeed a new development. The paper is organized as follows. Section 2 contains some preliminaries needed for the sequel. In Section 3, we establish the existence results for the problem (1.1)–(1.2) which are well illustrated with the aid of examples. We also discuss the Riemann–Stieltjes integral conditions case in this section. The results for the problem (1.1)–(1.3) are presented in Section 4.

2 Preliminaries

In this section, we recall some basic concepts of fractional calculus [23,28] and multi-valued analysis [16,21]. We also prove an auxiliary lemma which plays a key role in defining a fixed point problem related to the problem (1.1)–(1.2).

Definition 2.1. The Riemann–Liouville fractional integral of order q for a continuous function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}}\, ds, \qquad q > 0,$$

provided the integral exists.

Definition 2.2. For an at least n times continuously differentiable function $g\colon [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)\, ds, \qquad n-1 < q < n,\ n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Lemma 2.3 ([23, 28]).

(i) *If* $\alpha > 0,\ \beta > 0,\ \beta > \alpha,\ f \in L(0,1)$ *then*

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t), \qquad D^\alpha I^\alpha f(t) = f(t), \qquad D^\alpha I^\beta f(t) = I^{\beta-\alpha} f(t).$$

(ii)

$$^cD^\alpha t^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\alpha)} t^{\lambda-\alpha-1}, \qquad \lambda > [\alpha] \quad \text{and} \quad {}^cD^\alpha t^{\lambda-1} = 0,\ \lambda < [\alpha].$$

Lemma 2.4. *Let* $a \neq 2$, $b \neq \Gamma(2-\beta)$. *Let* $y \in C([0,1],\mathbb{R})$ *and* $x \in C^2([0,1],\mathbb{R})$ *be a solution of the linear boundary value problem*

$$\begin{cases} ^cD^\alpha x(t) = y(t), \quad 0 < t < 1,\ 1 < \alpha \le 2 \\ x(0) + x(1) = a \displaystyle\int_0^1 x(s)\,ds, \quad x'(0) = b\, {}^cD^\beta x(1), \quad 0 < \beta \le 1. \end{cases} \tag{2.1}$$

Then

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s)\,ds \\ & - \frac{1}{2-a} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + \frac{a}{2-a} \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} y(s)\,ds. \end{aligned} \tag{2.2}$$

Proof. It is well known that the general solution of the fractional differential equation in (2.1) can be written as

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + c_1 t + c_0, \tag{2.3}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants.

Using the boundary condition $x'(0) = b\, {}^cD^\beta x(1)$ in (2.3), we find that

$$c_1 = \frac{b\Gamma(2-\beta)}{\Gamma(2-\beta)-b} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s)\,ds.$$

In view of the condition $x(0) + x(1) = a \int_0^1 x(s)\,ds$, (2.3) yields

$$2c_0 + c_1 + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds = a \int_0^1 \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,du + \frac{ac_1}{2} + ac_0,$$

which, on inserting the value of c_1, and using the first relation in part *(i)* of Lemma 2.3, gives

$$\begin{aligned} c_0 = {} & -\frac{1}{2} \frac{b\Gamma(2-\beta)}{[\Gamma(2-\beta)-b]} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s)\,ds \\ & + \frac{a}{2-a} \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} y(s)\,ds - \frac{1}{2-a} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds. \end{aligned}$$

Substituting the values of c_0, c_1 in (2.3) we get (2.2). This completes the proof. □

Let $C([0,1],\mathbb{R})$ denote the Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$. Let $L^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $x\colon [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|\,dt$.

Definition 2.5. A function $x \in C^2([0,1],\mathbb{R})$ is called a solution of problem (1.1)–(1.2) if there exists a function $v \in L^1([0,1],\mathbb{R})$ with $v(t) \in F(t,x(t))$, a.e. $[0,1]$ such that $x(0)+x(1) = a\int_0^1 x(s)\,ds,\ x'(0) = b\,{}^cD^\beta x(1)$ and

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v(s)\,ds \\ - \frac{1}{2-a}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds + \frac{a}{2-a}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} v(s)\,ds.$$

Next we recall some basic definitions of multivalued analysis.

For a normed space $(\mathcal{A},\|\cdot\|)$, let $\mathcal{P}_{cl}(\mathcal{A}) = \{\mathcal{A}_1 \in \mathcal{P}(\mathcal{A}) : \mathcal{A}_1 \text{ is closed}\}$, $\mathcal{P}_b(\mathcal{A}) = \{\mathcal{A}_1 \in \mathcal{P}(\mathcal{A}) : \mathcal{A}_1 \text{ is bounded}\}$, $\mathcal{P}_{cp}(\mathcal{A}) = \{\mathcal{A}_1 \in \mathcal{P}(\mathcal{A}) : \mathcal{A}_1 \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(\mathcal{A}) = \{\mathcal{A}_1 \in \mathcal{P}(\mathcal{A}) : \mathcal{A}_1 \text{ is compact and convex}\}$. A multi-valued map $G\colon \mathcal{A} \to \mathcal{P}(\mathcal{A})$ is convex (closed) valued if $G(a)$ is convex (closed) for all $a \in \mathcal{A}$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in $\mathcal{A}$ for all $\mathbb{B} \in \mathcal{P}_b(\mathcal{A})$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on $\mathcal{A}$ if for each $a_0 \in \mathcal{A}$, the set $G(a_0)$ is a nonempty closed subset of $\mathcal{A}$, and if for each open set $\mathcal{N}$ of $\mathcal{A}$ containing $G(a_0)$, there exists an open neighborhood $\mathcal{N}_0$ of a_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(\mathcal{A})$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $a_n \to a_*$, $b_n \to b_*$, $b_n \in G(a_n)$ imply $b_* \in G(a_*)$. G has a fixed point if there is $a \in \mathcal{A}$ such that $a \in G(a)$. The fixed point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$. A multivalued map $G\colon [0,1] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $b \in \mathbb{R}$, the function $t \longmapsto d(b,G(t)) = \inf\{|b-c| : c \in G(t)\}$ is measurable.

3 Existence results for the boundary value problem (1.1)–(1.2)

In this section, we study the existence of solutions for the problem (1.1)–(1.2).

3.1 The upper semicontinuous case

Definition 3.1. A multivalued map $F\colon [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$.

Further, a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_\rho(t)$$

for all $\|x\| \le \rho$ and for a.e. $t \in [0,1]$.

For each $y \in C([0,1],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,1]\}.$$

For the forthcoming analysis, we need the following lemmas.

Lemma 3.2 (Nonlinear alternative for Kakutani maps [22])**.** *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F\colon \overline{U} \to \mathcal{P}_{cp,c}(C)$ is an upper semicontinuous compact map. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.

Lemma 3.3 ([25])**.** *Let X be a Banach space. Let $F\colon [0,1] \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\Theta \circ S_F\colon C([0,1],X) \to \mathcal{P}_{cp,c}(C([0,1],X)), \qquad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.

Now we are in a position to prove the existence of the solutions for the boundary value problem (1.1)–(1.2) when the right-hand side is convex valued.

Theorem 3.4. *Assume that:*

(H_1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is L^1-Carathéodory and has nonempty compact and convex values;*

(H_2) *there exists a continuous nondecreasing function $\psi\colon [0,\infty) \to (0,\infty)$ and a function $p \in C([0,1],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \quad \textit{for each } (t,x) \in [0,1] \times \mathbb{R};$$

(H_3) *there exists a constant $M > 0$ such that*

$$\frac{M}{\psi(M)\|p\|\Lambda} > 1,$$

where

$$\Lambda = \frac{1+|2-a|}{|2-a|\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)}. \tag{3.1}$$

Then the boundary value problem (1.1)–(1.2) has at least one solution on $[0,1]$.

Proof. Define the operator $\Omega_F\colon C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ by

$$\Omega_F(x) = \left\{ \begin{aligned} & h \in C([0,1],\mathbb{R}) : \\ & h(t) = \left\{ \begin{aligned} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\, ds \\ & + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v(s)\, ds \\ & + \frac{1}{2-a} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\, ds \\ & - \frac{a}{2-a} \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} v(s) ds, \end{aligned} \right. \end{aligned} \right\}$$

for $v \in S_{F,x}$. We will show that Ω_F satisfies the assumptions of the nonlinear alternative of Leray–Schauder type. The proof consists of several steps. As a first step, we show that Ω_F *is convex for each* $x \in C([0,1],\mathbb{R})$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

In the second step, we show that Ω_F *maps bounded sets (balls) into bounded sets in* $C([0,1],\mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \leq r\}$ be a bounded ball in $C([0,1],\mathbb{R})$. Then, for each $h \in \Omega_F(x), x \in B_r$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v(s)\,ds \\ & + \frac{1}{2-a}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds - \frac{a}{2-a}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} v(s)\,ds. \end{aligned}$$

Then for $t \in [0,1]$ we have

$$\begin{aligned} |h(t)| \leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|\,ds + \frac{|b(2t-1)|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |v(s)|\,ds \\ & + \frac{1}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|\,ds + \frac{|a|}{|2-a|}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} |v(s)|\,ds \\ \leq & \ \psi(\|x\|)\|p\| \left[\frac{1}{\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)} \right. \\ & \left. + \frac{1}{|2-a|\Gamma(\alpha+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)} \right] \\ = & \ \psi(\|x\|)\|p\|\Lambda. \end{aligned}$$

Consequently

$$\|h\| \leq \psi(r)\|p\|\Lambda.$$

Now we show that Ω_F *maps bounded sets into equicontinuous sets of* $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_r$. For each $h \in \Omega_F(x)$, we obtain

$$\begin{aligned} & |h(t_2) - h(t_1)| \\ & \leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} v(s)\,ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} v(s)\,ds \right| \\ & \quad + \frac{2|b|\Gamma(2-\beta)|t_2-t_1|}{2|\Gamma(2-\beta)-b|} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v(s)\,ds \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] p(s)\psi(r)\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} p(s)\psi(r)\,ds \\ & \quad + \frac{2|b|\Gamma(2-\beta)|t_2-t_1|}{2|\Gamma(2-\beta)-b|} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} p(s)\psi(r)\,ds \\ & \leq \frac{\|p\|\psi(r)}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha) + \frac{2\|p\|\psi(r)|b|\Gamma(2-\beta)|t_2-t_1|}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)}. \end{aligned}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \to 0$. As Ω_F satisfies the above three assumptions, therefore it follows by the Ascoli–Arzelà theorem that $\Omega_F \colon C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

In our next step, we show that Ω_F is upper semicontinuous. It is known [16, Proposition 1.2] that Ω_F will be upper semicontinuous if we prove that it has a closed graph, since Ω_F is already shown to be completely continuous. Thus we will prove that Ω_F *has a closed graph.* Let $x_n \to x_*, h_n \in \Omega_F(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega_F(x_*)$. Associated with $h_n \in \Omega_F(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$h_n(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)\, ds + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v_n(s)\, ds$$
$$+ \frac{1}{2-a}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)\, ds - \frac{a}{2-a}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} v_n(s)\, ds.$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$h_*(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)\, ds + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v_*(s)\, ds$$
$$+ \frac{1}{2-a}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)\, ds - \frac{a}{2-a}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} v_*(s)\, ds.$$

Let us consider the linear operator $\Theta\colon L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$f \mapsto \Theta(v)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\, ds + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v(s)\, ds$$
$$+ \frac{1}{2-a}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\, ds - \frac{a}{2-a}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} v(s)\, ds.$$

Observe that

$$\|h_n(t) - h_*(t)\| = \Bigg\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (v_n(s) - v_*(s))\, ds$$
$$+ \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} (v_n(s) - v_*(s))\, ds$$
$$+ \frac{1}{2-a}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} (v_n(s) - v_*(s))\, ds$$
$$- \frac{a}{2-a}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} (v_n(s) - v_*(s))\, ds \Bigg\| \to 0, \quad \text{as } n \to \infty.$$

Thus, it follows by Lemma 3.3 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s) ds + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v_*(s)\, ds$$
$$+ \frac{1}{2-a}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s) ds - \frac{a}{2-a}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} v_*(s)\, ds,$$

for some $v_* \in S_{F,x_*}$.

Finally, we show there exists an open set $U \subseteq C([0,1],\mathbb{R})$ with $x \notin \Omega_F(x)$ for any $\lambda \in (0,1)$ and all $x \in \partial U$. Let $\lambda \in (0,1)$ and $x \in \lambda\Omega_F(x)$. Then there exists $v \in L^1([0,1],\mathbb{R})$ with $v \in S_{F,x}$

such that, for $t \in [0,1]$, we have

$$x(t) = \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds + \lambda \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v(s)\,ds$$
$$+ \lambda \frac{1}{2-a} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds - \lambda \frac{a}{2-a} \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} v(s)\,ds.$$

Using the computations of the second step above we have

$$\|x\| \le \psi(\|x\|)\|p\|\Lambda,$$

which implies that

$$\frac{\|x\|}{\psi(\|x\|)\|p\|\Lambda} \le 1.$$

In view of (H_3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\| < M\}.$$

Note that the operator $\Omega_F \colon \overline{U} \to \mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda\Omega_F(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type (Lemma 3.2), we deduce that Ω_F has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1)–(1.2). This completes the proof. □

3.2 The Lipschitz case

Now we prove the existence of solutions for the problem (1.1)–(1.2) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler.

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d\colon \mathcal{P}(X)\times\mathcal{P}(X) \to \mathbb{R}\cup\{\infty\}$ given by

$$H_d(A,B) = \max\left\{ \sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b) \right\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{b,cl}(X),H_d)$ is a metric space and $(\mathcal{P}_{cl}(X),H_d)$ is a generalized metric space (see [24]).

Definition 3.5. A multivalued operator $N\colon X \to \mathcal{P}_{cl}(X)$ is called:

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x),N(y)) \le \gamma d(x,y) \quad \text{for each } x,y \in X;$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 3.6 ([15])**.** *Let (X,d) be a complete metric space. If $N\colon X \to \mathcal{P}_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

Theorem 3.7. *Assume that:*

(H_4) *$F\colon [0,1]\times\mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot,x) : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;*

(H_5) *$H_d(F(t,x),F(t,\bar{x})) \le m(t)|x-\bar{x}|$ for almost all $t \in [0,1]$ and $x,\bar{x} \in \mathbb{R}$ with $m \in C([0,1],\mathbb{R}^+)$ and $d(0,F(t,0)) \le m(t)$ for almost all $t \in [0,1]$.*

Then the boundary value problem (1.1)–(1.2) has at least one solution on $[0,1]$ if $\|m\|\Lambda < 1$, i.e.

$$\|m\|\left\{\frac{1+|2-a|}{|2-a|\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)}\right\} < 1.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0,1],\mathbb{R})$ by the assumption (H_4), so F has a measurable selection (see [10, Theorem III.6]). Now we show that the operator Ω_F, defined in the beginning of proof of Theorem 3.4, satisfies the assumptions of Lemma 3.6. To show that $\Omega_F(x) \in \mathcal{P}_{cl}((C[0,1],\mathbb{R}))$ for each $x \in C([0,1],\mathbb{R})$, let $\{u_n\}_{n\ge 0} \in \Omega_F(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0,1],\mathbb{R})$. Then $u \in C([0,1],\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$,

$$\begin{aligned} u_n(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)\,ds + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v_n(s)\,ds \\ & + \frac{1}{2-a}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)\,ds - \frac{a}{2-a}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} v_n(s)\,ds. \end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,1]$, we have

$$\begin{aligned} v_n(t) \to v(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v(s)\,ds \\ & + \frac{1}{2-a}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds - \frac{a}{2-a}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} v(s)\,ds. \end{aligned}$$

Hence, $u \in \Omega(x)$.

Next we show that there exists $\delta < 1$ such that

$$H_d(\Omega_F(x),\Omega_F(\bar{x})) \le \delta\|x-\bar{x}\| \quad \text{for each } x,\bar{x} \in C^2([0,1],\mathbb{R}).$$

Let $x,\bar{x} \in C^2([0,1],\mathbb{R})$ and $h_1 \in \Omega_F(x)$. Then there exists $v_1(t) \in F(t,x(t))$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)\,ds + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v_1(s)\,ds \\ & + \frac{1}{2-a}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)\,ds - \frac{a}{2-a}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} v_1(s)\,ds. \end{aligned}$$

By (H_5), we have

$$H_d(F(t,x),F(t,\bar{x})) \le m(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w \in F(t,\bar{x}(t))$ such that

$$|v_1(t)-w| \le m(t)|x(t)-\bar{x}(t)|, \qquad t \in [0,1].$$

Define $U\colon [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t)-w| \le m(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable [10, Proposition III.4], there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \leq m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0,1]$, let us define

$$h_2(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)\, ds + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v_2(s)\, ds + \frac{1}{2-a} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)\, ds - \frac{a}{2-a} \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} v_2(s)\, ds.$$

Thus,

$$\begin{aligned} &|h_1(t) - h_2(t)| \\ &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |v_1(s) - v_2(s)|\, ds \\ &\quad + \frac{|b(2t-1)|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |v_1(s) - v_2(s)|\, ds \\ &\quad + \frac{1}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |v_1(s) - v_2(s)| ds + \frac{|a|}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} |v_1(s) - v_2(s)|\, ds \\ &\leq \|m\| \left[\frac{1+|2-a|}{|2-a|\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\beta)}{2|\Gamma(2-\beta)-b|\Gamma(\alpha-\beta+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)} \right] \|x - \bar{x}\| \\ &= \|m\|\Lambda\|x - \bar{x}\|. \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \leq \|m\|\Lambda\|x - \bar{x}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(\Omega_F(x), \Omega_F(\bar{x})) \leq \delta\|x - \bar{x}\|,$$

where $\delta = \|m\|\Lambda < 1$. So Ω_F is a contraction. Hence it follows by Lemma 3.6 that Ω_F has a fixed point x which is a solution of (1.1)–(1.2). This completes the proof. □

3.3 The lower semicontinuous case

Here we study the case when F is not necessarily convex valued in the problem (1.1)–(1.2). We apply the nonlinear alternative of Leray–Schauder type and a selection theorem by Bressan and Colombo for lower semi-continuous maps with decomposable values to establish this result. Let us begin with some preliminary concepts.

Let X be a nonempty closed subset of a Banach space E and $G\colon X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,1], \mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 3.8. Let Y be a separable metric space and let $N\colon Y \to \mathcal{P}(L^1([0,1], \mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F\colon [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,1] \times \mathbb{R}) \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,1]\},$$

which is called the Nemytskii operator associated with F.

Definition 3.9. Let $F\colon [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Lemma 3.10 ([18]). *Let Y be a separable metric space and let $N\colon Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g\colon Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 3.11. *Assume that (H_2), (H_3) and the following condition holds:*

(H_6) *$F\colon [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*

- (a) *$(t,x) \longmapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,*
- (b) *$x \longmapsto F(t,x)$ is lower semicontinuous for each $t \in [0,1]$.*

Then the boundary value problem (1.1)–(1.2) has at least one solution on $[0,1]$.

Proof. It follows from (H_2) and (H_6) that F is of l.s.c. type. Then from Lemma 3.10, there exists a continuous function $f\colon C^2([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$\begin{cases} D^\alpha x(t) = f(x(t)), \quad 0 < t < 1, \quad 1 < \alpha \le 2, \\ x(0) + x(1) = a\displaystyle\int_0^1 x(s)\,ds, \quad x'(0) = bD^\beta x(1), \quad 0 < \beta \le 1. \end{cases} \tag{3.2}$$

Observe that if $x \in C^2([0,1],\mathbb{R})$ is a solution of (3.2), then x is a solution to the problem (1.1)–(1.2). In order to transform the problem (3.2) into a fixed point problem, we define the operator $\overline{\Omega}_F$ as

$$\begin{aligned} \overline{\Omega}_F x(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(x(s))\,ds + \frac{b(2t-1)\Gamma(2-\beta)}{2(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} f(x(s))\,ds \\ &+ \frac{1}{2-a}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(x(s))\,ds - \frac{a}{2-a}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} f(x(s))\,ds. \end{aligned}$$

It can easily be shown that $\overline{\Omega}_F$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.4. So we omit it. This completes the proof. □

3.4 Examples

Consider the problem

$$\begin{cases} D^{3/2} x(t) \in F(t,x(t)), \quad 0 < t < 1, \\ x(0) + x(1) = 4\displaystyle\int_0^1 x(s)\,ds, \quad x'(0) = \dfrac{1}{2}\, {}^c D^{1/2} x(1). \end{cases} \tag{3.3}$$

Here $\alpha = 3/2, \beta = 1/2,\ a = 1/10,\ b = 1/6$. With the given values, we find that

$$\Lambda = 0.865264.$$

(i) Let $F\colon [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map given by

$$x \to F(t,x) = \left[|x| + \frac{|x|^5}{|x|^5+3} + t^3 + t^2 + 4,\ \frac{|x|^3}{|x|^3+1} + t + 2 \right]. \tag{3.4}$$

For $f \in F$, we have

$$|f| \le \max\left(\frac{|x|^5}{|x|^5+3} + t^3 + t^2 + 4,\ \frac{|x|^3}{|x|^3+1} + t + 2 \right) \le 7 + \|x\|, \qquad x \in \mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le 7 = p(t)\psi(\|x\|), \qquad x \in \mathbb{R}$$

with $p(t) = 1,\ \psi(\|x\|) = 7 + \|x\|$. By the condition

$$\frac{M}{\psi(M)\|p\|\Lambda} > 1,$$

we find that $M > 44.95345$. Hence by Theorem 3.4, the problem (3.3) with F given by (3.4) has a solution on $[0,1]$.

(ii) Consider the multivalued map $F\colon [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ given by

$$F(t,x) = \left[0, \frac{1}{12}(t+1)\sin x + \frac{x}{\sqrt{t+36}} + \frac{1}{6} \right]. \tag{3.5}$$

Clearly

$$\sup\{|v| : v \in F(t,x)\} \le \frac{1}{12}(t+1)|\sin x| + \frac{|x|}{\sqrt{t+36}} + \frac{1}{6}$$

and

$$H_d(F(t,x), F(t,\bar{x}) \le \left(\frac{1}{12}(t+1) + \frac{1}{\sqrt{t+36}} \right) |x - \bar{x}|.$$

Let $m(t) = \frac{1}{12}(t+1) + \frac{1}{\sqrt{t+36}}$. Then $\|m\| = \frac{1}{3}$ and $\|m\|\Lambda \approx 0.288421 < 1$. Hence by Theorem 3.7, the problem (3.3) with F given by (3.5) has a solution.

3.5 Extension to Riemann–Stieltjes integral conditions case

The concept of Riemann–Stieltjes integral conditions is quite old, see the reviews by Whyburn [33] and Conti [13]. It provides a unified approach for dealing with a variety of boundary conditions such as multipoint and integral boundary conditions. For some recent works involving Riemann–Stieltjes integral conditions, we refer the reader to the papers [5, 30, 32, 34] and the references cited therein.

Let us now consider fractional differential inclusion (1.1) supplemented with the boundary data involving Riemann–Stieltjes integral condition given by

$$x(0) + x(1) = a \int_0^1 x(s)\, d\mu(s), \qquad x'(0) = b\, {}^cD^{\beta} x(1), \qquad 0 < \beta \le 1, \tag{3.6}$$

where μ is a function of bounded variation. In this case, the solution $x(t)$ given by Lemma 2.4 takes the form:

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + \left(t - \frac{\nu_2}{\nu_1}\right) \frac{b\Gamma(2-\beta)}{(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s)\,ds \\ & - \frac{1}{\nu_1} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + \frac{a}{\nu_1} \int_0^1 \int_0^s \left(\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,du \right) d\mu(s), \end{aligned} \tag{3.7}$$

where

$$\nu_1 = 2 - a \int_0^1 d\mu(s) \neq 0, \qquad \nu_2 = 1 - a \int_0^1 s\,d\mu(s).$$

Observe that the solution (3.7) reduces to (2.2) by taking $\mu(t) = t$

In this case, the operator $\Theta_F \colon C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ (analogue to Ω_F) takes the form:

$$\Theta_F(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds - \frac{1}{\nu_1} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds \\ \displaystyle + \left(t - \frac{\nu_2}{\nu_1}\right) \frac{b\Gamma(2-\beta)}{(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v(s)\,ds \\ \displaystyle + \frac{a}{\nu_1} \int_0^1 \int_0^s \left(\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,du \right) d\mu(s), \qquad v \in S_{F,x}, \end{array} \right. \end{array} \right\}$$

and the constant analogue to Λ given by (3.1) is

$$\Delta = \frac{1 + |\nu_1|}{|\nu_1| \Gamma(\alpha+1)} + \frac{|b| \omega \Gamma(2-\beta)}{|\Gamma(2-\beta) - b| \Gamma(\alpha-\beta+1)} + \left| \frac{a}{\nu_1} \right| \int_0^1 \frac{s^\alpha}{\Gamma(\alpha+1)}\, d\mu(s),$$

where $\omega = \max_{t \in [0,1]} \left| t - \frac{\nu_2}{\nu_1} \right|$.

Using the operator Θ_F and the constant Δ, we can prove the existence results for the problem (1.1)–(3.6) as we have done for the problem (1.1)–(1.2).

4 Existence results for the boundary value problem (1.1)–(1.3)

Lemma 4.1. *Let $2\Gamma(\gamma+1) - a\eta^\gamma \neq 0$ and $\Gamma(2-\beta) \neq b$. Let $y \in C([0,1],\mathbb{R})$ and $x \in C^2([0,1],\mathbb{R})$ be the solution of the linear boundary value problem*

$$\begin{cases} D^\alpha x(t) = y(t), \quad 0 < t < 1, \;\; 1 < \alpha \leq 2 \\ x(0) + x(1) = aI^\gamma x(\eta), \quad x'(0) = bD^\beta x(1), \;\; 0 < \beta, \gamma \leq 1, \;\; 0 < \eta < 1. \end{cases} \tag{4.1}$$

Then

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds \\ & + \frac{\Gamma(\gamma+1)}{(2\Gamma(\gamma+1) - a\eta^\gamma)} \left[a \int_0^\eta \frac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} y(s)\,ds - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds \right] \\ & + \left(t - \frac{[\Gamma(\gamma+2) - a\eta^{\gamma+1}]}{(\gamma+1)(2\Gamma(\gamma+1) - a\eta^\gamma)} \right) \frac{b\Gamma(2-\beta)}{(\Gamma(2-\beta)-b)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s)\,ds. \end{aligned} \tag{4.2}$$

Proof. We omit the proof as it is similar to that of Lemma 2.4. □

In relation to problem (1.1)–(1.3), we define $G_F\colon C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ as

$$G_F(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ \displaystyle + \frac{\Gamma(\gamma+1)}{(2\Gamma(\gamma+1) - a\eta^\gamma)} \\ \displaystyle \times \left[a \int_0^\eta \frac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} v(s)ds - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \right] \\ \displaystyle + \left(t - \frac{[\Gamma(\gamma+2) - a\eta^{\gamma+1}]}{(\gamma+1)(2\Gamma(\gamma+1) - a\eta^\gamma)} \right) \frac{b\Gamma(2-\beta)}{(\Gamma(2-\beta) - b)} \\ \displaystyle \times \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} v(s)ds, \end{array} \right. \end{array} \right\}$$

for $v \in S_{F,x}$, and set

$$\begin{aligned} \bar{\Lambda} = {} & \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(\gamma+1)}{|2\Gamma(\gamma+1) - a\eta^\gamma|} \left[\frac{a\eta^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + \frac{1}{\Gamma(\alpha+1)} \right] \\ & + \left(1 + \left| \frac{[\Gamma(\gamma+2) - a\eta^{\gamma+1}]}{(\gamma+1)(2\Gamma(\gamma+1) - a\eta^\gamma)} \right| \right) \frac{b\Gamma(2-\beta)}{|\Gamma(2-\beta) - b|\Gamma(\alpha-\beta+1)}. \end{aligned} \tag{4.3}$$

With the above operator and the estimate (4.3), we can reproduce all the existence results obtained in Section 3 for the boundary value problem (1.1)–(1.3).

5 Conclusions

We have established some existence results for the inclusion problems with non-separated local and nonlocal integral-flux boundary conditions. Our results in the given configuration yield many known and new results for different values of the parameters involved in the problems at hand and are listed below.

(i) The existence results for anti-periodic fractional inclusion problems [8] follow in the limit $\beta \to 1$ by fixing $a = 0,\ b = -1$.

(ii) Letting $a = 0,\ b = -1$ in the present results, we obtain the new results for fractional differential inclusions with new anti-periodic type boundary conditions: $x(0) = -x(1),\ x'(0) = -{}^cD^\beta x(1)$.

(iii) Fixing $a = 0,\ b > 0$, the results of this paper correspond to a fractional boundary value problem with anti-periodic boundary condition $x(0) = -x(1)$ and a periodic like condition $x'(0) = b\ {}^cD^\beta x(1)$. Such conditions can be regarded as fractional analogue of source type flux conditions $x'(0) = bx'(1)$ occurring in thermodynamic phenomena. On the other hand, for $a = 0,\ b < 0$, our results correspond to a fractional inclusion boundary value problem with sink type flux conditions. Clearly the choice $b = 0$ gives us the results associated with anti-periodic boundary condition $x(0) = -x(1)$ and zero flux condition $x'(0) = 0$.

(iv) The nonlocal Riemann–Liouville integral boundary condition in (1.3) is a generalization of the classical integral condition considered in (1.2) in the sense that it makes the length of the interval flexible from $(0,1)$ to $(0,\eta)$, $0<\eta<1$ and modifies the integrand $x(s)$ by $(\eta-s)^{\gamma-1}x(s)/\Gamma(\gamma)$. Thus the problem (1.1)–(1.3) is a generalization of the problem (1.1)–(1.2).

Acknowledgements

The authors thank the editor and the reviewers for their constructive remarks that led to the improvement of the original manuscript. In particular, we appreciate the editor for pointing out references [13, 33] on Riemann–Stieltjes integral conditions.

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