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Complementary equations: a fractional differential equation and a Volterra integral equation

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Abstract. It is shown that a continuous, absolutely integrable function satisfies the initial value problem

$$D^q x(t) = f(t, x(t)), \qquad \lim_{t \to 0^+} t^{1-q} x(t) = x^0 \qquad (0 < q < 1)$$

on an interval $(0, T]$ if and only if it satisfies the Volterra integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds$$

on this same interval. In contradistinction to established existence theorems for these equations, no Lipschitz condition is imposed on $f(t, x)$. Examples with closed-form solutions illustrate this result.

Keywords: fractional differential equations, Riemann–Liouville operators, singular kernels, Volterra integral equations.

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1 Introduction

This is the first in a series of papers that will deal with the interplay between the scalar fractional differential equation

$$D^q x(t) = f(t, x(t)) \tag{1.1}$$

and the scalar Volterra integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \tag{1.2}$$

where $q \in (0, 1)$, D^q is the Riemann–Liouville fractional differential operator of order q, $x^0 \in \mathbb{R}$ with $x^0 \neq 0$, and for an unbounded interval $I \subseteq \mathbb{R}$ the function $f\colon (0, T] \times I \to \mathbb{R}$ is continuous.

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The purpose of this paper is to relate continuous solutions of the fractional equation (1.1) when it is subject to the initial condition

$$\lim_{t\to 0^+} t^{1-q}x(t) = x^0 \tag{1.3}$$

to those of the integral equation (1.2). In short, the relationship is this: the initial value problem

$$D^q x(t) = f(t, x(t)), \qquad \lim_{t\to 0^+} t^{1-q}x(t) = x^0 \tag{1.4}$$

and the integral equation (1.2) are equivalent in the sense that a continuous, absolutely integrable function $x(t)$ satisfying one of them also satisfies the other provided that $f(t, x(t))$ is absolutely integrable. The precise statement is the content of the following theorem, which is the main result in this paper. Its proof will be the culmination of three theorems in Sections 4–6.

Theorem. Let $q \in (0, 1)$ and $x^0 \neq 0$. Let $f(t, x)$ be a function that is continuous on the set

$$\mathcal{B} := \{(t, x) \in \mathbb{R}^2 : 0 < t \leq T, x \in I\}$$

where $I \subseteq \mathbb{R}$ denotes an unbounded interval. Suppose a function $x\colon (0, T] \to I$ is continuous and that both $x(t)$ and $f(t, x(t))$ are absolutely integrable on $(0, T]$. Then $x(t)$ satisfies the initial value problem (1.4) on the interval $(0, T]$ if and only if it satisfies the Volterra integral equation (1.2) on this same interval.

The significance of this theorem can be seen relative to an existence theorem: imagine a theorem stating that a continuous, absolutely integrable function $x(t)$ exists satisfying (1.2) on an interval $(0, T]$ if $f(t, x)$ belongs to a set of functions with certain desired properties and that $f(t, x(t))$ is itself continuous and absolutely integrable on $(0, T]$. Then, according to the above theorem, the function $x(t)$ will also satisfy the initial value problem (1.4). In fact, this paper lays the groundwork for such an existence theorem, which will be introduced and discussed in future papers.

An important aspect of the theorem, especially in light of the existence theorem just described, is that unlike other results of this kind, such as in [4, 6, 8], no Lipschitz condition is imposed on the function $f(t, x)$. It was the study of typical inversion theorems (e.g., such as in [4, p. 78]) for transforming (1.4) into (1.2) that prompted this paper. Such results impose not only a Lipschitz condition on $f(t, x)$ but also ask that it be bounded in a certain region, a condition which is troubling in view of the obvious unboundedness of x in (1.2).

The condition asking that f be bounded for unbounded x has a history which is detailed in part in [5, pp. 136–7]. For a long time that boundedness condition was required in existence theory both with and without a Lipschitz condition. (See Lemma 5.3 of Diethelm [4, p. 80], Theorems 2.4.1 and 2.5.1 of Lakshmikantham et al. [6, pp. 30, 34], and Theorem 3.4 of Podlubny [8, p. 127].)

Kilbas et al. prove an existence result in [5, Thm. 3.11, p. 165] requiring a Lipschitz condition but without asking that f be bounded for x unbounded. The main theorem of this paper and the existence theorem of which we spoke earlier will extend such a result by dropping both the Lipschitz condition and the boundedness of f requirement.

2 Notation

First some words about notation and terminology: $\mathbb{R}^+$ denotes the set of all strictly positive real numbers. For $T > 0$, $C[0, T]$ denotes the set of all continuous functions on $[0, T]$. $L^1[0, T]$ denotes the set of all measurable functions f on $[0, T]$ for which $|f|$ is Lebesgue integrable on $[0, T]$. However, generally speaking, we use Riemann integrals (both proper and improper) since most of the functions dealt with in this paper are either continuous on the closed interval $[0, T]$ or on the half-open interval $(0, T]$. Consequently, $\int_0^T f(t)\, dt$ usually refers to a proper or improper Riemann integral. We use the phrase "f is absolutely integrable on the interval $(0, T]$" to convey that $\int_0^T |f(t)|\, dt$ is an improper Riemann integral (unless f is defined and bounded on $[0, T]$) and that it converges. That is, f is *absolutely integrable* on $(0, T]$ if f is Riemann integrable on every closed interval $[\eta, T]$, where $\eta \in (0, T]$, and $\lim_{\eta\to 0^+} \int_\eta^T |f(t)|\, dt$ exists and is finite, in which case $\int_0^T |f(t)|\, dt$ is defined to be

$$\int_0^T |f(t)|\, dt := \lim_{\eta\to 0^+} \int_\eta^T |f(t)|\, dt.$$

The following proposition relating improper Riemann integrals and their Lebesgue counterparts will aid in completing the proofs of some of the results in subsequent sections.

Proposition 2.1. *Let f be a function that is defined on the half-open interval $(0, T]$, and let f have a singularity at $t = 0$.*

(i) *If f is absolutely integrable on $(0, T]$, then $f \in L^1[0, T]$.*

(ii) *If $f \in L^1[0, T]$ is continuous on $(0, T]$, then f is absolutely integrable on $(0, T]$.*

In both (i) *and* (ii)*, the improper Riemann integral of f on $(0, T]$ is equal to the Lebesgue integral of f on $(0, T]$. Also, the Lebesgue and improper Riemann integrals of $|f|$ are equal.*

Part (i) follows from an adaptation of Theorem 10.33 in [2, p. 276] for integrals on unbounded intervals to integrals of unbounded functions on a finite interval. Part (ii) follows from a similar adaptation of Theorem 10.31 in [2, p. 274]. Details are left to the reader.

3 Initial conditions

In such works as [4, p. 77] and [5, p. 137], we find the initial condition

$$\lim_{t\to 0^+} \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} x(s)\, ds = b$$

associated with the fractional equation (1.1). But in place of $b \in \mathbb{R}$, we prefer to write $x^0\Gamma(q)$. Then the initial condition becomes

$$\lim_{t\to 0^+} \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} x(s)\, ds = x^0\Gamma(q). \tag{3.1}$$

In point of fact, for continuous, absolutely integrable functions x, it is equivalent to the initial condition (1.3) (cf. Thm. 6.1). This type of initial condition is not only of mathematical interest but is also important in physical applications. For example, the classical tautochrone problem can be modeled by a fractional differential equation of the form

$$D^q x(t) = g(t)$$

with $q = 1/2$ and subject to an initial condition of the form (3.1).

4 Inversion of the fractional differential equation

The main result in this section is Theorem 4.10. It states that a continuous function $x\colon (0,T] \to I$, where I represents some interval, that satisfies the fractional differential equation (1.1) on $(0,T]$ and the initial condition (3.1) will also satisfy the integral equation (1.2), provided some conditions of absolute integrability are met. What is noteworthy is that we ask f to be neither Lipschitz nor bounded; this result is to be compared with [4, p. 78], which in part motivated this paper. Theorem 4.10 is preceded by Lemmas 4.1, 4.3, 4.5, 4.6, 4.8, 4.9 upon which its proof rests. Lemma 4.8 states an important property of the fractional integral operator J^n, which is defined next.

For $n \in \mathbb{R}^+$, let J^n denote the *Riemann–Liouville fractional integral operator of order n*, which for $h \in L^1[0,T]$ is defined by (cf. [4, p. 13])

$$\mathrm{J}^n h(t) := \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} h(s)\, ds. \tag{4.1}$$

Let $\mathrm{J} := \mathrm{J}^1$.

Let D^q denote the *Riemann–Liouville fractional differential operator of order q*. For $q \in (0,1)$ and $h \in L^1[0,T]$, it is defined by (cf. [4, p. 27])

$$D^q h := D\mathrm{J}^{1-q} h, \tag{4.2}$$

where $D := d/dt$. Thus, if (4.2) exists at a given $t \in [0,T]$, its value is given by

$$D^q h(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} h(s)\, ds$$

where $\Gamma\colon (0,\infty) \to \mathbb{R}$ is *Euler's Gamma function*, namely,

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt.$$

It readily follows from this that D^q is a linear operator. That is, if for a pair of functions h_1, h_2, the fractional derivatives $D^q h_1(t)$ and $D^q h_2(t)$ exist at a given t, then

$$D^q(c_1 h_1 + c_2 h_2)(t) = c_1 D^q h_1(t) + c_2 D^q h_2(t) \tag{4.3}$$

for $c_1, c_2 \in \mathbb{R}$.

Lemma 4.1. *Let $k \in \mathbb{R}$ and $q \in (0,1)$. For $t \geq 0$,*

$$\mathrm{J}^q k = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} k\, ds = \frac{k}{\Gamma(q+1)} t^q.$$

For $t > 0$,

$$\mathrm{J}^{1-q} t^{q-1} = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} s^{q-1}\, ds = \Gamma(q).$$

Proof. Since

$$\int_0^t (t-s)^{q-1}\, ds = \frac{t^q}{q},$$

we have

$$\mathrm{J}^q k = \frac{k}{\Gamma(q)} \cdot \frac{t^q}{q} = \frac{k}{\Gamma(q+1)} t^q,$$

which is the first result of this lemma.

The second result can be derived from the *Beta function*, namely, the function $B(p,q)$ that is defined by

$$B(p,q) := \int_0^1 v^{p-1}(1-v)^{q-1}\,dv.$$

$B(p,q)$ converges if and only if both p and q are positive. Using the change of variable $s = tv$, where $t > 0$, and the well-known formula

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

we obtain

$$\begin{aligned}\int_0^t (t-s)^{-q}s^{q-1}\,ds &= \int_0^1 (t-tv)^{-q}(tv)^{q-1}t\,dv = \int_0^1 (1-v)^{-q}v^{q-1}\,dv \\ &= \int_0^1 v^{q-1}(1-v)^{(1-q)-1}\,dv = B(q,1-q) \\ &= \frac{\Gamma(q)\Gamma(1-q)}{\Gamma(q+1-q)} = \Gamma(q)\Gamma(1-q).\end{aligned}$$

The second result follows from this equation. □

Remark 4.2. Let $p,q > 0$. With the same change of variable, i.e. $s = tv$, the integration formula

$$\int_0^t (t-s)^{p-1}s^{q-1}\,ds = t^{p+q-1}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad (t>0) \tag{4.4}$$

can also be derived from the Beta function.

Lemma 4.3. *Let φ be a continuous function on the compact interval $[a,b]$ and $n \in \mathbb{R}^+$. Then, for each $t \in [a,b]$, the Riemann integral*

$$H_a(t) := \int_a^t (t-s)^{n-1}\varphi(s)\,ds \tag{4.5}$$

converges absolutely. Furthermore, H_a is continuous on $[a,b]$.

Remark 4.4. Note that the integral in (4.5) is improper when $0 < n < 1$. It is convenient here to say H_a is absolutely convergent when $n \geq 1$ even though the integral is not improper.

Proof. It suffices to show that

$$H(t) := \int_0^t (t-s)^{n-1}\varphi(s)\,ds \tag{4.6}$$

is continuous on an interval $[0,T]$ for any given $\varphi \in C[0,T]$ since with an appropriate change of variable we can translate $[a,b]$ to $[0,T]$, where $T = b-a$, and at the same time change the form of (4.5) to (4.6). Thus we begin with (4.6) but rewritten as

$$H(t) = \int_0^t s^{n-1}\varphi(t-s)\,ds, \tag{4.7}$$

where $\varphi \in C[0,T]$.

First let us view H as a Lebesgue integral. Define the function $h\colon [0,T]\times[0,T]\to\mathbb{R}$ by

$$h(t,s) := \begin{cases} s^{n-1}\varphi(t-s), & \text{if } \ (t,s)\in\Omega_T \\ 0, & \text{if } \ (t,s)\notin\Omega_T \end{cases}$$

where $\Omega_T := \{(t,s) \mid 0 < s \le t \le T\}$. Then, for each fixed $t\in[0,T]$, define the function $h_t\colon [0,T]\to\mathbb{R}$ by

$$h_t(s) := h(t,s).$$

Observe that

$$h_t(s) = f(s)g_t(s),$$

where

$$f(s) := \begin{cases} s^{n-1}, & \text{if } \ 0 < s \le T \\ 0, & \text{if } \ s = 0 \end{cases}$$

and

$$g_t(s) := \begin{cases} \varphi(t-s), & \text{if } \ 0 \le s \le t \\ 0, & \text{if } \ t < s \le T. \end{cases}$$

If $n\in(0,1)$, the improper Riemann integral $\int_0^T s^{n-1}\,ds$ converges; so, by Proposition 2.1, $s^{n-1}\in L^1[0,T]$. Apart from $s=0$, the functions $f(s)$ and s^{n-1} are equal. Thus, $f\in L^1[0,T]$. Clearly $f\in L^1[0,T]$ if $n=1$. If $n>1$, observe that $f(s)=s^{n-1}$ for $0\le s\le T$, a proper Riemann integrable function. So once again $f\in L^1[0,T]$. In sum, for all $n>0$, $f\in L^1[0,T]$ and, a fortiori, measurable on $[0,T]$.

As for the function g_t, it is defined and bounded on $[0,T]$ and continuous everywhere except at $s=t$ unless $\varphi(0)=0$. Thus, by Lebesgue's criterion for integrability, g_t is Riemann integrable on $[0,T]$. Hence, $g_t\in L^1[0,T]$. So it too is measurable on $[0,T]$.

Note the function $h(t,s)$ has the following properties:

(a) For each fixed $t\in[0,T]$, the function $h_t(s)$, being the product of the measurable functions $f(s)$ and $g_t(s)$, is measurable on $[0,T]$.

(b) Since $\varphi(s)$ is continuous on $[0,T]$, there is a constant $M>0$ such that $|g_t(s)|\le M$ for $0\le s\le T$. Thus, for each $t\in[0,T]$,

$$|h(t,s)| = |h_t(s)| = |f(s)||g_t(s)| \le Mf(s)$$

on $[0,T]$, where $Mf\in L^1[0,T]$.

(c) For each fixed $t\in[0,T]$, it is clear that

$$\lim_{u\to t} h(u,s) = h(t,s)$$

for almost all $s\in[0,T]$. That is, depending on how φ is defined, this may or may not be the case at $s=t$.

Then, because h has these properties, we can invoke a theorem for integrals whose integrands depend on a parameter, such as the theorem in [2, p. 281]. For this situation, it states that the Lebesgue integral $\int_0^T h(t,s)\,ds$ exists for each $t \in [0,T]$ and is a continuous function of t on $[0,T]$. Since $h(t,s) = 0$ for $t < s \le T$, it is equal to $H(t)$ as

$$\int_0^T h(t,s)\,ds = \int_0^t h(t,s)\,ds + \int_t^T h(t,s)\,ds = \int_0^t s^{n-1}\varphi(t-s)\,ds.$$

Hence H is continuous on $[0,T]$.

Finally, for $0 < n < 1$, let us show for each $t \in (0,T]$ that the value of the Lebesgue integral $H(t)$ is equal to the improper Riemann integral of $s^{n-1}\varphi(t-s)$ on $(0,t]$. For such a fixed t, we see from the definition of $h(t,s)$ that

$$h_t(s) = s^{n-1}\varphi(t-s) \quad \text{for } 0 < s \le t.$$

Since

$$\int_0^t h_t(s)\,ds = \int_0^t s^{n-1}\varphi(t-s)\,ds = H(t) < \infty,$$

$h_t(s)$ is Lebesgue integrable on $[0,t]$. Consequently, so is $|h_t(s)|$. From (a) we see that $h_t(s)$ is measurable on $[0,t]$. Hence, $h_t \in L^1[0,t]$. Thus, as $h_t(s)$ is continuous on $(0,t]$, it follows from Proposition 2.1 that it is absolutely integrable on $(0,t]$ and that the improper Riemann integral of $h_t(s)$ on $(0,t]$ is equal to the Lebesgue integral of $h_t(s)$. □

If we replace $(t-s)^{n-1}$ in (4.5) with $\psi(t-s)$ where $\psi \in L^1[0,T]$, we obtain the following generalization of Lemma 4.3.

Lemma 4.5. *Let $\varphi \in C[a,b]$ and $\psi \in L^1[a,b]$. Then the Lebesgue integral*

$$\mathcal{H}_a(t) := \int_a^t \psi(t-s)\varphi(s)\,ds \tag{4.8}$$

defines a function that is continuous on $[a,b]$.

Proof. The proof is the same as that of Lemma 4.3, aside from some minor details, if in that proof s^{n-1} is replaced with $\psi(t-s)$. □

In Lemma 4.3 the integrand of the integral (4.6) has a singularity at the upper limit of integration if $n \in (0,1)$. In the next lemma, we add to that a singularity at the lower limit of integration by supposing that the function $\varphi(s)$ has a singularity at $s = 0$.

Lemma 4.6. *Let $n \in \mathbb{R}^+$. If a function φ is continuous and absolutely integrable on $(0,T]$, then the improper Riemann integral*

$$H(t) = \int_0^t (t-s)^{n-1}\varphi(s)\,ds \tag{4.9}$$

defines a function that is also continuous and absolutely integrable on $(0,T]$.

Remark 4.7. Whether or not H is continuous at $t = 0$ depends on the particular function φ. For example, let $\varphi(s) = s^{m-1}$ with $m > 0$. Then from (4.4) we have

$$H(t) = t^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

for $t > 0$. Thus, as $H(0) = 0$, H is continuous at $t = 0$ if $m + n > 1$ but discontinuous if $m + n \le 1$.

Proof. Choose an arbitrary $a \in (0, T)$. Then, for each fixed $t \in (a, T]$, we can rewrite $H(t)$ as

$$H(t) = \mathcal{I}(t) + H_a(t),$$

where H_a is defined by (4.5) and

$$\mathcal{I}(t) := \int_0^a (t-s)^{n-1} \varphi(s)\, ds.$$

According to Lemma 4.3, the integral $H_a(t)$ converges absolutely. Since $t > a$, $(t-s)^{n-1}$ is continuous for $0 \le s \le a$, hence, bounded. Thus, as $\varphi(s)$ is absolutely integrable on $(0, a]$, so is $(t-s)^{n-1}\varphi(s)$. In other words, the integral $\mathcal{I}(t)$ also converges absolutely. Consequently, $H(t)$ converges absolutely. Therefore, H defines a function on $(a, T]$.

As to continuity, we have already established in Lemma 4.3 that $H_a(t)$ is continuous on $[a, T]$. So let us now show that $\mathcal{I}(t)$ is continuous on $(a, T]$. To that end, choose any $t_1 \in (a, T]$. Then for $t \in (a, T]$, we have

$$\begin{aligned} |\mathcal{I}(t) - \mathcal{I}(t_1)| &= \left| \int_0^a (t-s)^{n-1}\varphi(s)\, ds - \int_0^a (t_1 - s)^{n-1}\varphi(s)\, ds \right| \\ &\le \int_0^a \left|(t-s)^{n-1} - (t_1 - s)^{n-1}\right| \left|\varphi(s)\right| ds \\ &\le \max_{0 \le s \le a} \left|(t-s)^{n-1} - (t_1 - s)^{n-1}\right| \int_0^a \left|\varphi(s)\right| ds. \end{aligned}$$

Consider the three cases: (i) $n \in (0, 1)$, (ii) $n = 1$, and (iii) $n > 1$.

If $0 < n < 1$, then

$$\max_{0 \le s \le a} \left|(t-s)^{n-1} - (t_1 - s)^{n-1}\right| = \left|(t-a)^{n-1} - (t_1 - a)^{n-1}\right|.$$

Consequently,

$$|\mathcal{I}(t) - \mathcal{I}(t_1)| \le M\left|(t-a)^{n-1} - (t_1 - a)^{n-1}\right|,$$

where $M := \int_0^a \left|\varphi(s)\right| ds < \infty$. Since $(t-a)^{n-1} \in C(a, T]$, for each $\epsilon > 0$ there is a $\delta > 0$ such that $|t - t_1| < \delta$ implies that

$$\left|(t-a)^{n-1} - (t_1 - a)^{n-1}\right| < \frac{\epsilon}{M}.$$

As a result,

$$|\mathcal{I}(t) - \mathcal{I}(t_1)| < M \cdot \frac{\epsilon}{M} = \epsilon,$$

which shows that $\mathcal{I}(t)$ is continuous at t_1. Since t_1 is an arbitrary point in $(a, T]$, we conclude $\mathcal{I}(t)$ is continuous on this interval. Consequently, H is continuous on $(a, T]$. In point of fact, H is continuous on the entire interval $(0, T]$ since at the outset of this proof we chose an arbitrary $a \in (0, T]$.

For $n > 1$, a similar analysis again leads to the conclusion that H is continuous on $(a, T]$—and so on $(0, T]$.

If $n = 1$, then $H(t) = \int_0^t \varphi(s)\, ds$. By hypothesis, φ is absolutely integrable on $(0, T]$; so $\varphi \in L^1[0, T]$. Therefore, H is (absolutely) continuous on the closed interval $[0, T]$ (cf. [10, p. 319]).

To prove H is absolutely integrable on $(0, T]$, define the set

$$\Omega_T := \{(t, s) \mid 0 \le s < t \le T\}$$

and rewrite H as

$$H(t) = \int_0^T s^{n-1}\widetilde{\varphi}(t-s)\,ds$$

where

$$\widetilde{\varphi}(t-s) := \begin{cases} \varphi(t-s), & \text{if} \quad (t,s) \in \Omega_T \\ 0, & \text{if} \quad (t,s) \notin \Omega_T. \end{cases}$$

With the aid of results in [3, p. 121] and [10, (6.126), p. 355], one can show that the integrand $s^{n-1}\widetilde{\varphi}(t-s)$ is a measurable function on the rectangle $[0,T] \times [0,T]$.

Now consider the following iterated integral of $|s^{n-1}\widetilde{\varphi}(t-s)|$:

$$\begin{aligned} \int_0^T \int_0^T |s^{n-1}\widetilde{\varphi}(t-s)|\,dt\,ds &= \int_0^T s^{n-1} \left(\int_0^T |\widetilde{\varphi}(t-s)|\,dt \right) ds \\ &= \int_0^T s^{n-1} \left(\int_0^s |\widetilde{\varphi}(t-s)|\,dt + \int_s^T |\widetilde{\varphi}(t-s)|\,dt \right) ds \\ &= \int_0^T s^{n-1} \left(\int_s^T |\varphi(t-s)|\,dt \right) ds. \end{aligned}$$

As φ is absolutely integrable on $(0,T]$,

$$\begin{aligned} \int_0^T \int_0^T |s^{n-1}\widetilde{\varphi}(t-s)|\,dt\,ds &= \int_0^T s^{n-1} \left(\int_0^{T-s} |\varphi(u)|\,du \right) ds \\ &\le \int_0^T s^{n-1} \left(\int_0^T |\varphi(u)|\,du \right) ds = \frac{T^n}{n} \int_0^T |\varphi(u)|\,du < \infty. \end{aligned}$$

The finiteness of this iterated integral implies $s^{n-1}\widetilde{\varphi}(t-s)$ is Lebesgue integrable on $[0,T] \times [0,T]$ and that

$$\int_0^T \int_0^T s^{n-1}\widetilde{\varphi}(t-s)\,dt\,ds = \int_0^T \int_0^T s^{n-1}\widetilde{\varphi}(t-s)\,ds\,dt.$$

(Cf. the Tonelli–Hobson test in [2, p. 415] or [9, p. 93].) As a result, H is Lebesgue integrable on $[0,T]$ as

$$\int_0^T H(t)\,dt = \int_0^T \int_0^T s^{n-1}\widetilde{\varphi}(t-s)\,ds\,dt < \infty.$$

Thus, H is measurable on $[0,T]$ and $|H|$ is Lebesgue integrable on $[0,T]$; so $H \in L^1[0,T]$.

Therefore, as H is continuous on $(0,T]$ and $H \in L^1[0,T]$, it follows from Proposition 2.1 that H is absolutely integrable on $(0,T]$. □

Lemma 4.8. *Let φ be continuous and absolutely integrable on $(0,T]$. Let $m,n \in \mathbb{R}^+$. If $m+n \ge 1$, then*

$$\mathsf{J}^{m+n}\varphi(t) = \frac{1}{\Gamma(m+n)} \int_0^t (t-s)^{m+n-1}\varphi(s)\,ds \tag{4.10}$$

is continuous on the closed interval $[0,T]$. Moreover,

$$\mathsf{J}^{m+n}\varphi(t) = \mathsf{J}^m\,\mathsf{J}^n\varphi(t) \tag{4.11}$$

at each $t \in [0,T]$.

Proof. It follows from Lemma 4.6 that

$$\mathrm{J}^n\varphi(t) = \frac{1}{\Gamma(n)}\int_0^t (t-s)^{n-1}\varphi(s)\,ds$$

and $\mathrm{J}^{m+n}\varphi$ are continuous and absolutely integrable on $(0, T]$. By that same lemma,

$$\mathrm{J}^m\mathrm{J}^n\varphi(t) = \frac{1}{\Gamma(m)}\int_0^t (t-s)^{m-1}\mathrm{J}^n\varphi(s)\,ds \tag{4.12}$$

is also continuous and absolutely integrable on $(0, T]$.

Now suppose $m + n \geq 1$. Then, as $s^{m+n-1} \in C[0, T]$ and $\varphi \in L^1[0, T]$, we see from Lemma 4.5 that $\mathrm{J}^{m+n}\varphi$ is continuous not only on $(0, T]$ but at $t = 0$ as well. This concludes the proof of the first statement.

Observe that (4.11) holds at $t = 0$ since both (4.10) and (4.12) are equal to 0. To prove (4.11) holds on the entire interval, select an arbitrary $t \in (0, T]$. Then, because of (4.4), we can rewrite (4.10) as

$$\begin{aligned}\mathrm{J}^{m+n}\varphi(t) &= \frac{1}{\Gamma(m)\Gamma(n)}\int_0^t (t-u)^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\,\varphi(u)\,du \\ &= \frac{1}{\Gamma(m)\Gamma(n)}\int_0^t \left[\int_0^{t-u}(t-u-v)^{m-1}v^{n-1}\,dv\right]\varphi(u)\,du.\end{aligned}$$

With the change of variable $s = u + v$, this becomes

$$\begin{aligned}\mathrm{J}^{m+n}\varphi(t) &= \frac{1}{\Gamma(m)\Gamma(n)}\int_0^t \left[\int_u^t (t-s)^{m-1}(s-u)^{n-1}\,ds\right]\varphi(u)\,du \\ &= \frac{1}{\Gamma(m)\Gamma(n)}\int_0^t\int_u^t f_t(s,u)\,ds\,du,\end{aligned}$$

where $f_t(s,u) := (t-s)^{m-1}(s-u)^{n-1}\varphi(u)$.

With the intent of justifying interchanging the order of integration, let us first rewrite this as

$$\mathrm{J}^{m+n}\varphi(t) = \frac{1}{\Gamma(m)\Gamma(n)}\int_0^t\int_0^t F_t(s,u)\,ds\,du,$$

where $F_t\colon [0,t]\times[0,t] \to \mathbb{R}$ is the function defined by

$$F_t(s,u) := \begin{cases} f_t(s,u), & \text{if} \quad (s,u) \in \Omega_t \\ 0, & \text{if} \quad (s,u) \notin \Omega_t \end{cases}$$

and $\Omega_t \subset [0,t]\times[0,t]$ is defined by

$$\Omega_t := \{(s,u) : 0 < u < s < t\}.$$

It can be shown that F_t is measurable on $[0,t]\times[0,t]$ using results from [3, p. 121] and [10, (6.126), p. 355].

It once again follows from Lemma 4.5 that

$$\int_0^t (t-u)^{m+n-1}|\varphi(u)|\,du < \infty$$

since $u^{m+n-1} \in C[0,T]$ and $|\varphi| \in L^1[0,T]$. From the previous work, we see that this can also be written as

$$\int_0^t \int_0^t |F_t(s,u)|\, ds\, du < \infty.$$

As a result, the Tonelli–Hobson test justifies the following interchange in the order of integration:

$$\begin{aligned}
\mathsf{J}^{m+n}\varphi(t) &= \frac{1}{\Gamma(m)\Gamma(n)} \int_0^t \int_0^t F_t(s,u)\, ds\, du \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \int_0^t \int_0^t F_t(s,u)\, du\, ds = \frac{1}{\Gamma(m)\Gamma(n)} \int_0^t \int_0^s f_t(s,u)\, du\, ds \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \int_0^t \int_0^s (t-s)^{m-1}(s-u)^{n-1}\varphi(u)\, du\, ds \\
&= \frac{1}{\Gamma(m)} \int_0^t (t-s)^{m-1} \left(\frac{1}{\Gamma(n)} \int_0^s (s-u)^{n-1}\varphi(u)\, du \right) ds \\
&= \frac{1}{\Gamma(m)} \int_0^t (t-s)^{m-1} \mathsf{J}^n \varphi(s)\, ds = \mathsf{J}^m \mathsf{J}^n \varphi(t).
\end{aligned}$$

Since t denotes an arbitrary point in $(0,T]$, $\mathsf{J}^{m+n}\varphi(t) = \mathsf{J}^m\mathsf{J}^m\varphi(t)$ holds at every $t \in (0,T]$. And, as was pointed out earlier, it holds at $t = 0$ as well. This concludes the proof of the second statement. □

Lemma 4.9. *Let $\varphi \in L^1[0,T]$ and $m,n \in \mathbb{R}^+$. If $m+n \geq 1$, then the Lebesgue integral*

$$\mathsf{J}^{m+n}\varphi(t) = \frac{1}{\Gamma(m+n)} \int_0^t (t-s)^{m+n-1}\varphi(s)\, ds \tag{4.13}$$

is continuous on $[0,T]$ and (4.11) holds at each $t \in [0,T]$.

Proof. With (4.13) rewritten as

$$\mathsf{J}^{m+n}\varphi(t) = \frac{1}{\Gamma(m+n)} \int_0^t \varphi(t-s)s^{m+n-1}\, ds,$$

we see that the integral is the convolution of $s^{m+n-1} \in C[0,T]$ and $\varphi \in L^1[0,T]$. It then follows from Lemma 4.5 that $J^{m+n}\varphi \in C[0,T]$.

Because the proof of (4.11) is based on $|\varphi| \in L^1[0,T]$, which by hypothesis is the case here, it is also valid here. □

With these lemmas we can at last prove the main result of this section.

Theorem 4.10. *Let $q \in (0,1)$. Let $f(t,x)$ be a function that is continuous on the set*

$$\mathcal{B} = \{(t,x) \in \mathbb{R}^2 : 0 < t \leq T, x \in I\},$$

where I denotes an unbounded interval in $\mathbb{R}$. Suppose there is a continuous function $x\colon (0,T] \to I$ such that both $x(t)$ and $f(t,x(t))$ are absolutely integrable on $(0,T]$. Suppose further that $x(t)$ satisfies the fractional differential equation

$$D^q x(t) = f(t,x(t)) \tag{1.1}$$

on $(0,T]$ and the initial condition

$$\lim_{t\to 0^+} \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q}x(s)\, ds = x^0\Gamma(q), \tag{3.1}$$

where $x^0 \neq 0$. *Then* $x(t)$ *also satisfies the integral equation*

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \tag{1.2}$$

on $(0, T]$.

Remark 4.11. The interval I must be unbounded. It can be left-unbounded, right-unbounded, or it can be $\mathbb{R}$ (cf. Cor. 6.3 in Sec. 6). For instance, we will see in Example 4.12 that $I = [0, \infty)$ while $I = \mathbb{R}$ in Example 7.1.

Proof. Suppose a continuous function $x\colon (0, T] \to I$ exists satisfying the fractional equation (1.1) on $(0, T]$ and the initial condition (3.1) and that it is absolutely integrable on $(0, T]$. Then, as f is continuous on the set $\mathcal{B}$, the function φ defined by

$$\varphi(t) := f(t, x(t)) \tag{4.14}$$

is continuous on $(0, T]$. From (1.1) and (4.2), we have

$$\varphi(t) = D^q x(t) = D\mathrm{J}^{1-q} x(t). \tag{4.15}$$

Consequently,

$$\mathrm{J}^{1-q} x(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} x(s)\, ds \tag{4.16}$$

is continuously differentiable on $(0, T]$. An integration of (4.15) yields

$$\int_\eta^t \varphi(s)\, ds = \mathrm{J}^{1-q} x(t) - \mathrm{J}^{1-q} x(\eta)$$

for $0 < \eta < t \leq T$. Taking the right-hand limit

$$\lim_{\eta \to 0^+} \int_\eta^t \varphi(s)\, ds = \mathrm{J}^{1-q} x(t) - \lim_{\eta \to 0^+} \mathrm{J}^{1-q} x(\eta)$$

and using (3.1), we obtain

$$\int_0^t \varphi(s)\, ds = \mathrm{J}^{1-q} x(t) - x^0 \Gamma(q).$$

Therefore,

$$\mathrm{J}^{1-q} x(t) = x^0 \Gamma(q) + \mathrm{J}\varphi(t) \tag{4.17}$$

for $0 < t \leq T$.

Now apply J^q to both sides of (4.17). Since $\varphi(t)$ is continuous and absolutely integrable on $(0, T]$, Lemma 4.8 allows us to simplify the right-hand side to

$$\begin{aligned} \mathrm{J}^q \mathrm{J}^{1-q} x(t) &= \mathrm{J}^q \left[x^0 \Gamma(q) \right] + \mathrm{J}^q \mathrm{J} \varphi(t) \\ &= \frac{x^0 \Gamma(q)}{\Gamma(q+1)} t^q + \mathrm{J}^{q+1} \varphi(t) = \frac{x^0}{q} t^q + \mathrm{J}^1 \mathrm{J}^q \varphi(t), \end{aligned}$$

where we have also used Lemma 4.1. And as $\mathrm{J}^q \mathrm{J}^{1-q} x(t) = \mathrm{J}^1 x(t)$, this further simplifies to

$$\int_0^t x(s)\, ds = \frac{x^0}{q} t^q + \int_0^t \mathrm{J}^q \varphi(s)\, ds. \tag{4.18}$$

Moreover from Lemma 4.6 we see that the integrand

$$\mathrm{J}^q\varphi(s) = \frac{1}{\Gamma(q)}\int_0^s (s-u)^{q-1}\varphi(u)\,du$$

is continuous on $(0, T]$. Differentiating (4.18), we obtain

$$x(t) = x^0 t^{q-1} + \mathrm{J}^q\varphi(t) = x^0 t^{q-1} + \mathrm{J}^q f(t, x(t))$$

for $0 < t \leq T$, which is (1.2). $\square$

It is challenging to come up with an example of a nonlinear fractional differential equation or a nonlinear Volterra integral equation with a closed-form solution to illustrate the previous theorem. However, here is an example.

Example 4.12. A continuous, absolutely integrable function $x(t)$ that satisfies the fractional differential equation

$$D^{1/2}x(t) = -\frac{\sqrt{\pi}}{2}\big(\sqrt{t}x(t)\big)^{3/2} \tag{4.19}$$

on $(0, \infty)$ and the initial condition

$$\lim_{t\to 0^+} \frac{1}{\sqrt{\pi}}\int_0^t (t-s)^{-1/2}x(s)\,ds = \sqrt{\pi} \tag{4.20}$$

is

$$x(t) = \frac{1}{\sqrt{t}(1+t)}. \tag{4.21}$$

Furthermore, it also satisfies the integral equation

$$x(t) = \frac{1}{\sqrt{t}} - \frac{1}{2}\int_0^t (t-s)^{-1/2}\big(\sqrt{s}x(s)\big)^{3/2}\,ds \tag{4.22}$$

on $(0, \infty)$.

Proof. Comparing (4.19) with (1.1), we see that $q = 1/2$ and

$$f(t, x) = -\frac{\sqrt{\pi}}{2}\big(\sqrt{t}\,x\big)^{3/2}. \tag{4.23}$$

And comparing (4.20) with (3.1), we see that $x^0 = 1$ as $\Gamma(1/2) = \sqrt{\pi}$. Choose any $T > 0$. Note that $f(t, x)$ is continuous on the set

$$\mathcal{B} = \big\{(t, x) \in \mathbb{R}^2 : 0 < t \leq T,\ 0 \leq x < \infty\big\}.$$

The function $x\colon (0, T] \to (0, \infty)$ defined by (4.21) has a singularity at $t = 0$, but it is continuous on $(0, T]$. Integrating, with the change of variable $u = \sqrt{t}$, we obtain

$$\begin{aligned}\int_0^T |x(t)|\,dt &= \lim_{\eta\to 0^+}\int_\eta^T \frac{dt}{\sqrt{t}(1+t)} = 2\lim_{\eta\to 0^+}\int_{\sqrt{\eta}}^{\sqrt{T}} \frac{du}{1+u^2} \\ &= 2\tan^{-1}\sqrt{T} < \infty,\end{aligned}$$

showing that x is absolutely integrable on $(0,T]$. Likewise the same is true of $f(t,x(t))$ as

$$\int_0^T |f(t,x(t))|\,dt = \frac{\sqrt{\pi}}{2}\int_0^T \left(\sqrt{t}x(t)\right)^{3/2} dt = \frac{\sqrt{\pi}}{2}\int_0^T \frac{1}{(1+t)^{3/2}}\,dt$$
$$= \sqrt{\pi}\left(1-\frac{1}{\sqrt{1+T}}\right) < \infty.$$

Let us now directly verify that (4.21) satisfies (4.19). Substituting it into the left-hand side of (4.19), we get

$$\begin{aligned} D^{1/2}x(t) &= \frac{1}{\Gamma(1/2)}\frac{d}{dt}\int_0^t (t-s)^{-1/2}x(s)\,ds \\ &= \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t (t-s)^{-1/2}\cdot\frac{1}{\sqrt{s}(1+s)}\,ds = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\mathcal{I}(t) \end{aligned} \tag{4.24}$$

where

$$\mathcal{I}(t) := \int_0^t \frac{1}{\sqrt{t-s}}\cdot\frac{1}{\sqrt{s}(1+s)}\,ds.$$

With the change of variable $u=\sqrt{s}$, this integral becomes

$$\mathcal{I}(t) = 2\int_0^{\sqrt{t}} \frac{du}{(1+u^2)\sqrt{t-u^2}}.$$

Using an integration formula from [1, (3.3.49), p. 13], we find that for $\eta < \sqrt{t}$:

$$\int_0^{\eta} \frac{du}{(1+u^2)\sqrt{t-u^2}} = \frac{1}{\sqrt{t+1}}\tan^{-1}\frac{\eta(t+1)}{\sqrt{t-\eta^2}}.$$

Hence,

$$\int_0^{\sqrt{t}} \frac{du}{(1+u^2)\sqrt{t-u^2}} = \lim_{\eta\to\sqrt{t}^-}\frac{1}{\sqrt{t+1}}\tan^{-1}\frac{\eta(t+1)}{\sqrt{t-\eta^2}} = \frac{1}{\sqrt{t+1}}\left(\frac{\pi}{2}\right).$$

And so

$$\mathcal{I}(t) = \frac{\pi}{\sqrt{t+1}}$$

for $t>0$. Then, for $0<t\le T$, we see from (4.24) and (4.21) that

$$\begin{aligned} D^{1/2}x(t) &= \frac{1}{\sqrt{\pi}}\frac{d}{dt}\frac{\pi}{\sqrt{t+1}} = -\frac{\sqrt{\pi}}{2}\frac{1}{(1+t)^{3/2}} = -\frac{\sqrt{\pi}}{2}\left(\frac{1}{1+t}\right)^{3/2} \\ &= -\frac{\sqrt{\pi}}{2}\left(\sqrt{t}\cdot\frac{1}{\sqrt{t}(1+t)}\right)^{3/2} = -\frac{\sqrt{\pi}}{2}\left(\sqrt{t}x(t)\right)^{3/2}, \end{aligned}$$

which verifies that (4.21) satisfies (4.19) on $(0,T]$.

The initial condition (4.20) is also satisfied because

$$\begin{aligned} \lim_{t\to 0^+}\frac{1}{\sqrt{\pi}}\int_0^t (t-s)^{-1/2}x(s)\,ds &= \lim_{t\to 0^+}\frac{1}{\sqrt{\pi}}\int_0^t (t-s)^{-1/2}\frac{1}{\sqrt{s}(1+s)}\,ds \\ &= \lim_{t\to 0^+}\frac{1}{\sqrt{\pi}}\mathcal{I}(t) = \lim_{t\to 0^+}\frac{1}{\sqrt{\pi}}\frac{\pi}{\sqrt{t+1}} = \sqrt{\pi}. \end{aligned}$$

Since all of the conditions of Theorem 4.10 are met, we conclude (4.21) satisfies not only the differential equation (4.19) on $(0,T]$ but also the integral equation (4.22). Since this is true for every $T>0$, (4.21) satisfies (4.19) and (4.22) for all $t>0$. $\square$

5 Complementary integral equation

We now show that the converse of Theorem 4.10 is also true.

Theorem 5.1. *Let $f(t,x)$ be a function that is continuous on the set*

$$\mathcal{B} = \{(t,x) \in \mathbb{R}^2 : 0 < t \leq T, x \in I\},$$

where I denotes an unbounded interval in $\mathbb{R}$. If there is a continuous function $x\colon (0,T] \to I$ such that both $x(t)$ and $f(t,x(t))$ are absolutely integrable on $(0,T]$ and if $x(t)$ satisfies the integral equation (1.2)*, namely*

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))\, ds,$$

on $(0,T]$, where $q \in (0,1)$ and $x^0 \neq 0$, then it also satisfies the fractional differential equation (1.1) *on $(0,T]$ and the initial condition* (3.1).

We remark that condition (3.1) is shown to be equivalent to condition (1.3) in Section 6.

Proof. By hypothesis, a function $x(t)$ satisfies the integral equation (1.2). In other words, this function satisfies the equation

$$x(t) = x^0 t^{q-1} + \mathsf{J}^q f(t,x(t)) \tag{5.1}$$

for $0 < t \leq T$. To show that it also satisfies (1.1), apply the operator D^q to both of the terms on the right-hand side of (5.1).

Beginning with the first term, we have

$$D^q \left(x^0 t^{q-1}\right) = D \mathsf{J}^{1-q} \left(x^0 t^{q-1}\right),$$

where, from Lemma 4.1,

$$\mathsf{J}^{1-q}\left(x^0 t^{q-1}\right) = x^0 \mathsf{J}^{1-q} t^{q-1} = x^0 \Gamma(q).$$

Therefore,

$$D^q \left(x^0 t^{q-1}\right) = \frac{d}{dt}\left(x^0 \Gamma(q)\right) = 0 \tag{5.2}$$

for $t \in (0,T]$.

Now consider the second term of (5.1):

$$\mathsf{J}^q f(t,x(t)) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))\, ds.$$

Since $f(t,x(t))$ is continuous and absolutely integrable on $(0,T]$, we see from Lemma 4.8 that

$$\mathsf{J}^{1-q} \mathsf{J}^q f(t,x(t)) = \mathsf{J}^{(1-q)+q} f(t,x(t)) = \mathsf{J}^1 f(t,x(t))$$

for each $t \in [0,T]$. Consequently,

$$\begin{aligned} D^q \mathsf{J}^q f(t,x(t)) &= D \mathsf{J}^{1-q} \mathsf{J}^q f(t,x(t)) = D \mathsf{J}^1 f(t,x(t)) \\ &= \frac{d}{dt} \int_0^t f(s,x(s))\, ds = f(t,x(t)) \end{aligned} \tag{5.3}$$

for $t \in (0, T]$. It then follows from (5.2), (5.3), and the linearity property (4.3) that

$$D^q\big(x^0 t^{q-1} + \mathrm{J}^q f(t, x(t))\big) = D^q\big(x^0 t^{q-1}\big) + D^q \mathrm{J}^q f(t, x(t)) = f(t, x(t)).$$

This, together with (5.1), yields

$$D^q x(t) = f(t, x(t))$$

for $0 < t \le T$.

Finally we prove that x satisfies the initial condition (3.1). Applying the integral operator J^{1-q} to both sides of (5.1), we obtain

$$\begin{aligned} \mathrm{J}^{1-q} x(t) &= \mathrm{J}^{1-q}\big(x^0 t^{q-1}\big) + \mathrm{J}^{1-q}\mathrm{J}^q f(t, x(t)) \\ &= x^0 \Gamma(q) + \mathrm{J}^1 f(t, x(t)). \end{aligned}$$

Consequently,

$$\frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} x(s)\, ds = x^0 \Gamma(q) + \int_0^t f(s, x(s))\, ds.$$

Therefore,

$$\lim_{t \to 0^+} \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} x(s)\, ds = x^0 \Gamma(q) + \lim_{t \to 0^+} \int_0^t f(s, x(s))\, ds = x^0 \Gamma(q). \qquad \square$$

The next example has already been dealt with in Example 4.12. However, let us consider it from the point of view of Theorem 5.1.

Example 5.2. A function that satisfies the integral equation

$$\begin{aligned} x(t) &= t^{-1/2} + \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-s)^{-1/2} \left[-\frac{\sqrt{\pi}}{2} \big(\sqrt{s}\, x(s)\big)^{3/2} \right] ds \\ &= \frac{1}{\sqrt{t}} - \frac{1}{2} \int_0^t (t-s)^{-1/2} \big(\sqrt{s}\, x(s)\big)^{3/2} ds \end{aligned} \tag{5.4}$$

on $(0, \infty)$ is

$$x(t) = \frac{1}{\sqrt{t}(1+t)}. \tag{5.5}$$

Furthermore, it also satisfies the differential equation

$$D^{1/2} x(t) = -\frac{\sqrt{\pi}}{2} \big(\sqrt{t}\, x(t)\big)^{3/2} \tag{5.6}$$

on $(0, \infty)$ and the initial condition

$$\lim_{t \to 0^+} \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-1/2} x(s)\, ds = \sqrt{\pi}. \tag{5.7}$$

Proof. For an arbitrary $T > 0$, we have already shown in Example 4.12 that $x(t)$ and

$$f(t, x(t)) = -\frac{\sqrt{\pi}}{2} \big(\sqrt{t}\, x(t)\big)^{3/2}$$

are absolutely integrable on $(0, T]$. What remains to be shown is that (5.5) satisfies (5.4). Then we can simply invoke Theorem 5.1 to assert that (5.5) also satisfies (5.6) and (5.7).

From (5.5) we have

$$\left(\sqrt{t}x(t)\right)^{3/2} = \left(\frac{1}{1+t}\right)^{3/2} = \frac{1}{(1+t)^{3/2}}.$$

Consequently, the integral term in (5.4) is

$$\mathcal{I}(t) := \frac{1}{2}\int_0^t (t-s)^{-1/2}\left(\sqrt{s}x(s)\right)^{3/2} ds = \frac{1}{2}\int_0^t \frac{1}{\sqrt{t-s}}\cdot\frac{1}{(1+s)\sqrt{1+s}}\, ds.$$

With the change of variable $u = \sqrt{1+s}$, the function $\mathcal{I}(t)$ becomes

$$\mathcal{I}(t) = \int_1^{\sqrt{1+t}} \frac{1}{u^2\sqrt{1+t-u^2}}\, du.$$

Then, letting $a = \sqrt{1+t}$ and using the trig substitution $u = a\sin\theta$, we obtain

$$\begin{aligned}\mathcal{I}(t) &= -\left[\frac{\sqrt{a^2-u^2}}{a^2u}\right]_1^{\sqrt{1+t}} = -\frac{1}{1+t}\left[\frac{\sqrt{1+t-u^2}}{u}\right]_1^{\sqrt{1+t}}\\ &= -\frac{1}{1+t}\left(\frac{\sqrt{1+t-(1+t)}}{\sqrt{1+t}} - \sqrt{t}\right) = \frac{\sqrt{t}}{1+t}.\end{aligned}$$

Therefore, for $t > 0$ the right-hand of (5.4) is

$$\frac{1}{\sqrt{t}} - \mathcal{I}(t) = \frac{1}{\sqrt{t}} - \frac{\sqrt{t}}{1+t} = \frac{1}{\sqrt{t}(1+t)} = x(t),$$

which verifies that (5.5) satisfies (5.4) on $(0, T]$.

Since all of the conditions of Theorem 5.1 are fulfilled, it then follows that (5.5) also satisfies the differential equation (5.6) on $(0, T]$ and the initial condition (5.7). Moreover, as $T > 0$ is arbitrary, (5.5) satisfies both (5.4) and (5.6) on $(0, \infty)$. Recall that a direct verification of this is given in Example 4.12. □

6 Equivalent initial conditions and equations

We now prove, under the hypotheses of Theorems 4.10 and 5.1, that the two initial conditions (1.3) and (3.1) are equivalent.

Theorem 6.1. *Let $x^0 \in \mathbb{R}$ and $q \in (0,1)$. Suppose a function x is continuous and absolutely integrable on $(0, T]$. Then*

$$\lim_{t\to 0^+} t^{1-q}x(t) = x^0 \tag{1.3}$$

if and only if

$$\lim_{t\to 0^+} \mathrm{J}^{1-q}x(t) = x^0\Gamma(q),$$

to wit:

$$\lim_{t\to 0^+} \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}x(s)\, ds = x^0\Gamma(q). \tag{3.1}$$

Proof. Assume (1.3). Then for each $\epsilon > 0$ there is a $\delta \in (0, T]$ such that $0 < t < \delta$ implies

$$\left|t^{1-q}x(t) - x^0\right| < \frac{\epsilon}{2\Gamma(q)}$$

or

$$\left(x^0 - \frac{\epsilon}{2\Gamma(q)}\right) t^{q-1} < x(t) < \left(x^0 + \frac{\epsilon}{2\Gamma(q)}\right) t^{q-1}. \tag{6.1}$$

Then this implies that

$$|x(t)| < \left(|x^0| + \frac{\epsilon}{2\Gamma(q)}\right) t^{q-1}$$

for $0 < t < \delta$. Consequently,

$$(t-s)^{-q}|x(s)| < \left(|x^0| + \frac{\epsilon}{2\Gamma(q)}\right) s^{q-1}(t-s)^{-q} \tag{6.2}$$

for $0 < s < t < \delta$. From Lemma 4.1, we have

$$\int_0^t s^{q-1}(t-s)^{-q}\, ds = \Gamma(q)\Gamma(1-q).$$

Thus the improper integral of the right-hand side of (6.2) over $[0, t]$ converges. It then follows from the comparison test that the integral

$$\int_0^t (t-s)^{-q}x(s)\, ds$$

converges absolutely for each $t \in (0, \delta)$. As a result, we see from (6.1) that

$$\begin{aligned}\left(x^0 - \frac{\epsilon}{2\Gamma(q)}\right) \int_0^t (t-s)^{-q}s^{q-1}\, ds &\le \int_0^t (t-s)^{-q}x(s)\, ds \\ &\le \left(x^0 + \frac{\epsilon}{2\Gamma(q)}\right) \int_0^t (t-s)^{-q}s^{q-1}\, ds.\end{aligned}$$

Using Lemma 4.1 again, we obtain

$$\begin{aligned}\left(x^0 - \frac{\epsilon}{2\Gamma(q)}\right) \Gamma(q)\Gamma(1-q) &\le \int_0^t (t-s)^{-q}x(s)\, ds \\ &\le \left(x^0 + \frac{\epsilon}{2\Gamma(q)}\right) \Gamma(q)\Gamma(1-q)\end{aligned}$$

or

$$x^0\Gamma(q) - \frac{\epsilon}{2} \le \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q}x(s)\, ds \le x^0\Gamma(q) + \frac{\epsilon}{2}.$$

Therefore,

$$\left|\frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q}x(s)\, ds - x^0\Gamma(q)\right| < \epsilon$$

if $0 < t < \delta$. This concludes the "only if" part of the proof.

Now consider the "if" part of the proof. By hypothesis, x is continuous and absolutely integrable on $(0, T]$. As $-q > -1$, it follows from Lemma 4.6 that

$$\mathrm{J}^{1-q}x(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q}x(s)\, ds$$

exists at each $t \in [0, T]$. With that established, let us assume (3.1) holds. Then for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\mathrm{J}^{1-q}x(t) - x^0\Gamma(q)\right| < \frac{\epsilon}{2}\Gamma(q)$$

or

$$\left(x^0 - \frac{\epsilon}{2}\right)\Gamma(q) < \mathrm{J}^{1-q}x(t) < \left(x^0 + \frac{\epsilon}{2}\right)\Gamma(q) \tag{6.3}$$

for $0 < t < \delta$. Applying the operator J^q, we have

$$\mathrm{J}^q\left[\left(x^0 - \frac{\epsilon}{2}\right)\Gamma(q)\right] \le \mathrm{J}^q\mathrm{J}^{1-q}x(t) \le \mathrm{J}^q\left[\left(x^0 + \frac{\epsilon}{2}\right)\Gamma(q)\right]$$

for $0 < t < \delta$. By Lemma 4.8,

$$\mathrm{J}^q\mathrm{J}^{1-q}x(t) = \mathrm{J}^1x(t) = \int_0^t x(s)\,ds$$

at every $t \in [0, T]$. Therefore, due to this and Lemma 4.1, the previous pair of inequalities simplify to

$$\left(x^0 - \frac{\epsilon}{2}\right)\frac{t^q}{q} \le \int_0^t x(s)\,ds \le \left(x^0 + \frac{\epsilon}{2}\right)\frac{t^q}{q}$$

for $0 < t < \delta$. In sum, we have shown that for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|qt^{-q}\int_0^t x(s)\,ds - x^0\right| < \epsilon$$

for $0 < t < \delta$. In other words,

$$\lim_{t\to0^+}\frac{q\int_0^t x(s)\,ds}{t^q} = x^0. \tag{6.4}$$

Since x is improperly integrable on $(0, T]$, $\int_0^t x(s)\,ds \to 0$ as $t \to 0^+$. And, as $q \in (0, 1)$, $t^q \to 0$ as $t \to 0^+$. Applying L'Hôpital's rule, we have

$$\lim_{t\to0^+}\frac{q\int_0^t x(s)\,ds}{t^q} = \lim_{t\to0^+}\frac{x(t)}{t^{q-1}}.$$

Combining this with (6.4), we obtain (1.3). This concludes the "if" part of the proof. □

By virtue of Theorems 4.10 and 5.1, we see under the conditions of Theorem 6.1 that the initial value problem (1.4) and the Volterra integral equation (1.2) are equivalent in the sense that they share the same set of solutions from the space of continuous, absolutely integrable functions. Consequently we have the following theorem, which is the main result of this paper that was previewed in Section 1.

Theorem 6.2. *Let $q \in (0, 1)$ and $x^0 \neq 0$. Let $f(t, x)$ be a function that is continuous on the set*

$$\mathcal{B} = \left\{(t, x) \in \mathbb{R}^2 : 0 < t \le T, x \in I\right\}$$

where $I \subseteq \mathbb{R}$ denotes an unbounded interval. Suppose a function $x\colon (0, T] \to I$ is continuous and that both $x(t)$ and $f(t, x(t))$ are absolutely integrable on $(0, T]$. Then $x(t)$ satisfies the initial value problem

$$D^q x(t) = f(t, x(t)), \qquad \lim_{t\to0^+} t^{1-q}x(t) = x^0 \tag{1.4}$$

on $(0, T]$ *if and only if it satisfies the Volterra integral equation*

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \tag{1.2}$$

on $(0, T]$.

Corollary 6.3. *Let* $x\colon (0, T] \to I$ *and* $f\colon \mathcal{B} \to \mathbb{R}$ *represent functions that satisfy the continuity and integrability conditions of Theorem 6.2. If x satisfies either the initial value problem (1.4) or the integral equation (1.2), then it has the following properties:*

(i) $\lim_{t\to 0^+} t^{1-q} x(t) = x^0$.

(ii) *The sign of* $x(t)$ *is the same as that of* x^0 *for each* $t \in (0, \delta)$ *if* δ *is sufficiently small.*

(iii) *If* $x^0 > 0$, *then*

$$\lim_{t\to 0^+} t^r x(t) = \infty$$

for $r < 1 - q$. *In particular,* $\lim_{t\to 0^+} x(t) = \infty$.

(iv) *If* $x^0 < 0$, *then*

$$\lim_{t\to 0^+} t^r x(t) = -\infty$$

for $r < 1 - q$. *In particular,* $\lim_{t\to 0^+} x(t) = -\infty$.

(v) $\lim_{t\to 0^+} t^{1-q} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds = 0$.

(vi) *If* $\lim_{t\to 0^+} f(t, x(t))$ *exists and is finite, then*

$$\lim_{t\to 0^+} \int_0^t (t-s)^{n-1} f(s, x(s))\, ds = 0$$

for $n > 0$. *In particular, this is true for* $n = q$.

Proof. Item (i) is an immediate consequence of the function x satisfying either the initial value problem (1.4) or the integral equation (1.2). This in turn implies (ii)–(iv). In order to see this, let $\epsilon = \frac{1}{2}|x^0|$. Then there is a $\delta > 0$ such that

$$|t^{1-q} x(t) - x^0| < \epsilon$$

if $0 < t < \delta$. Consequently,

$$(x^0 - \epsilon) t^{q-1} < x(t) < (x^0 + \epsilon) t^{q-1}$$

for all $t \in (0, \delta)$.

If $x^0 > 0$, then $\epsilon = \frac{1}{2} x^0$ and

$$t^r x(t) > t^r (x^0 - \epsilon) t^{q-1} = \frac{x^0}{2t^{1-q-r}} > 0$$

for $t \in (0, \delta)$. Thus $t^r x(t) \to \infty$ as $t \to 0^+$ if $r < 1 - q$. On the other hand, if $x^0 < 0$, then $\epsilon = -\frac{1}{2} x^0$ and

$$t^r x(t) < t^r (x^0 + \epsilon) t^{q-1} = \frac{x^0}{2t^{1-q-r}} < 0$$

for $t \in (0, \delta)$. So for this case, $t^r x(t) \to -\infty$ as $t \to 0^+$. This concludes the proof of (ii)–(iv).

Item (i) and (1.2) imply that

$$\lim_{t\to 0^+} t^{1-q} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds = \Gamma(q) \lim_{t\to 0^+} \left[t^{1-q} x(t) - x^0\right] = 0,$$

which proves (v).

As for (vi), the hypotheses of Theorem 6.2 imply $f(t, x(t))$ is continuous on $(0, T]$. By the hypothesis of (vi),

$$\lim_{t\to 0^+} f(t, x(t)) = l$$

for some $l \in \mathbb{R}$. Thus we can extend the domain of $f(t, x(t))$ to the closed interval $[0, T]$ by defining its value at $t = 0$ to be l. Now that the extension of f is continuous on $[0, T]$, it follows from Lemma 4.3 that the integral in (vi) defines a continuous function on $[0, T]$. Consequently (vi) obtains. □

Example 6.4. The function

$$x(t) = \frac{1}{\sqrt{t}(1+t)}$$

in Examples 4.12 and 5.2 satisfies the relevant items in Corollary 6.3.

Proof. First observe that

$$\lim_{t\to 0^+} x(t) = \lim_{t\to 0^+} \frac{1}{\sqrt{t}(1+t)} = \infty$$

and, as $q = 1/2$,

$$\lim_{t\to 0^+} t^{1-q} x(t) = \lim_{t\to 0^+} \sqrt{t} x(t) = \lim_{t\to 0^+} \frac{1}{1+t} = 1.$$

Recall that $x^0 = 1$, so this illustrates (i) and (iii) when $r = 0$.

Since

$$\begin{aligned}\lim_{t\to 0+} f(t, x(t)) &= -\frac{\sqrt{\pi}}{2} \lim_{t\to 0+} \left(\sqrt{t}\, x(t)\right)^{3/2} \\ &= -\frac{\sqrt{\pi}}{2} \lim_{t\to 0+} \frac{1}{(1+t)^{3/2}} = -\frac{\sqrt{\pi}}{2},\end{aligned}$$

the premise of (vi) is met. As a result, the limit of the integral in (vi) is equal to zero, which implies the limit in (v) is also zero. A direct confirmation of (vi) when $n = q$ is easy to obtain from the calculations in Example 5.2:

$$\begin{aligned}\lim_{t\to 0+} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds &= -\frac{\sqrt{\pi}}{2} \lim_{t\to 0+} \int_0^t (t-s)^{-1/2} \left(\sqrt{s}\, x(s)\right)^{3/2} ds \\ &= -\sqrt{\pi} \lim_{t\to 0+} I(t) = -\sqrt{\pi} \lim_{t\to 0+} \frac{\sqrt{t}}{1+t} = 0.\end{aligned}$$

□

7 Concluding example

Example 7.1. The function

$$x(t) = \frac{\cos(\sqrt{t}\,)}{\sqrt{t}} \tag{7.1}$$

is a solution of the initial value problem

$$D^{1/2}x(t) = -\frac{\sqrt{\pi}\,\mathfrak{J}_1(\sqrt{t})}{2\cos(\sqrt{t}\,)}\,x(t), \qquad \lim_{t\to 0^+} \sqrt{t}\,x(t) = 1 \tag{7.2}$$

on the interval $(0, \pi^2/4)$, where $\mathfrak{J}_1(t)$ denotes the Bessel function of the first kind of order 1. It is also a solution of the integral equation

$$x(t) = \frac{1}{\sqrt{t}} - \frac{1}{2}\int_0^t \frac{1}{\sqrt{t-s}} \cdot \frac{\mathfrak{J}_1(\sqrt{s})}{\cos(\sqrt{s}\,)}\,x(s)\,ds \tag{7.3}$$

on $(0, \pi^2/4)$.

Proof. An important equation of mathematical physics is Bessel's equation of order ν:

$$t^2x'' + tx' + (t^2 - \nu^2)x = 0, \tag{7.4}$$

where ν is a constant. If $\nu = 1$, the function $\mathfrak{J}_1(t)$ is a solution of (7.4) for $t > 0$ ([7, p. 487]). Thus, $\mathfrak{J}_1(t)$ is continuously differentiable on $(0, \infty)$. Hence, for any fixed $T \in (0, \pi^2/4)$, the function f defined by

$$f(t,x) := -\frac{\sqrt{\pi}\,\mathfrak{J}_1(\sqrt{t})}{2\cos(\sqrt{t}\,)}\,x$$

is continuous on $(0, T] \times \mathbb{R}$.

Since $|\mathfrak{J}_1(\sqrt{t})| \le 1$ ([1, (9.1.60), p. 362]),

$$|f(t,x(t))| = \left| -\frac{\sqrt{\pi}\,\mathfrak{J}_1(\sqrt{t})}{2\cos(\sqrt{t}\,)} \cdot \frac{\cos(\sqrt{t}\,)}{\sqrt{t}} \right| = \frac{\sqrt{\pi}\,|\mathfrak{J}_1(\sqrt{t})|}{2\sqrt{t}} \le \frac{\sqrt{\pi}}{2\sqrt{t}}$$

for $t \in (0, T]$. Thus, as $\sqrt{\pi}/(2\sqrt{t})$ dominates $|f(t, x(t))|$ and the improper integral of $\sqrt{\pi}/(2\sqrt{t})$ exists on $(0, T]$, it follows from the comparison test that $f(t, x(t))$ is absolutely integrable on $(0, T]$. Likewise, the same is true of $x(t)$.

The function x in (7.1) clearly satisfies the initial condition in (7.2). Let us verify that it also satisfies the fractional equation on the interval $(0, T]$. Applying the fractional differential operator (4.2) with $q = 1/2$ to the alleged solution x, we obtain

$$\begin{aligned} D^{1/2}x(t) = D\mathrm{J}^{1/2}x(t) &= \frac{d}{dt}\left[\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-1/2}x(s)\,ds\right] \\ &= \frac{d}{dt}\left[\frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-s}} \cdot \frac{\cos(\sqrt{s}\,)}{\sqrt{s}}\,ds\right]. \end{aligned}$$

With the change of variable $u := \sqrt{s/t}$, the function $\mathrm{J}^{1/2}x(t)$ can be rewritten as

$$\begin{aligned} \mathrm{J}^{1/2}x(t) &= \frac{1}{\sqrt{\pi}}\int_0^t \frac{\cos(\sqrt{s}\,)}{\sqrt{s}\,\sqrt{t-s}}\,ds = \frac{2}{\sqrt{\pi}}\int_0^1 \frac{\cos(u\sqrt{t}\,)}{\sqrt{1-u^2}}\,du \\ &= \sqrt{\pi}\left[\frac{2}{\pi}\int_0^1 \frac{\cos(u\sqrt{t}\,)}{\sqrt{1-u^2}}\,du\right]. \end{aligned}$$

From formula (9.1.20) in [1, p. 360], we see that the bracketed quantity is an integral representation of $\mathfrak{J}_0(\sqrt{t})$, where $\mathfrak{J}_0(t)$ denotes the Bessel function of the first kind of order 0, which is a solution of (7.4) when $\nu = 0$. Thus,

$$\mathrm{J}^{1/2}x(t) = \sqrt{\pi}\,\mathfrak{J}_0(\sqrt{t}\,). \tag{7.5}$$

Since $\mathfrak{J}_0'(t) = -\mathfrak{J}_1(t)$ ([1, (9.1.28), p. 361]),

$$D^{1/2}x(t) = \frac{d}{dt}\big(\sqrt{\pi}\,\mathfrak{J}_0(\sqrt{t}\,)\big) = -\sqrt{\pi}\,\mathfrak{J}_1(\sqrt{t})\frac{d}{dt}\sqrt{t} = -\frac{\sqrt{\pi}}{2\sqrt{t}}\mathfrak{J}_1(\sqrt{t}\,).$$

Therefore,

$$D^{1/2}x(t) = -\frac{\sqrt{\pi}}{2\sqrt{t}}\mathfrak{J}_1(\sqrt{t}\,) = -\frac{\sqrt{\pi}\,\mathfrak{J}_1(\sqrt{t}\,)}{2\cos(\sqrt{t})}\cdot\frac{\cos(\sqrt{t})}{\sqrt{t}} = -\frac{\sqrt{\pi}\,\mathfrak{J}_1(\sqrt{t}\,)}{2\cos(\sqrt{t})}\,x(t),$$

which confirms that the function x defined by (7.1) is a solution of the fractional differential equation on $(0, T]$.

We have established that the function x defined by (7.1) is continuous and absolutely integrable on $(0, T]$ and that it satisfies the initial problem (7.2) on this interval. Now observe that all of the conditions of Theorem 6.2 are met. Consequently, x also satisfies the integral equation (7.3) on $(0, T]$. Finally, the arbitrary choice of T shows that it is a solution of both (7.2) and (7.3) on $(0, \pi^2/4)$. □

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