# Positive solutions for a system of higher-order singular nonlinear fractional differential equations with nonlocal boundary conditions 

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#### Abstract

The paper deals with the existence and multiplicity of positive solutions for a system of higher-order singular nonlinear fractional differential equations with nonlocal boundary conditions. The main tool used in the proof is fixed point index theory in cone. Some limit type conditions for ensuring the existence of positive solutions are given.


Keywords: higher-order singular fractional differential equations, positive solution, cone, fixed point index.

2010 Mathematics Subject Classification: 26A33, 34B10, 34B16, 34 B18.

## 1 Introduction

In this paper, we discuss the following system of higher-order singular nonlinear fractional differential equations with nonlocal boundary conditions:

$$
\begin{gather*}
D_{0+}^{\alpha} u(x)+h_{1}(x) f_{1}(x, u(x), v(x)=0, \\
D_{0+}^{\beta} v(x)+h_{2}(x) f_{2}(x, u(x), v(x))=0,  \tag{1.1}\\
u^{(i)}(0)=0, \quad v^{(i)}(0)=0, \quad 0 \leq i \leq n-2, \\
D_{0+}^{\mu} u(1)=\eta_{1} D_{0+}^{\mu} u\left(\xi_{1}\right), \quad D_{0+}^{v} v(1)=\eta_{2} D_{0+}^{v} v\left(\xi_{2}\right), \tag{1.2}
\end{gather*}
$$

where $x \in(0,1), D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives of order $\alpha, \beta \in(n-1, n], 1 \leq \mu, v \leq n-3$ for $n>3$ and $n \in \mathbb{N}^{+}, \xi_{1}, \xi_{2} \in(0,1), 0 \leq \eta_{1} \xi_{1}^{\alpha-\mu-1}<1$, $0 \leq \eta_{2} \xi_{2}^{\beta-v-1}<1, f_{j} \in C\left([0,1] \times \mathbb{R}^{+} \times \mathbb{R}^{+}, \mathbb{R}^{+}\right), h_{j} \in C\left((0,1), \mathbb{R}^{+}\right)(j=1,2), \mathbb{R}^{+}=[0,+\infty)$, $h_{j}(x)$ is allowed to be singular at $x=0$ and/or $x=1$.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback

[^0]amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so forth. Recently, the existence and multiplicity of positive solutions for the nonlinear fractional differential equations have been researched, see [ $3,5,6,12,18,22,23,25,31]$ and the references therein. Such as, C. F. Li et al. [16] studied the existence and multiplicity of positive solutions of the following boundary value problem for nonlinear fractional differential equations:
\[

\left\{$$
\begin{array}{l}
D_{0+}^{\alpha} u(t)+f(t, u(t))=0, \quad t \in(0,1), \\
u(0)=0, D_{0+}^{\beta} u(1)=a D_{0+}^{\beta} u(\xi),
\end{array}
$$\right.
\]

where $D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order $\alpha \in(1,2], \beta, a \in$ $[0,1], \xi \in(0,1), a \xi^{\alpha-\beta-1} \leq 1-\beta, \alpha-\beta-1 \geq 0$.

The existence and uniqueness of some systems for nonlinear fractional differential equations have been studied by using fixed point theory or coincidence degree theory, see [ $1,10,21,24,25,34]$ and references therein. In [7,17,29,30], authors studied the existence and multiplicity of positive solutions of two types of systems for nonlinear fractional differential equations with boundary conditions:

$$
\left\{\begin{array}{l}
D_{0+}^{\alpha} u(t)+\lambda f(t, u(t), v(t))=0, \\
D_{0+}^{\beta} v(t)+\mu g(t, u(t), v(t))=0, \quad t \in(0,1), \\
u(0)=u^{\prime}(0)=\cdots=u^{(n-2)}(0)=0, u(1)=\int_{0}^{1} v(t) d H(t), \\
v(0)=v^{\prime}(0)=\cdots=v^{(n-2)}(0)=0, v(1)=\int_{0}^{1} u(t) d K(t),
\end{array}\right.
$$

and

$$
\begin{cases}D_{0+}^{\alpha} u(t)+\lambda a_{1}(t) f(u(t), v(t))=0, & \\ D_{0+}^{\beta} v(t)+\mu a_{2}(t) g(u(t), v(t))=0, & t \in[0,1], \\ u^{(i)}(0)=v^{(i)}(0)=0, & 0 \leq i \leq n-2, \\ D_{0+}^{\gamma} u(1)=\phi_{1}(u), D_{0+}^{\gamma} v(1)=\phi_{2}(v), & 1 \leq \gamma \leq n-2,\end{cases}
$$

where $D_{0+}^{\alpha}$ and $D_{0+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives, $\alpha, \beta \in$ ( $n-1, n]$ for $n \geq 3, \lambda, \mu>0$. The sublinear or superlinear condition is used in [7,17,29,30,33]. Another example, the following extreme limits:

$$
\begin{array}{ll}
f_{\delta}^{s}=: \limsup _{u+v \rightarrow \delta} \max _{t \in[0,1]} \frac{f(t, u, v)}{u+v}, & g_{\delta}^{s}=: \limsup _{u+v \rightarrow \delta} \max _{t \in[0,1]} \frac{g(t, u, v)}{u+v}, \\
f_{\delta}^{i}=: \liminf _{u+v \rightarrow \delta} \min _{t \in[\theta, 1-\theta]} \frac{f(t, u, v)}{u+v}, & g_{\delta}^{i}=: \liminf _{u+v \rightarrow \delta} \min _{t \in[\theta, 1-\theta]} \frac{g(t, u, v)}{u+v},
\end{array}
$$

are used in $[9,10]$, where $\theta \in\left(0, \frac{1}{2}\right), \delta=0^{+}$or $+\infty$. For the existence of positive solutions for systems of Hammerstein integral equations, see $[4,11,15,28]$ and their references.

Motivated by the above mentioned works and continuing the paper [27], in this paper, we present some limit type conditions and discuss the existence and multiplicity of positive solutions of the singular system (1.1)-(1.2) by using of fixed point index theory in cone. Our conditions are applicable for more functions, and the results obtained here are different from those in $[7,9,10,17,24,29,30,33]$. Some examples are also provided to illustrate our main results.

## 2 Preliminaries

Definition 2.1 ([19]). The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $u:(0,+\infty) \rightarrow \mathbb{R}$ is given by

$$
I_{0+}^{\alpha} u(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} u(s) d s
$$

provided the right side is pointwise defined on $(0,+\infty)$. The Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $u:(0,+\infty) \rightarrow \mathbb{R}$ is given by

$$
D_{0+}^{\alpha} u(t)=\frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{d t^{n}} \int_{0}^{t}(t-s)^{n-\alpha-1} u(s) d s
$$

where $n=[\alpha]+1,[\alpha]$ denotes the integer part of number $\alpha$, provided the right side is pointwise defined on $(0,+\infty)$.

Lemma 2.2 ([13]). (i) If $x \in L^{1}[0,1], \rho>\sigma>0$, then

$$
I_{0+}^{\rho} I_{0+}^{\sigma} x(t)=I_{0+}^{\rho+\sigma} x(t), \quad D_{0+}^{\sigma} I_{0+}^{\rho} x(t)=I_{0+}^{\rho-\sigma} x(t), \quad D_{0+}^{\sigma} I_{0+}^{\sigma} x(t)=x(t)
$$

(ii) If $\rho>\sigma>0$, then $D_{0+}^{\sigma} t^{\rho-1}=\Gamma(\rho) t^{\rho-\sigma-1} / \Gamma(\rho-\sigma)$.

Lemma 2.3. Let $\xi_{1} \in(0,1), \eta_{1} \xi_{1}^{\alpha-\mu-1} \neq 1, n-1<\alpha \leq n, 1 \leq \mu \leq n-3(n>3)$. Then for any $g \in C[0,1]$, the unique solution of the following boundary value problem:

$$
\begin{gather*}
D_{0+}^{\alpha} u(t)+g(t)=0, \quad t \in(0,1)  \tag{2.1}\\
u^{(i)}(0)=0 \quad(0 \leq i \leq n-2), \quad D_{0+}^{\mu} u(1)=\eta_{1} D_{0+}^{\mu} u\left(\xi_{1}\right) \tag{2.2}
\end{gather*}
$$

is given by

$$
\begin{equation*}
u(t)=\int_{0}^{1} G_{1}(t, s) g(s) d s \tag{2.3}
\end{equation*}
$$

where $d_{1}=1-\eta_{1} \xi_{1}^{\alpha-\mu-1}$,

$$
G_{1}(t, s)= \begin{cases}\frac{t^{\alpha-1}\left[(1-s)^{\alpha-\mu-1}-\eta_{1}\left(\xi_{1}-s\right)^{\alpha-\mu-1}\right]-d_{1}(t-s)^{\alpha-1}}{d_{1} \Gamma(\alpha)}, & 0 \leq s \leq \min \left\{t, \xi_{1}\right\}  \tag{2.4}\\ \frac{t^{\alpha-1}(1-s)^{\alpha-\mu-1}-d_{1}(t-s)^{\alpha-1}}{d_{1} \Gamma(\alpha)}, & 0<\xi_{1} \leq s \leq t \leq 1 \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\mu-1}-t^{\alpha-1} \eta_{1}\left(\xi_{1}-s\right)^{\alpha-\mu-1}}{d_{1} \Gamma(\alpha)}, & 0 \leq t \leq s \leq \xi_{1}<1 \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\mu-1}}{d_{1} \Gamma(\alpha)}, & \max \left\{t, \xi_{1}\right\} \leq s \leq 1\end{cases}
$$

is the Green's function of the integral equation (2.3).
Proof. The equation (2.1) is equivalent to an integral equation:

$$
\begin{equation*}
u(t)=\frac{-1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} g(s) d s+c_{1} t^{\alpha-1}+c_{2} t^{\alpha-2}+\cdots+c_{n} t^{\alpha-n} \tag{2.5}
\end{equation*}
$$

By $u(0)=0$, we have $c_{n}=0$. Then

$$
\begin{equation*}
u(t)=\frac{-1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} g(s) d s+c_{1} t^{\alpha-1}+c_{2} t^{\alpha-2}+\cdots+c_{n-1} t^{\alpha-n+1} . \tag{2.6}
\end{equation*}
$$

Differentiating (2.6), we have

$$
\begin{equation*}
u^{\prime}(t)=\frac{1-\alpha}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-2} g(s) d s+c_{1}(\alpha-1) t^{\alpha-2}+\cdots+c_{n-1}(\alpha-n+1) t^{\alpha-n} \tag{2.7}
\end{equation*}
$$

By (2.7) and $u^{\prime}(0)=0$, we have $c_{n-1}=0$. Similarly, we can get that $c_{2}=c_{3}=\cdots=c_{n-2}=0$. Thus

$$
\begin{equation*}
u(t)=\frac{-1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} g(s) d s+c_{1} t^{\alpha-1} . \tag{2.8}
\end{equation*}
$$

By $D_{0+}^{\mu} u(1)=\eta_{1} D_{0+}^{\mu} u\left(\xi_{1}\right)$ and Lemma 2.2,

$$
D_{0+}^{\mu} u(t)=\frac{1}{\Gamma(\alpha-\mu)}\left[c_{1} \Gamma(\alpha) t^{\alpha-\mu-1}-\int_{0}^{t}(t-s)^{\alpha-\mu-1} g(s) d s\right],
$$

we get

$$
c_{1}=\frac{1}{d_{1} \Gamma(\alpha)} \int_{0}^{1}(1-s)^{\alpha-\mu-1} g(s) d s-\frac{\eta_{1}}{d_{1} \Gamma(\alpha)} \int_{0}^{\xi_{1}}\left(\xi_{1}-s\right)^{\alpha-\mu-1} g(s) d s .
$$

Therefore, the unique solution of the problem (2.1)-(2.2) is

$$
\begin{align*}
u(t)= & \frac{t^{\alpha-1}}{d_{1} \Gamma(\alpha)}\left[\int_{0}^{1}(1-s)^{\alpha-\mu-1} g(s) d s-\eta_{1} \int_{0}^{\xi_{1}}\left(\xi_{1}-s\right)^{\alpha-\mu-1} g(s) d s\right]  \tag{2.9}\\
& -\int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s) d s=\int_{0}^{1} G_{1}(t, s) g(s) d s .
\end{align*}
$$

Similar to the proof of Lemma 2.3 in [20], we can get the following lemma.
Lemma 2.4. Let $0<\eta_{1} \xi_{1}^{\alpha-\mu-1}<1$. The function $G_{1}(t, s)$ defined by (2.4) satisfies
(i) $G_{1}(t, s) \geq 0$ is continuous for any $t, s \in[0,1]$.
(ii) $\max _{t \in[0,1]} G_{1}(t, s)=G_{1}(1, s), G_{1}(t, s) \geq t^{\alpha-1} G_{1}(1, s)$ for $t, s \in[0,1]$, where

$$
G_{1}(1, s)= \begin{cases}\frac{(1-s)^{\alpha-\mu-1}-\eta_{1}\left(\xi_{1}-s\right)^{\alpha-\mu-1}-d_{1}(1-s)^{\alpha-1}}{d_{1} \Gamma(\alpha)}, & 0 \leq s \leq \xi_{1}  \tag{2.10}\\ \frac{(1-s)^{\alpha-\mu-1}-d_{1}(1-s)^{\alpha-1}}{d_{1} \Gamma(\alpha)}, & \xi_{1} \leq s \leq 1\end{cases}
$$

(iii) There are $\theta \in\left(0, \frac{1}{2}\right)$ and $\gamma_{\alpha} \in(0,1)$ such that $\min _{t \in J_{\theta}} G_{1}(t, s) \geq \gamma_{\alpha} G_{1}(1, s)$ for $s \in[0,1]$, where $J_{\theta}=[\theta, 1-\theta], \gamma_{\alpha}=\theta^{\alpha-1}$.

$$
\begin{aligned}
& \text { Let } \xi_{2} \in(0,1), 0<\eta_{2} \xi_{2}^{\beta-v-1}<1, d_{2}=1-\eta_{2} \xi_{2}^{\beta-v-1}, \\
& G_{2}(t, s)= \begin{cases}\frac{t^{\beta-1}\left[(1-s)^{\beta-v-1}-\eta_{2}\left(\xi_{2}-s\right)^{\beta-v-1}\right]-d_{2}(t-s)^{\beta-1}}{d_{2} \Gamma(\beta)}, & 0 \leq s \leq \min \left\{t, \xi_{2}\right\}, \\
\frac{t^{\beta-1}(1-s)^{\beta-v-1}-d_{2}(t-s)^{\beta-1}}{d_{2} \Gamma(\beta)}, & 0<\xi_{2} \leq s \leq t \leq 1, \\
\frac{t^{\beta-1}(1-s)^{\beta-v-1}-t^{\beta-1} \eta_{2}\left(\xi_{2}-s\right)^{\beta-v-1}}{d_{2} \Gamma(\beta)}, & 0 \leq t \leq s \leq \xi_{2}<1, \\
\frac{t^{\beta-1}(1-s)^{\beta-v-1}}{d_{2} \Gamma(\beta)}, & \max \left\{t, \xi_{2}\right\} \leq s \leq 1\end{cases}
\end{aligned}
$$

From Lemma 2.4 we know that $G_{1}(t, s)$ and $G_{2}(t, s)$ have the same properties, and there exists $\gamma_{\beta}=\theta^{\beta-1}$ such that $\min _{t \in J_{\theta}} G_{2}(t, s) \geq \gamma_{\beta} G_{2}(1, s)$. Let $\gamma=\min \left\{\gamma_{\alpha}, \gamma_{\beta}\right\}$,

$$
\delta_{j}=\int_{\theta}^{1-\theta} G_{j}(1, y) h_{j}(y) d y, \mu_{j}=\int_{0}^{1} G_{j}(1, y) h_{j}(y) d y \quad(j=1,2)
$$

For convenience we list the following assumptions:
$\left(H_{1}\right) f_{j} \in C\left([0,1] \times \mathbb{R}^{+} \times \mathbb{R}^{+}, \mathbb{R}^{+}\right) \quad(j=1,2)$.
$\left(H_{2}\right) h_{j} \in C\left((0,1), \mathbb{R}^{+}\right), h_{j}(x) \not \equiv 0$ on any subinterval of $(0,1)$ and $0<\int_{0}^{1} G_{j}(1, y) h_{j}(y) d y$ $<+\infty(j=1,2)$.
$\left(H_{3}\right)$ There exist $a, b \in C\left(\mathbb{R}^{+}, \mathbb{R}^{+}\right)$such that
(1) $a(\cdot)$ is concave and strictly increasing on $\mathbb{R}^{+}$with $a(0)=0$;
(2) $f_{10}=\liminf _{v \rightarrow 0+} \frac{f_{1}(x, u, v)}{a(v)}>0, f_{20}=\liminf _{u \rightarrow 0+} \frac{f_{2}(x, u, v)}{b(u)}>0$ uniformly with respect to $(x, u) \in J_{\theta} \times \mathbb{R}^{+}$and $(x, v) \in J_{\theta} \times \mathbb{R}^{+}$, respectively (specifically, $f_{10}=f_{20}=+\infty$ );
(3) $\lim _{u \rightarrow 0+} \frac{a(C b(u))}{u}=+\infty$ for any constant $C>0$.
$\left(H_{4}\right)$ There exists $t \in(0,+\infty)$ such that

$$
f_{1}^{\infty}=\limsup _{v \rightarrow+\infty} \frac{f_{1}(x, u, v)}{v^{t}}<+\infty, \quad f_{2}^{\infty}=\limsup _{u \rightarrow+\infty} \frac{f_{2}(x, u, v)}{u^{\frac{1}{t}}}=0
$$

uniformly with respect to $(x, u) \in[0,1] \times \mathbb{R}^{+}$and $(x, v) \in[0,1] \times \mathbb{R}^{+}$, respectively (specifically, $f_{1}^{\infty}=f_{2}^{\infty}=0$ ).
$\left(H_{5}\right)$ There exist $p, q \in C\left(\mathbb{R}^{+}, \mathbb{R}^{+}\right)$such that
(1) $p$ is concave and strictly increasing on $\mathbb{R}^{+}$;
(2) $f_{1 \infty}=\liminf _{v \rightarrow+\infty} \frac{f_{1}(x, u, v)}{p(v)}>0, f_{2 \infty}=\liminf _{u \rightarrow+\infty} \frac{f_{2}(x, u, v)}{q(u)}>0$ uniformly with respect to $(x, u) \in J_{\theta} \times \mathbb{R}^{+}$and $(x, v) \in J_{\theta} \times \mathbb{R}^{+}$, respectively (specifically, $f_{1 \infty}=$ $\left.f_{2 \infty}=+\infty\right)$;
(3) $\lim _{u \rightarrow+\infty} \frac{p(C q(u))}{u}=+\infty$ for any constant $C>0$.
$\left(H_{6}\right)$ There exists $s \in(0,+\infty)$ such that

$$
f_{1}^{0}=\limsup _{v \rightarrow 0+} \frac{f_{1}(x, u, v)}{v^{s}}<+\infty, \quad f_{2}^{0}=\limsup _{u \rightarrow 0+} \frac{f_{2}(x, u, v)}{u^{\frac{1}{s}}}=0
$$

uniformly with respect to $(x, u) \in[0,1] \times \mathbb{R}^{+}$and $(x, v) \in[0,1] \times \mathbb{R}^{+}$, respectively (specifically, $f_{1}^{0}=f_{2}^{0}=0$ ).
$\left(H_{7}\right)$ There exists $r>0$ such that

$$
f_{1}(x, u, v) \geq\left(\gamma \delta_{1}\right)^{-1} r, \quad f_{2}(x, u, v) \geq\left(\gamma \delta_{2}\right)^{-1} r, \quad \forall x \in J_{\theta}, \gamma r \leq u+v \leq r .
$$

$\left(H_{8}\right) f_{1}(x, u, v)$ and $f_{2}(x, u, v)$ are increasing with respect to $u$ and $v$, there exists $R>r>0$ such that

$$
4 \mu_{1} f_{1}(x, R, R)<R, \quad 4 \mu_{2} f_{2}(x, R, R)<R, \quad \forall x \in[0,1] .
$$

Let $E=C[0,1],\|u\|=\max _{t \in[0,1]}|u(t)|$, the product space $E \times E$ be equipped with norm $\|(u, v)\|=\|u\|+\|v\|$ for $(u, v) \in E \times E$, and

$$
P=\left\{u \in E: u(t) \geq 0, t \in[0,1], \min _{t \in J_{\theta}} u(t) \geq \gamma\|u\|\right\} .
$$

Then $E \times E$ is a real Banach space and $P \times P$ is a positive cone of $E \times E$. By $\left(H_{1}\right),\left(H_{2}\right)$, we can define operators

$$
A_{j}(u, v)(x)=\int_{0}^{1} G_{j}(x, y) h_{j}(y) f_{j}(y, u(y), v(y)) d y \quad(j=1,2)
$$

$A(u, v)=\left(A_{1}(u, v), A_{2}(u, v)\right)$. Similar to the proof of Lemma 3.1 in [2], it follows from $\left(H_{1}\right),\left(H_{2}\right)$ that $A_{j}: P \times P \rightarrow P$ is a completely continuous operator and $A(P \times P) \subset P \times P$. Clearly $(u, v)$ is a positive solution of the system (1.1) if and only if $(u, v) \in P \times P \backslash\{(0,0)\}$ is a fixed point of $A$ (refer $[9,27]$ ).

Lemma 2.5 ([8]). Let $E$ be a Banach space, $P$ be a cone in $E$ and $\Omega \subset E$ be a bounded open set. Assume that $A: \bar{\Omega} \cap P \rightarrow P$ is a completely continuous operator. If there exists $u_{0} \in P \backslash\{0\}$ such that

$$
u \neq A u+\lambda u_{0}, \quad \forall \lambda \geq 0, u \in \partial \Omega \cap P,
$$

then the fixed point index $i(A, \Omega \cap P, P)=0$.
Lemma 2.6 ( $[8,14])$. Let $E$ be a Banach space, $P$ be a cone in $E$ and $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Assume that $A: \bar{\Omega} \cap P \rightarrow P$ is a completely continuous operator.
(1) If $u \notin A u$ for all $u \in \partial \Omega \cap P$, then the fixed point index $i(A, \Omega \cap P, P)=1$.
(2) If $u \not \geqq A u$ for all $u \in \partial \Omega \cap P$, then the fixed point index $i(A, \Omega \cap P, P)=0$.

In the following, we adopt the convention that $C_{1}, C_{2}, C_{3}, \ldots$ stand for different positive constants. Let $\Omega_{\rho}=\{(u, v) \in E \times E:\|(u, v)\|<\rho\}$ for $\rho>0$.

## 3 Existence of a positive solution

Theorem 3.1. Assume that the conditions $\left(H_{1}\right),\left(H_{2}\right)$ are satisfied and that $\left(H_{3}\right),\left(H_{4}\right)$ or $\left(H_{7}\right),\left(H_{8}\right)$ hold. Then the system (1.1)-(1.2) has at least one positive solution.

Proof. Case 1. The conditions $\left(H_{3}\right)$ and $\left(H_{4}\right)$ hold. By $\left(H_{3}\right)$, there are $\xi_{1}>0, \eta_{1}>0$ and a sufficiently small $\rho>0$ such that

$$
\begin{array}{ll}
f_{1}(x, u, v) \geq \xi_{1} a(v), & \forall(x, u) \in J_{\theta} \times \mathbb{R}^{+}, 0 \leq v \leq \rho  \tag{3.1}\\
f_{2}(x, u, v) \geq \eta_{1} b(u), & \forall(x, v) \in J_{\theta} \times \mathbb{R}^{+}, 0 \leq u \leq \rho
\end{array}
$$

and

$$
\begin{equation*}
a\left(K_{1} b(u)\right) \geq \frac{2 K_{1}}{\xi_{1} \eta_{1} \delta_{1} \delta_{2} \gamma^{3}} u, \quad \forall u \in[0, \rho] \tag{3.2}
\end{equation*}
$$

where $K_{1}=\max \left\{\eta_{1} \gamma G_{2}(1, y) h_{2}(y): y \in J_{\theta}\right\}$. We claim that

$$
(u, v) \neq A(u, v)+\lambda(\varphi, \varphi), \quad \forall \lambda \geq 0,(u, v) \in \partial \Omega_{\rho} \cap(P \times P)
$$

where $\varphi \in P \backslash\{0\}$. If not, there are $\lambda \geq 0$ and $(u, v) \in \partial \Omega_{\rho} \cap(P \times P)$ such that $(u, v)=$ $A(u, v)+\lambda(\varphi, \varphi)$, then $u \geq A_{1}(u, v), v \geq A_{2}(u, v)$. By using the monotonicity and concavity of $a(\cdot)$, Jensen's inequality and Lemma 2.4, we have by (3.1) and (3.2),

$$
\begin{align*}
u(x) & \geq \int_{0}^{1} G_{1}(x, y) h_{1}(y) f_{1}(y, u(y), v(y)) d y \\
& \geq \xi_{1} \gamma_{\alpha} \int_{0}^{1} G_{1}(1, y) h_{1}(y) a(v(y)) d y \\
& \geq \xi_{1} \gamma_{\alpha} \int_{0}^{1} G_{1}(1, y) h_{1}(y) a\left(\int_{0}^{1} \eta_{1} G_{2}(y, z) h_{2}(z) b(u(z)) d z\right) d y \\
& \geq \xi_{1} \gamma \int_{\theta}^{1-\theta} G_{1}(1, y) h_{1}(y) \int_{0}^{1} a\left(\eta_{1} \gamma G_{2}(1, z) h_{2}(z) b(u(z))\right) d z d y  \tag{3.3}\\
& \geq \xi_{1} \gamma \int_{\theta}^{1-\theta} G_{1}(1, y) h_{1}(y) \int_{0}^{1} a\left(K_{1}^{-1} \eta_{1} \gamma G_{2}(1, z) h_{2}(z) K_{1} b(u(z))\right) d z d y \\
& \geq \xi_{1} \eta_{1} \gamma^{2} K_{1}^{-1} \int_{\theta}^{1-\theta} \int_{\theta}^{1-\theta} G_{1}(1, y) h_{1}(y) G_{2}(1, z) h_{2}(z) a\left(K_{1} b(u(z))\right) d z d y \\
& \geq \xi_{1} \eta_{1} \gamma^{2} \delta_{1} K_{1}^{-1} \int_{\theta}^{1-\theta} G_{2}(1, z) h_{2}(z) a\left(K_{1} b(u(z))\right) d z \\
& \geq \frac{2}{\delta_{2} \gamma} \int_{\theta}^{1-\theta} G_{2}(1, z) h_{2}(z) u(z) d z \geq 2\|u\|, \quad x \in J_{\theta}
\end{align*}
$$

Consequently, $\|u\|=0$. Next, (3.1) and (3.2) yield that

$$
\begin{aligned}
a(v(x)) & \geq a\left(\int_{0}^{1} G_{2}(x, y) h_{2}(y) f_{2}(y, u(y), v(y)) d y\right) \\
& \geq \int_{0}^{1} a\left(\eta_{1} \gamma G_{2}(1, y) h_{2}(y) b(u(y))\right) d y \\
& \geq \eta_{1} \gamma K_{1}^{-1} \int_{\theta}^{1-\theta} G_{2}(1, y) h_{2}(y) a\left(K_{1} b(u(y))\right) d y \\
& \geq \frac{2}{\xi_{1} \delta_{1} \delta_{2} \gamma^{2}} \int_{\theta}^{1-\theta} G_{2}(1, y) h_{2}(y) u(y) d y
\end{aligned}
$$

$$
\begin{align*}
& \geq \frac{2}{\delta_{1} \delta_{2} \gamma} \int_{\theta}^{1-\theta} G_{2}(1, y) h_{2}(y) d y \int_{0}^{1} G_{1}(1, z) h_{1}(z) a(v(z)) d z \\
& \geq \frac{2}{\delta_{1} \gamma} \int_{\theta}^{1-\theta} G_{1}(1, z) h_{1}(z) a(v(z)) d z \geq 2 a(\|v\|), \quad x \in J_{\theta}, \tag{3.4}
\end{align*}
$$

this means that $a(\|v\|)=0$. It follows from strict monotonicity of $a(v)$ and $a(0)=0$ that $\|v\|=0$. Hence $\|(u, v)\|=0$, which is a contradiction. Lemma 2.5 implies that

$$
\begin{equation*}
i\left(A, \Omega_{\rho} \cap(P \times P), P \times P\right)=0 \tag{3.5}
\end{equation*}
$$

On the other hand, by $\left(H_{4}\right)$, there exist $\zeta>0$ and $C_{1}>0, C_{2}>0$ such that

$$
\begin{array}{ll}
f_{1}(x, u, v) \leq \zeta v^{t}+C_{1}, & \forall(x, u, v) \in[0,1] \times \mathbb{R}^{+} \times \mathbb{R}^{+}, \\
f_{2}(x, u, v) \leq \varepsilon_{2} u^{\frac{1}{t}}+C_{2}, & \forall(x, u, v) \in[0,1] \times \mathbb{R}^{+} \times \mathbb{R}^{+}, \tag{3.6}
\end{array}
$$

where

$$
\varepsilon_{2}=\min \left\{\frac{1}{\mu_{2}\left(8 \zeta \mu_{1}\right)^{\frac{1}{t}}}, \frac{1}{8 \mu_{2}\left(\zeta \mu_{1}\right)^{\frac{1}{t}}}\right\} .
$$

Let

$$
W=\{(u, v) \in P \times P:(u, v)=\lambda A(u, v), 0 \leq \lambda \leq 1\} .
$$

We prove that $W$ is bounded. Indeed, for any $(u, v) \in W$, there exists $\lambda \in[0,1]$ such that $u=\lambda A_{1}(u, v), v=\lambda A_{2}(u, v)$. Then (3.6) implies that

$$
\begin{aligned}
& u(x) \leq A_{1}(u, v)(x) \leq \zeta \int_{0}^{1} G_{1}(1, y) h_{1}(y) v^{t}(y) d y+C_{3} \\
& v(x) \leq A_{2}(u, v)(x) \leq \varepsilon_{2} \int_{0}^{1} G_{2}(1, y) h_{2}(y) u^{\frac{1}{t}}(y) d y+C_{4} .
\end{aligned}
$$

Consequently,

$$
\begin{align*}
u(x) & \leq \zeta \int_{0}^{1} G_{1}(1, y) h_{1}(y) d y\left(\varepsilon_{2} \int_{0}^{1} G_{2}(1, z) h_{2}(z) u^{\frac{1}{t}}(z) d z+C_{4}\right)^{t}+C_{3} \\
& \leq \zeta \mu_{1}\left(\varepsilon_{2} \int_{0}^{1} G_{2}(1, z) h_{2}(z)\|u\|^{\frac{1}{t}} d z+C_{4}\right)^{t}+C_{3}  \tag{3.7}\\
& \leq \zeta \mu_{1}\left[\left(\frac{\|(u, v)\|}{8 \zeta \mu_{1}}\right)^{\frac{1}{t}}+C_{4}\right]^{t}+C_{3}, \\
v(x) & \leq \varepsilon_{2} \int_{0}^{1} G_{2}(1, y) h_{2}(y) d y\left(\zeta \int_{0}^{1} G_{1}(1, z) h_{1}(z) v^{t}(z) d z+C_{3}\right)^{\frac{1}{t}}+C_{4} \\
& \leq \varepsilon_{2} \mu_{2}\left(\zeta \int_{0}^{1} G_{1}(1, z) h_{1}(z)\|v\|^{t} d z+C_{3}\right)^{\frac{1}{t}}+C_{4}  \tag{3.8}\\
& \leq \frac{1}{8\left(\zeta \mu_{1}\right)^{\frac{1}{t}}}\left(\zeta \mu_{1}\|(u, v)\|^{t}+C_{3}\right)^{\frac{1}{t}}+C_{4} .
\end{align*}
$$

Since

$$
\lim _{w \rightarrow+\infty} \frac{\zeta \mu_{1}\left[\left(\frac{w}{8 \zeta \mu_{1}}\right)^{\frac{1}{t}}+C_{4}\right]^{t}}{w}=\frac{1}{8}, \quad \lim _{w \rightarrow+\infty} \frac{\left(\zeta \mu_{1} w^{t}+C_{3}\right)^{\frac{1}{t}}}{8\left(\zeta \mu_{1}\right)^{\frac{1}{t}} w}=\frac{1}{8}
$$

there exists $r_{1}>r$, when $\|(u, v)\|>r_{1}$, (3.7) and (3.8) yield that

$$
u(x) \leq \frac{1}{4}\|(u, v)\|+C_{3}, \quad v(x) \leq \frac{1}{4}\|(u, v)\|+C_{4} .
$$

Hence $\|(u, v)\| \leq 2\left(C_{3}+C_{4}\right)$ and $W$ is bounded.
Select $G>\sup W$. We obtain from the homotopic invariant property of fixed point index that

$$
\begin{equation*}
i\left(A, \Omega_{G} \cap(P \times P), P \times P\right)=i\left(\theta, \Omega_{G} \cap(P \times P), P \times P\right)=1 \tag{3.9}
\end{equation*}
$$

(3.5) and (3.9) yield that

$$
\begin{aligned}
& i\left(A,\left(\Omega_{G} \backslash \bar{\Omega}_{\rho}\right) \cap(P \times P), P \times P\right) \\
& \quad=i\left(A, \Omega_{G} \cap(P \times P), P \times P\right)-i\left(A, \Omega_{\rho} \cap(P \times P), P \times P\right)=1
\end{aligned}
$$

So $A$ has at least one fixed point on $\left(\Omega_{G} \backslash \bar{\Omega}_{\rho}\right) \cap(P \times P)$. This means that the system (1.1)-(1.2) has at least one positive solution.

Case 2. The conditions $\left(H_{7}\right)$ and $\left(H_{8}\right)$ hold. First, we prove that

$$
\begin{equation*}
i\left(A, \Omega_{r} \cap(P \times P), P \times P\right)=0 \tag{3.10}
\end{equation*}
$$

We claim that

$$
(u, v) \nsupseteq A(u, v), \quad \forall(u, v) \in \partial \Omega_{r} \cap(P \times P) .
$$

If not, there is $(u, v) \in \partial \Omega_{r} \cap(P \times P)$ such that $(u, v) \geq A(u, v)$. Since $\gamma r \leq u(x)+v(x) \leq r$ for $(u, v) \in \partial \Omega_{r} \cap(P \times P), x \in[\theta, 1-\theta]$, we know from $\left(H_{7}\right)$ that

$$
\begin{align*}
u(x) & \geq \int_{0}^{1} G_{1}(x, y) h_{1}(y) f_{1}(y, u(y), v(y)) d y \\
& \geq \delta_{1}^{-1} r \int_{\theta}^{1-\theta} G_{1}(1, y) h_{1}(y) d y=r, \quad x \in J_{\theta}  \tag{3.11}\\
v(x) & \geq \int_{0}^{1} G_{2}(x, y) h_{2}(y) f_{2}(y, u(y), v(y)) d y \\
& \geq \delta_{2}^{-1} r \int_{\theta}^{1-\theta} G_{2}(1, y) h_{2}(y) d y=r, \quad x \in J_{\theta} . \tag{3.12}
\end{align*}
$$

Hence $\|(u, v)\| \geq 2 r$, which is a contradiction. As a result (3.10) is true.
It remains to prove

$$
\begin{equation*}
i\left(A, \Omega_{R} \cap(P \times P), P \times P\right)=1 \tag{3.13}
\end{equation*}
$$

$\left(H_{8}\right)$ implies that

$$
\begin{equation*}
f_{1}(x, u, v) \leq f_{1}(x, R, R) \leq \frac{R}{4 \mu_{1}}, \quad f_{2}(x, u, v) \leq f_{2}(x, R, R) \leq \frac{R}{4 \mu_{2}} \tag{3.14}
\end{equation*}
$$

for any $x \in[0,1],(u, v) \in \bar{\Omega}_{R}$. We claim that

$$
(u, v) \not \leq A(u, v), \quad \forall(u, v) \in \partial \Omega_{R} \cap(P \times P)
$$

If not, there is $(u, v) \in \partial \Omega_{R} \cap(P \times P)$ such that $(u, v) \leq A(u, v)$, then we have by (3.14),

$$
\begin{aligned}
& u(x) \leq \int_{0}^{1} G_{1}(1, y) h_{1}(y) f_{1}(y, u(y), v(y)) d y \leq \frac{R}{4} \\
& v(x) \leq \int_{0}^{1} G_{2}(1, y) h_{2}(y) f_{2}(y, u(y), v(y)) d y \leq \frac{R}{4}
\end{aligned}
$$

for $x \in[0,1]$. Hence $R=\|(u, v)\|=\|u\|+\|v\| \leq \frac{R}{2}$, which is a contradiction. As a result (3.13) is true. We have by (3.10) and (3.13),

$$
\begin{aligned}
& i\left(A,\left(\Omega_{R} \backslash \bar{\Omega}_{r}\right) \cap(P \times P), P \times P\right) \\
& \quad=i\left(A, \Omega_{R} \cap(P \times P), P \times P\right)-i\left(A, \Omega_{r} \cap(P \times P), P \times P\right)=1 .
\end{aligned}
$$

So $A$ has a fixed point on $\left(\Omega_{R} \backslash \bar{\Omega}_{r}\right) \cap(P \times P)$. This means that the system (1.1)-(1.2) has at least one positive solution.

Theorem 3.2. Assume that the conditions $\left(H_{1}\right),\left(H_{2}\right),\left(H_{5}\right)$ and $\left(H_{6}\right)$ are satisfied. Then the system (1.1)-(1.2) has at least one positive solution.

Proof. By $\left(H_{5}\right)$, there are $\xi_{2}>0, \eta_{2}>0, C_{5}>0, C_{6}>0$ and $C_{7}>0$ such that

$$
f_{1}(x, u, v) \geq \xi_{2} p(v)-C_{5}, \quad f_{2}(x, u, v) \geq \eta_{2} q(u)-C_{6}, \quad(x, u, v) \in J_{\theta} \times \mathbb{R}^{+} \times \mathbb{R}^{+},
$$

and

$$
\begin{equation*}
p\left(K_{2} q(u)\right) \geq \frac{2 K_{2}}{\xi_{2} \eta_{2} \delta_{1} \delta_{2} \gamma^{3}} u-C_{7}, \quad u \in \mathbb{R}^{+}, \tag{3.15}
\end{equation*}
$$

where $K_{2}=\max \left\{\eta_{2} \gamma G_{2}(1, y) h_{2}(y): y \in J_{\theta}\right\}$. Then we have

$$
\begin{array}{ll}
A_{1}(u, v)(x) \geq \xi_{2} \int_{0}^{1} G_{1}(x, y) h_{1}(y) p(v(y)) d y-C_{8}, & x \in J_{\theta}, \\
A_{2}(u, v)(x) \geq \eta_{2} \int_{0}^{1} G_{2}(x, y) h_{2}(y) q(u(y)) d y-C_{9}, & x \in J_{\theta} . \tag{3.16}
\end{array}
$$

We affirm that the set

$$
W=\{(u, v) \in P \times P:(u, v)=A(u, v)+\lambda(\varphi, \varphi), \lambda \geq 0\}
$$

is bounded, where $\varphi \in P \backslash\{0\}$. Indeed, $(u, v) \in W$ implies that $u \geq A_{1}(u, v), v \geq A_{2}(u, v)$ for some $\lambda \geq 0$. We have by (3.16),

$$
\begin{array}{ll}
u(x) \geq \xi_{2} \int_{0}^{1} G_{1}(x, y) h_{1}(y) p(v(y)) d y-C_{8}, & x \in J_{\theta} \\
v(x) \geq \eta_{2} \int_{0}^{1} G_{2}(x, y) h_{2}(y) q(u(y)) d y-C_{9}, & x \in J_{\theta} . \tag{3.18}
\end{array}
$$

By the monotonicity and concavity of $p(\cdot)$ as well as Jensen's inequality, (3.18) implies that

$$
\begin{align*}
p\left(v(x)+C_{9}\right) & \geq p\left(\int_{0}^{1} \eta_{2} G_{2}(x, y) h_{2}(y) q(u(y)) d y\right) \\
& \geq \int_{0}^{1} p\left(\eta_{2} \gamma G_{2}(1, y) h_{2}(y) q(u(y))\right) d y  \tag{3.19}\\
& \geq \eta_{2} \gamma K_{2}^{-1} \int_{\theta}^{1-\theta} G_{2}(1, y) h_{2}(y) p\left(K_{2} q(u(y))\right) d y, \quad x \in J_{\theta} .
\end{align*}
$$

Since $p(v(x)) \geq p\left(v(x)+C_{9}\right)-p\left(C_{9}\right)$, we have by (3.15), (3.17) and (3.19),

$$
\begin{align*}
u(x) & \geq \xi_{2} \gamma \int_{0}^{1} G_{1}(1, y) h_{1}(y)\left[p\left(v(y)+C_{9}\right)-p\left(C_{9}\right)\right] d y-C_{8} \\
& \geq \xi_{2} \gamma \int_{\theta}^{1-\theta} G_{1}(1, y) h_{1}(y) p\left(v(y)+C_{9}\right) d y-C_{10} \\
& \geq \xi_{2} \eta_{2} \gamma^{2} K_{2}^{-1} \int_{\theta}^{1-\theta} G_{1}(1, y) h_{1}(y) \int_{\theta}^{1-\theta} G_{2}(1, z) h_{2}(z) p\left(K_{2} q(u(z))\right) d z d y-C_{10}  \tag{3.20}\\
& \geq \xi_{2} \eta_{2} \gamma^{2} \delta_{1} K_{2}^{-1} \int_{\theta}^{1-\theta} G_{2}(1, z) h_{2}(z) p\left(K_{2} q(u(z))\right) d z-C_{10} \\
& \geq 2\left(\delta_{2} \gamma\right)^{-1} \int_{\theta}^{1-\theta} G_{2}(1, z) h_{2}(z) u(z) d z-C_{11} \geq 2\|u\|-C_{11}, \quad x \in J_{\theta} .
\end{align*}
$$

Hence $\|u\| \leq C_{11}$.
Since $p(v(x)) \geq \gamma p(\|v\|)$ for $x \in J_{\theta}, v \in P$, it follows from (3.19), (3.15) and (3.17) that

$$
\begin{aligned}
p(v(x)) & \geq p\left(v(x)+C_{9}\right)-p\left(C_{9}\right) \\
& \geq \eta_{2} \gamma K_{2}^{-1} \int_{\theta}^{1-\theta} G_{2}(1, y) h_{2}(y) p\left(K_{2} q(u(y))\right) d y-p\left(C_{9}\right) \\
& \geq \frac{2}{\xi_{2} \delta_{1} \delta_{2} \gamma^{2}} \int_{\theta}^{1-\theta} G_{2}(1, y) h_{2}(y) u(y) d y-C_{12} \\
& \geq \frac{2}{\delta_{1} \delta_{2} \gamma} \int_{\theta}^{1-\theta} G_{2}(1, y) h_{2}(y) d y \int_{0}^{1} G_{1}(1, z) h_{1}(z) p(v(z)) d z-C_{13} \\
& \geq 2 \delta_{1}^{-1} \int_{\theta}^{1-\theta} G_{1}(1, z) h_{1}(z) p(\|v\|) d z-C_{13} \\
& =2 p(\|v\|)-C_{13}, \quad x \in J_{\theta} .
\end{aligned}
$$

Hence $p(\|v\|) \leq C_{13}$. By (1) and (3) of the condition ( $H_{5}$ ), we know that $\lim _{v \rightarrow+\infty} p(v)=+\infty$, thus there exists $C_{14}>0$ such that $\|v\| \leq C_{14}$. This shows $W$ is bounded. Then there exists a sufficiently large $Q>0$ such that

$$
(u, v) \neq A(u, v)+\lambda(\varphi, \varphi), \quad \forall(u, v) \in \partial \Omega_{Q} \cap(P \times P), \lambda \geq 0 .
$$

Lemma 2.5 yields that

$$
\begin{equation*}
i\left(A, \Omega_{Q} \cap(P \times P), P \times P\right)=0 \tag{3.21}
\end{equation*}
$$

On the other hand, by $\left(H_{6}\right)$, there is a $\sigma>0$ and sufficiently small $\rho>0$ such that

$$
\begin{array}{ll}
f_{1}(x, u, v) \leq \sigma v^{s}, & \forall(x, u) \in[0,1] \times \mathbb{R}^{+}, v \in[0, \rho], \\
f_{2}(x, u, v) \leq \varepsilon_{1} u^{\frac{1}{s}}, & \forall(x, v) \in[0,1] \times \mathbb{R}^{+}, u \in[0, \rho] . \tag{3.22}
\end{array}
$$

where

$$
\varepsilon_{1}=\min \left\{\left(2 \sigma \mu_{1} \mu_{2}^{s}\right)^{-\frac{1}{s}}, \mu_{2}^{-1}\right\} .
$$

We claim that

$$
\begin{equation*}
(u, v) \npreceq A(u, v), \quad \forall(u, v) \in \partial \Omega_{\rho} \cap(P \times P) . \tag{3.23}
\end{equation*}
$$

If not, there exists a $(u, v) \in \partial \Omega_{\rho} \cap(P \times P)$ such that $(u, v) \leq A(u, v)$, that is, $u \leq$ $A_{1}(u, v), v \leq A_{2}(u, v)$. Then (3.22) implies that

$$
\begin{align*}
u(x) & \leq \int_{0}^{1} G_{1}(x, y) h_{1}(y) f_{1}(y, u(y), v(y)) d y \\
& \leq \sigma \int_{0}^{1} G_{1}(1, y) h_{1}(y) v^{s}(y) d y \\
& \leq \sigma \int_{0}^{1} G_{1}(1, y) h_{1}(y)\left(\int_{0}^{1} G_{2}(y, z) h_{2}(z) f_{2}(z, u(z), v(z)) d z\right)^{s} d y \\
& \leq \sigma \int_{0}^{1} G_{1}(1, y) h_{1}(y) d y\left(\int_{0}^{1} G_{2}(1, z) h_{2}(z) f_{2}(z, u(z), v(z)) d z\right)^{s}  \tag{3.24}\\
& =\sigma \mu_{1}\left(\int_{0}^{1} G_{2}(1, z) h_{2}(z) f_{2}(z, u(z), v(z)) d z\right)^{s} \\
& \leq \sigma \mu_{1} \varepsilon_{1}^{s}\left(\int_{0}^{1} G_{2}(1, z) h_{2}(z) u^{\frac{1}{s}}(z) d z\right)^{s} \\
& \leq \sigma \mu_{1} \varepsilon_{1}^{s} \mu_{2}^{s}\|u\| \leq \frac{1}{2}\|u\|, \quad x \in[0,1],
\end{align*}
$$

and

$$
\begin{align*}
v(x) & \leq \int_{0}^{1} G_{2}(x, y) h_{2}(y) f_{2}(y, u(y), v(y)) d y \\
& \leq \varepsilon_{1} \int_{0}^{1} G_{2}(1, y) h_{2}(y) u^{\frac{1}{s}}(y) d y  \tag{3.25}\\
& \leq \varepsilon_{1} \mu_{2}\|u\|^{\frac{1}{s}} \leq\|u\|^{\frac{1}{s}}, \quad x \in[0,1] .
\end{align*}
$$

(3.24) and (3.25) imply that $\|(u, v)\|=0$, which contradicts $\|(u, v)\|=\rho$ and the inequality (3.23) holds. Lemma 2.6 yields that

$$
\begin{equation*}
i\left(A, \Omega_{\rho} \cap(P \times P), P \times P\right)=1 \tag{3.26}
\end{equation*}
$$

We have by (3.21) and (3.26),

$$
\begin{aligned}
& i\left(A,\left(\Omega_{Q} \backslash \bar{\Omega}_{\rho}\right) \cap(P \times P), P \times P\right) \\
& \quad=i\left(A, \Omega_{Q} \cap(P \times P), P \times P\right)-i\left(A, \Omega_{\rho} \cap(P \times P), P \times P\right)=-1 .
\end{aligned}
$$

Hence $A$ has a fixed point on $\left(\Omega_{Q} \backslash \bar{\Omega}_{\rho}\right) \cap(P \times P)$. This means that the system (1.1)-(1.2) has at least one positive solution.

## 4 Existence of multiple positive solutions

Theorem 4.1. Assume that the conditions $\left(H_{1}\right),\left(H_{2}\right),\left(H_{3}\right),\left(H_{5}\right)$ and $\left(H_{8}\right)$ hold. Then the system (1.1)-(1.2) has at least two positive solutions.

Proof. We may take $Q>R>\rho$ such that both (3.5), (3.13) and (3.21) hold. Then we have

$$
\begin{aligned}
& i\left(A,\left(\Omega_{Q} \backslash \bar{\Omega}_{R}\right) \cap(P \times P), P \times P\right) \\
& \quad=i\left(A, \Omega_{Q} \cap(P \times P), P \times P\right)-i\left(A, \Omega_{R} \cap(P \times P), P \times P\right)=-1 \\
& i\left(A,\left(\Omega_{R} \backslash \bar{\Omega}_{\rho}\right) \cap(P \times P), P \times P\right) \\
& \quad=i\left(A, \Omega_{R} \cap(P \times P), P \times P\right)-i\left(A, \Omega_{\rho} \cap(P \times P), P \times P\right)=1 .
\end{aligned}
$$

Hence $A$ has a fixed point on $\left(\Omega_{Q} \backslash \bar{\Omega}_{R}\right) \cap(P \times P)$ and $\left(\Omega_{R} \backslash \bar{\Omega}_{\rho}\right) \cap(P \times P)$, respectively. This means the system (1.1)-(1.2) has at least two positive solutions.

Theorem 4.2. Assume that the conditions $\left(H_{1}\right),\left(H_{2}\right),\left(H_{4}\right),\left(H_{6}\right)$ and $\left(H_{7}\right)$ hold. Then the system (1.1)-(1.2) has at least two positive solutions.

Proof. We may take $G>r>\rho$ such that both (3.9), (3.10) and (3.26) hold. Then we have

$$
\begin{aligned}
& i\left(A,\left(\Omega_{G} \backslash \bar{\Omega}_{r}\right) \cap(P \times P), P \times P\right) \\
& \quad=i\left(A, \Omega_{G} \cap(P \times P), P \times P\right)-i\left(A, \Omega_{r} \cap(P \times P), P \times P\right)=1 \\
& \quad i\left(A,\left(\Omega_{r} \backslash \bar{\Omega}_{\rho}\right) \cap(P \times P), P \times P\right) \\
& \quad=i\left(A, \Omega_{r} \cap(P \times P), P \times P\right)-i\left(A, \Omega_{\rho} \cap(P \times P), P \times P\right)=-1
\end{aligned}
$$

Hence $A$ has a fixed point on $\left(\Omega_{G} \backslash \bar{\Omega}_{r}\right) \cap(P \times P)$ and $\left(\Omega_{r} \backslash \bar{\Omega}_{\rho}\right) \cap(P \times P)$, respectively. This means the system (1.1)-(1.2) has at least two positive solutions.

## 5 The nonexistence of positive solutions

Theorem 5.1. Assume that the conditions $\left(H_{1}\right)$ and $\left(H_{2}\right)$ hold, and

$$
f_{1}(x, u, v)>\left(\gamma^{2} \delta_{1}\right)^{-1} v, \quad f_{2}(x, u, v)>\left(\gamma^{2} \delta_{2}\right)^{-1} u, \quad \forall x \in[0,1], u>0, v>0
$$

Then the system (1.1)-(1.2) has no positive solution.

Proof. Assume that $(u, v)$ is a positive solution of the system (1.1)-(1.2), then $(u, v) \in P \times P$, $u(x)>0, v(x)>0$ for $x \in(0,1)$, and for $x \in J_{\theta}$,

$$
\begin{aligned}
u(x) & =\int_{0}^{1} G_{1}(x, y) h_{1}(y) f_{1}(y, u(y), v(y)) d y \\
& \geq \gamma_{\alpha} \int_{0}^{1} G_{1}(1, y) h_{1}(y) f_{1}(y, u(y), v(y)) d y \\
& >\gamma\left(\gamma^{2} \delta_{1}\right)^{-1} \int_{0}^{1} G_{1}(1, y) h_{1}(y) v(y) d y \\
& \geq \gamma^{2}\left(\gamma^{2} \delta_{1}\right)^{-1} \int_{\theta}^{1-\theta} G_{1}(1, y) h_{1}(y) d y\|v\|=\|v\|
\end{aligned}
$$

Hence $\|u\|>\|v\|$. Similarly, $\|v\|>\|u\|$, which is a contradiction.

Similarly, we can obtain the following result.
Theorem 5.2. Assume that $\left(H_{1}\right),\left(H_{2}\right)$ hold, and $f_{1}(x, u, v)<\mu_{1}^{-1} v, f_{2}(x, u, v)<\mu_{2}^{-1} u$ for any $x \in[0,1], u>0, v>0$, then the system (1.1)-(1.2) has no positive solution.

Remark 5.3. If $h_{j} \in C\left([0,1], \mathbb{R}^{+}\right)(j=1,2)$ and $1 \leq \mu, v \leq n-2(n \geq 3)$ in the system (1.1)-(1.2), all our results are still true.

## 6 Examples

In the following examples 6.1-6.4, we select $\alpha=\beta \in(n-1, n], \mu=v \in[1, n-3]$ for $n>3$ and $\eta_{1}=\eta_{2}, \xi_{1}=\xi_{2}, d_{1}=1-\eta_{1} \xi_{1}^{\alpha-\mu-1}$ in the system (1.1)-(1.2).
Example 6.1. Let $h_{1}(x)=h_{2}(x)=d_{1} \Gamma(\alpha) /(1-x)^{\alpha-\mu-1}, x \in(0,1), f_{1}(x, u, v)=e^{x}\left(1+e^{-(u+v)}\right)$, $f_{2}(x, u, v)=1-e^{-(u+v)}, x \in[0,1], u, v \in \mathbb{R}^{+}, a(v)=v^{\frac{1}{2}}, b(u)=u^{\frac{1}{2}}, t=1 / 2$. Clearly,

$$
0<\int_{0}^{1} G_{j}(1, y) h_{j}(y) d y \leq 1
$$

but $\int_{0}^{1} h_{j}(y) d y=+\infty(j=1,2)$. The results of $[7,9,10,17,24,29,30]$ are not suitable for the problem. It is easy to verify that the conditions $\left(H_{1}\right)-\left(H_{4}\right)$ hold, hence Theorem 3.1 implies that the system (1.1)-(1.2) has at least one positive solution. Here $f_{1}(x, u, v)$ and $f_{2}(x, u, v)$ are sublinear on $u$ and $v$ at 0 and $+\infty$.

Example 6.2. Let $h_{j}(x)$ be as in the Example 6.1, $f_{1}(x, u, v)=e^{x}\left(1+e^{-(u+v)}\right), f_{2}(x, u, v)=$ $u^{\frac{3}{2}}, a(v)=v^{\frac{1}{3}}, b(u)=u^{2}, t=1 / 2$. It is easy to verify that the conditions $\left(H_{1}\right)-\left(H_{4}\right)$ hold, Theorem 3.1 implies that the system (1.1)-(1.2) has at least one positive solution. Here $f_{1}(x, u, v)$ is sublinear on $u$ and $v$ at 0 and $+\infty$, whereas $f_{2}(x, u, v)$ is superlinear on $u$ at 0 and $+\infty$.

Example 6.3. Let $h_{j}(x)$ be as in the Example 6.1, $f_{1}(x, u, v)=\left(1+e^{-u}\right) v^{3}, f_{2}(x, u, v)=u^{3}$, $p(v)=v^{\frac{1}{2}}, q(u)=u^{3}, s=3$. It is easy to verify that the conditions $\left(H_{1}\right),\left(H_{2}\right),\left(H_{5}\right)$ and $\left(H_{6}\right)$ hold. Theorem 3.2 yields that the system (1.1)-(1.2) has at least one positive solution. Here $f_{1}(x, u, v)$ is superlinear on $v$ at 0 and $+\infty, f_{2}(x, u, v)$ is superlinear on $u$ at 0 and $+\infty$.
Example 6.4. Let $h_{j}(x)$ be as in the Example 6.1, $f_{1}(x, u, v)=\left(1+e^{-u}\right) v^{\frac{2}{3}}, f_{2}(x, u, v)=$ $\left(1+e^{-v}\right) u^{5}, p(v)=v^{\frac{1}{3}}, q(u)=u^{4}, s=1 / 3$. It is easy to see that the conditions $\left(H_{1}\right),\left(H_{2}\right),\left(H_{5}\right)$ and $\left(H_{6}\right)$ hold. Theorem 3.2 yields that the system (1.1)-(1.2) has at least one positive solution. Here $f_{1}(x, u, v)$ is sublinear on $v$ at 0 and $+\infty$, whereas $f_{2}(x, u, v)$ is superlinear on $u$ at 0 and $+\infty$.

Remark 6.5. From the Examples 6.1-6.4 we know that the conditions $\left(H_{3}\right)-\left(H_{6}\right)$ are applicable for more general function and it is not included among the known differential system. Hence our results are different from those in [7,9,10,17,24,29,30,33].

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## References

[1] B. Ahmad, J. J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, Comput. Math. Appl. 58(2009), No. 9, 1838-1843. MR2557562; url
[2] Z. B. BAI, H. S. Lu, Positive solutions for boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl. 311(2005), No. 2, 495-505. MR2168413; url
[3] Z. B. Bai, On positive solutions of a nonlocal fractional boundary value problem, Nonlinear Anal. 72(2010), No. 2, 916-924. MR2579357; url
[4] A. Cabada, J. Á. Cid, G. Infante, A positive fixed point theorem with applications to systems of Hammerstein integral equations, Bound. Value Probl. 2014, No. 254, 10 pp. MR3286705; url
[5] M. Q. Feng, X. M. Zhang, W. G. Ge, New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions, Bound. Value Probl. 2011, Art. ID 720702, 20 pp. MR2679690; url
[6] C. S. Goodrich, Existence of a positive solution to a class of fractional differential equations, Appl. Math. Lett. 23(2010), No. 9, 1050-1055. MR2659137; url
[7] C. S. Goodrich, Existence of a positive solution to systems of differential equations of fractional order, Comput. Math. Appl. 62(2011), No. 3, 1251-1268. MR2824712; url
[8] D. Guo, V. Laksmikantham, Nonlinear problems in abstract cones, Academic Press, Boston, New York, 1988. MR0959889
[9] J. Henderson, R. Luca, Positive solution for a system of nonlocal fractional boundary value problems, Fract. Calc. Appl. Anal. 16(2013), No. 4, 985-1008. MR3124348; url
[10] J. Henderson, R. Luca, Positive solutions for a system of fractional differential equations with coupled integral boundary conditions, Appl. Math. Comput. 249(2014), 182-197. MR3279412; url
[11] G. Infante, P. Pietramala, Existence and multiplicity of non-negative solutions for systems of perturbed Hammerstein integral equations, Nonlinear Anal. 71(2009), No. 3-4, 1301-1310. MR2527550; url
[12] W. H. Jiang, J. Q. Qiu, W. W. Guo, The existence of positive solutions for fractional differential equations with sign changing nonlinearities, Abstr. Appl. Anal. 2012, Art. ID 180672, 13 pp. MR2947739; url
[13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies, Vol. 204, Elsevier, Amsterdam, 2006. MR2218073
[14] M. A. Krasnoselskir, Positive solution of operator equations, Noordhoff, Groningen, 1964. MR0181881
[15] K. Q. Lan, W. Lin, Multiple positive solutions of systems of Hammerstein integral equations with applications to fractional differential equations, J. Lond. Math. Soc. 83(2011), No. 2, 449-469. MR2776646; url
[16] C. F. Li, X. N. Luo, Y. Zhou, Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations, Comput. Math. Appl. 59(2010), No. 3, 1363-1375. MR2579500; url
[17] W. N. Liu, X. J. Yan, W. Qi, Positive solutions for coupled nonlinear fractional differential equations, J. Appl. Math. 2014, Art. ID 790862, 7 pp. MR3212514; url
[18] M. Ur Rehman, R. Ali Khan, Positive solutions to nonlinear higher-order nonlocal boundary value problems for fractional differential equations, Abstr. Appl. Anal. 2010, Art. ID 501230, 15 pp. MR2739683; url
[19] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional integrals and derivatives: theory and applications, Gordon and Breach, Yverdon, 1993. MR1347689
[20] M. El-Shahed, Existence of positive solutions of the boundary value problems for nonlinear fractional differential equations, Int. Math. Forum 6(2011), No. 25-28, 1371-1385. MR2820235; url
[21] X. W. Su, Boundary value problem for a coupled system of nonlinear fractional differential equations, Appl. Math. Lett. 22(2009), No. 1, 64-69. MR2483163; url
[22] Y. S. Tian, Y. Zhou, Positive solutions for multipoint boundary value problem of fractional differential equations, J. Appl. Math. Comput. 38(2012), 417-427. MR2886690; url
[23] Y. S. Tian, Positive solutions to m-point boundary value problem of fractional differential equation, Acta Math. Appl. Sin. Engl. Ser. 29(2013), No. 3, 661-672. MR3129832; url
[24] J. H. Wang, H. J. Xiang, Z. G. Liu, Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations, Int. J. Differ. Equ. 2010, Art. ID 186928, 12 pp. MR2525726; url
[25] L. Wang, X. Q. Zhang, Positive solutions of m-point boundary value problems for a class of nonlinear fractional differential equations, J. Appl. Math. Comput. 42(2013), 387-399. MR3061835; url
[26] Y. Wang, L. S. Liu, X. Q. Zhang, Y. H. Wu, Positive solutions for ( $n-1,1$ )-type singular fractional differential system with coupled integral boundary conditions, Abstr. Appl. Anal. 2014, Art. ID 142391, 14 pp. MR3278339; url
[27] S. L. XIE, J. ZhU, Positive solutions of the system for $n$ th-order singular nonlocal boundary value problems, J. Appl. Math. Comput. 37(2011), 119-132. MR2831528; url
[28] J. F. Xu, Z. L. Yang, Positive solutions for a system of nonlinear Hammerstein integral equations and applications, J. Integral Equations Appl. 24(2012), No. 1, 131-147. MR2911093; url
[29] W. G. Yang, Positive solutions for a coupled system of nonlinear fractional differential equations with integral boundary conditions, Comput. Math. Appl. 63(2012), No. 1, 288-297. MR2863500; url
[30] C. B. Zhai, M. R. Hao, Multiple positive solutions to nonlinear boundary value problems of a system for fractional differential equations, The Scientific World Journal 2014, Art. ID 817542, 11 pp. url
[31] S. Q. Zhang, Existence results of positive solutions to fractional differential equation with integral boundary conditions, Math. Bohem. 135(2010), No. 2, 299-317. MR2683641
[32] X. Z. Zhang, C. X. Zhu, Z. Q. Wu, Solvability for a coupled system of fractional differential equations with impulses at resonance, Bound. Value Probl. 2013, No. 80, 23 pp. MR3053805; url
[33] Y. G. Zhao, S. R. Sun, Z. L. Han, W. Q. Feng, Positive solutions for a coupled system of nonlinear differential equations of mixed fractional orders, Adv. Difference Equ. 2011, No. 10, 13 pp. MR2820284; url
[34] C. X. Zhu, X. Z. Zhang, Z. Q. Wu, Solvability for a coupled system of fractional differential equations with nonlocal integral boundary conditions, Taizanese J. Math. 17(2013), No. 6, 2039-2054. MR3141873; url


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