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# Nonoscillation of higher order half-linear differential equations

**Ondřej Došlý**✉ and **Vojtěch Růžička**

Department of Mathematics and Statistics, Masaryk University,
Kotlářská 2, CZ-611 37 Brno, Czech Republic

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**Abstract.** We establish nonoscillation criteria for even order half-linear differential equations. The principal tool we use is the Wirtinger type inequality combined with various perturbation techniques. Our results extend nonoscillation criteria known for linear higher order differential equations.

**Keywords:** even-order half-linear differential equation, Wirtinger inequality, nonoscillation, half-linear Euler equation.

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## 1 Introduction

In this paper we deal with the even order half-linear differential equation

$$\sum_{k=0}^{n}(-1)^k\Big(r_k(t)\Phi\big(y^{(k)}\big)\Big)^{(k)}=0 \tag{1.1}$$

where $\Phi(y) = |y|^{p-2}y$, $p > 1$, is the odd power function, $r_j$ are continuous functions, $j = 0,\dots,n$, and $r_n(t) > 0$ in the interval under consideration. The terminology *half-linear* equation was introduced by I. Bihari [3] and reflects the fact that the solution space of (1.1) is homogeneous, but not additive, i.e., it has just one half of the properties characterizing linearity. In the case $n = 1$, equation (1.1) reduces to the classical second order half-linear differential equation

$$-\big(r_1(t)\Phi(x')\big)'+r_0(t)\Phi(x)=0 \tag{1.2}$$

whose oscillation theory is relatively deeply developed, see [1, 16] and e.g. the recent papers [11, 13, 17, 19, 24, 27, 28].

The theory of (1.1) is much less developed and as far as we known only [16, Sec. 9.4] and the paper [25] deal with this problem. The reason is that we miss the so-called Reid's roundabout theorem in the higher order case, in particular, the Riccati technique is not available for

✉Corresponding author. Email: dosly@math.muni.cz

(1.1), in contrast to (1.2). Actually, necessary and sufficient conditions for (non)oscillation of (1.1) with $p = 2$, i.e., in the *linear case*, follow from the fact that this equation can be written as a linear Hamiltonian system (for which the Reid's roundabout theorem is well known, [26, Chap. V., Theorem 6.3]) and this enables to present oscillation and spectral theory of (1.1) with $p = 2$ as it is exhibited e.g. in the book [22], see also [20] and the references given therein.

The energy functional associated with (1.1) considered on the interval $[T, \infty)$ is

$$\mathcal{F}_n(y) = \int_T^\infty \left[ \sum_{k=0}^n r_k(t) |y^{(k)}|^p \right] dt \tag{1.3}$$

(equation (1.1) is the Euler–Lagrange equation of (1.3)). If there exists a nontrivial solution $\tilde{y}$ of (1.1) with two zeros of multiplicity $n$ in $[T, \infty)$, i.e.,

$$\tilde{y}^{(i)}(t_1) = 0 = \tilde{y}^{(i)}(t_2), \qquad i = 0, \dots, n-1, \tag{1.4}$$

for some $T \le t_1 < t_2$, then we define the function

$$y(t) = \begin{cases} \tilde{y}(t), & t \in [t_1, t_2] \\ 0 & t \in [T, \infty) \setminus [t_1, t_2], \end{cases}$$

and obviously $y \in W_0^{n,p}[T, \infty)$ (the definition of this Sobolev space will be recalled later). Multiplying (1.1) by $y$ and integrating by parts over $[T, \infty)$ gives $\mathcal{F}_n(y) = 0$. Hence, if we show that $\mathcal{F}_n(y) > 0$ for all nontrivial functions $y \in W_0^{n,p}[T, \infty)$, we eliminate the existence of a solution of (1.1) satisfying (1.4) for some $t_1, t_2 \in [T, \infty)$.

The paper is organized as follows. In the next section we concentrate our attention on basic properties of the higher order half-linear Euler differential equation and on the so-called Wirtinger inequality which is the principal tool in our investigation. Section 3 is devoted to nonoscillation criteria for Euler type even order differential equation. Section 4 deals with nonoscillation criteria for general two-term $2n$th order half-linear differential equations and in the last section we present some remarks and comments concerning possible further investigation.

## 2 Preliminaries and Euler equation

The higher order Euler type half-linear differential equation is the equation

$$(-1)^n \big(t^\alpha \Phi(y^{(n)})\big)^{(n)} + (-1)^{n-1} \beta_{n-1} \big(t^{\alpha-p} \Phi(y^{(n-1)})\big)^{(n-1)} + \dots + \beta_0 t^{\alpha-np} \Phi(y) = 0, \tag{2.1}$$

where $\alpha$, $\beta_i$, $i = 0, \dots, n-1$, are real constants. Moreover, it is supposed that $\alpha \notin \{p-1, 2p-1, \dots, np-1\}$ (this restriction will be explained later).

The "classical" Euler second order half-linear differential equation is the equation

$$-\big(\Phi(x')\big)' + \frac{\gamma}{t^p} \Phi(x) = 0. \tag{2.2}$$

This equation and its various perturbations were studied in detail in [18] and also in [11, 13, 14, 17, 24, 27]. It is known that the classical linear Sturmian oscillation theory extends almost verbatim to (1.2). Elbert [18] showed that (2.2) is oscillatory if and only if $\gamma < -\gamma_p$,

$\gamma_p := \left(\frac{p-1}{p}\right)^p$. In the critical case $\gamma = -\gamma_p$, equation (1.2) has a solution $x(t) = t^{\frac{p-1}{p}}$ as can be verified by a direct computation.

Concerning equation (2.1), similarly to the linear case, we look for a solution in the form $x(t) = t^\lambda$. Consider first the two-term equation

$$(-1)^n \left(t^\alpha \Phi(x^{(n)})\right)^{(n)} + \gamma t^{\alpha - np}\Phi(x) = 0, \tag{2.3}$$

with $\alpha \notin \{p-1, \dots, np-1\}$ and $\gamma \in \mathbb{R}$. Substituting into (2.3) we find that $\lambda$ must be a root of the algebraic equation $G(\lambda) + \gamma = 0$ with

$$G(\lambda) = (-1)^n \Phi\big(\lambda(\lambda-1)\cdots(\lambda-n+1)\big)\big[(p-1)(\lambda-n)+\alpha\big]\cdots\big[(p-1)(\lambda-n)+\alpha-n+1\big].$$

Next we show that the function $G$ has a stationary point $\lambda^* = \frac{np-1-\alpha}{p}$. We have the equality $\Phi'(x) = (p-1)\frac{\Phi(x)}{x}$, therefore, by a direct calculation we obtain that for $\lambda \neq j, n - \frac{\alpha - j}{p-1}$, $j = 0, \dots, n-1$,

$$G'(\lambda) = (-1)^n (p-1) G(\lambda) \left[\frac{1}{\lambda} + \frac{1}{\lambda - 1} + \cdots + \frac{1}{\lambda - (n-1)} + \frac{1}{(p-1)(\lambda-n)+\alpha} \right.$$
$$\left. + \frac{1}{(p-1)(\lambda-n)+\alpha-1} + \cdots + \frac{1}{(p-1)(\lambda-n)+\alpha-(n-1)}\right].$$

Because

$$\frac{1}{\lambda^* - k} = -\frac{1}{(p-1)(\lambda^*-n)+\alpha-(n-1-k)}$$

for each $k \in \{0, \dots, n-1\}$, we have

$$G'(\lambda^*) = 0.$$

Substituting the value $\lambda^*$ into $G$ gives the value of the so-called *critical constant* in the $2n$th order Euler half-linear differential equation (2.3). We denote

$$\gamma_{n,p,\alpha} := G(\lambda^*) = \prod_{j=1}^{n} \left(\frac{|jp-1-\alpha|}{p}\right)^p.$$

The previous computation shows that the equation $G(\lambda) - \gamma_{n,p,\alpha} = 0$ has a double root $\lambda^* = \frac{np-1-\alpha}{p}$.

The terminology critical constant is used by analogy with the linear case where its value is a "borderline" between oscillation and nonoscillation of equation (2.3) with $p = 2$. In the half-linear case, we are able to prove only "one half" of conditions for an oscillation constant yet, namely that (2.3) is nonoscillatory for $\gamma > -\gamma_{n,p,\alpha}$. The proof of an "oscillation counterpart" resists our effort till now, nevertheless, it is a subject of the present investigation. More details about this problem are given in the last section.

Therefore, (2.3) with $\gamma = -\gamma_{n,p,\alpha}$ has a solution $x(t) = t^{\lambda^*}$. Note that linearly independent solutions cannot be computed explicitly even in the case $n = 1$ and $\alpha = 0$ (i.e., for second order equation (2.2) with $\gamma = -\gamma_p$, because $\gamma_p = \gamma_{1,p,0}$). Nevertheless, as shown in [18], any solution of (2.2) with $\gamma = -\gamma_p$, which is linearly independent of $x(t) = t^{\frac{p-1}{p}}$ is asymptotically equivalent to the function $\tilde{x}(t) = Ct^{\frac{p-1}{p}} \log^{\frac{2}{p}} t$, $0 \neq C \in \mathbb{R}$. It is an open problem whether the function $\tilde{x}(t) = t^{\frac{np-1-\alpha}{p}} \log^{\frac{2}{p}} t$ is also an "approximate" solution of the equation

$$(-1)^n \left(t^\alpha \Phi(x^{(n)})\right)^{(n)} - \gamma_{n,p,\alpha} t^{\alpha-np}\Phi(x) = 0, \tag{2.4}$$

since if $p = 2$ in (2.4) then $\tilde{x}(t) = t^{\frac{2n-1-\alpha}{2}} \log t$ is a solution of this equation.

Now we recall the definition of the Sobolev space, consisting of functions with a compact support. We denote for $T \in \mathbb{R}$

$$W_0^{n,p}[T,\infty) = \Big\{ y\colon [T,\infty) \to \mathbb{R} \mid y^{(n-1)} \in \mathcal{AC}[T,\infty),\, y^{(n)} \in \mathcal{L}^p(T,\infty), \\ y^{(i)}(T) = 0 \text{ for } i = 0,1,\dots,n-1 \text{ and there exists } T_1 > T \\ \text{such that } y(t) = 0 \text{ for } t \geq T_1 \Big\},$$

where $\mathcal{AC}[T,\infty)$ is the set of absolutely continuous functions with the domain $[T,\infty)$.

We finish this section with a half-linear version of the classical Wirtinger inequality, which we use in the next sections. Its proof in the formulation presented here can be found in [7].

**Lemma 2.1.** *Let $M$ be a positive continuously differentiable function for which $M'(t) \neq 0$ in $[T,\infty)$ and let $y \in W_0^{1,p}[T,\infty)$. Then*

$$\int_T^\infty |M'(t)||y|^p \, dt \leq p^p \int_T^\infty \frac{M^p}{|M'(t)|^{p-1}} |y'|^p \, dt. \tag{2.5}$$

## 3 Euler equation

Following the linear terminology, we say that (1.1) is *nonoscillatory* if there exists $T \in \mathbb{R}$ such that no solution of this equation has two or more zeros of multiplicity $n$ in $[T,\infty)$. In the opposite case, i.e., when for every $T \in \mathbb{R}$ there exists a nontrivial solution of (1.1) with at least two zeros of multiplicity $n$ in $[T,\infty)$, then (1.1) is said to be *oscillatory*.

We start this section with a variational lemma which plays the fundamental role in our treatment, for its proof (whose outline we have already presented below (1.3)) see [16, Sec. 9.4].

**Lemma 3.1.** *Equation (1.1) is nonoscillatory if there exists $T \in \mathbb{R}$ such that*

$$\mathcal{F}_n(y) > 0$$

*for every $0 \not\equiv y \in W_0^{n,p}[T,\infty)$.*

The first statement of this section is a nonoscillation criterion which is essentially proved in [16, Theorem 9.4.5]. This criterion is formulated in [16] for the equation

$$(-1)^n \big(\Phi(x^{(n)})\big)^{(n)} + \frac{\gamma}{t^{np}} \Phi(x) = 0, \tag{3.1}$$

but a small modification of the proof (via Wirtinger inequality) shows that it can be extended to a more general equation (2.3).

**Theorem 3.2.** *Suppose that $\alpha \notin \{p-1,\dots,np-1\}$. If*

$$\gamma_{n,p,\alpha} + \gamma > 0, \qquad \gamma_{n,p,\alpha} = \prod_{j=1}^n \left( \frac{|jp-1-\alpha|}{p} \right)^p,$$

*then (2.3) is nonoscillatory.*

*Proof.* The proof is based on the application of the inequality

$$\int_T^\infty t^\alpha |y^{(n)}|^p\,dt \ge \gamma_{n,p,\alpha} \int_T^\infty t^{\alpha-np}|y|^p\,dt \tag{3.2}$$

for $y \in W_0^{n,p}[T,\infty)$, which is obtained by repeated application of the following Wirtinger inequality

$$\int_T^\infty t^\beta |x'|^p\,dt \ge \left(\frac{|p-1-\beta|}{p}\right)^p \int_T^\infty t^{\beta-p}|x|^p\,dt, \qquad x \in W_0^{1,p}[T,\infty) \tag{3.3}$$

for $\beta = \alpha, \alpha-p, \alpha-2p, \dots, \alpha-(n-1)p$ and for $x' = y^{(n)}, y^{(n-1)}, \dots, y'$ respectively. Inequality (3.3) follows from inequality (2.5) in Lemma 2.1 by taking $M(t) = (|p-1-\beta|)^{p-1}t^{\beta-p+1}$ for $\beta \ne p-1$. Then for any $y \in W_0^{n,p}[T,\infty)$ such that $y \not\equiv 0$ we have

$$\begin{aligned}\mathcal{F}_n(y) &= \int_T^\infty t^\alpha |y^{(n)}|^p\,dt + \gamma \int_T^\infty t^{\alpha-np}|y|^p\,dt \\ &\ge (\gamma_{n,p,\alpha}+\gamma)\int_T^\infty t^{\alpha-np}|y|^p\,dt > 0,\end{aligned}$$

what we needed to prove, due to Lemma 3.1. □

Note that the same statement (for $\alpha = 0$) is proved via the weighted Hardy inequality in [25], we will mention this result later in our paper.

Now we turn our attention to the "full term" $2n$th order Euler differential equation.

$$(-1)^n\left(t^\alpha\Phi(y^{(n)})\right)^{(n)} + (-1)^{n-1}\beta_{n-1}\left(t^{\alpha-p}\Phi(y^{(n-1)})\right)^{(n-1)} + \cdots + \beta_0 t^{\alpha-np}\Phi(y) = 0, \tag{3.4}$$

with $\alpha \notin \{p-1, 2p-1, \dots, np-1\}$.

**Theorem 3.3.** *Suppose that* $\alpha \notin \{p-1,\dots,np-1\}$ *and*

$$\sum_{k=0}^{n-1}\prod_{j=1}^{n-k}\left(\frac{|(k+j)p-1-\alpha|}{p}\right)^p \beta_{n-k} + \beta_0 > 0, \qquad \beta_n := 1,$$

*then equation (3.4) is nonoscillatory.*

*Proof.* We apply the Wirtinger inequality to each term (except that one for $k = n$) in the energy functional

$$\mathcal{F}_n(y) = \int_T^\infty \left(\sum_{k=0}^n t^{\alpha-kp}|y^{(n-k)}|^p\right) dt.$$

We obtain for any $y \in W_0^{n,p}[T,\infty)$ and for $k = 0,\dots,n-1$

$$\int_T^\infty t^{\alpha-pk}|y^{(n-k)}|^p\,dt \ge \prod_{j=1}^{n-k}\left(\frac{|(k+j)p-1-\alpha|}{p}\right)^p \int_T^\infty t^{\alpha-np}|y|^p\,dt.$$

Then we have

$$\mathcal{F}_n(y) \ge \left[\sum_{k=0}^{n-1}\prod_{j=1}^{n-k}\left(\frac{|(k+j)p-1-\alpha|}{p}\right)^p \beta_{n-k} + \beta_0\right]\int_T^\infty t^{\alpha-np}|y|^p\,dt > 0$$

for any nontrivial $y \in W_0^{n,p}[T,\infty)$. □

**Remark 3.4.** The reason why the case $\alpha \in \{p-1,\dots,np-1\}$ we needed to exclude from the previous considerations is the following. For $\alpha = p-1$ the Wirtinger inequality takes the form

$$\int_T^\infty t^{p-1}|y'|^p\,dt \ge \left(\frac{p-1}{p}\right)^p \int_T^\infty \frac{1}{t\log^p t}|y|^p\,dt, \tag{3.5}$$

so, a logarithmic term appears. This more difficult case is treated in the next part of this section.

We start with an auxiliary statement.

**Lemma 3.5.** *Let $\alpha = jp-1$ for some $j \in \{1,\dots,n\}$. Then, we have for any $y \in W_0^{n,p}[T,\infty)$*

$$\int_T^\infty t^\alpha |y^{(n)}|^p\,dt \ge \frac{[(n-j)!(j-1)!]^p}{\gamma_p^{n-j-1}} \int_T^\infty \frac{1+O(\log^{-1} t)}{t^{(n-j)p+1}\log^p t}|y|^p\,dt.$$

*Proof.* First we make some auxiliary computations. Integration by parts gives for $l \in \mathbb{N}$ and $q$ the conjugate exponent of $p$, i.e., $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}\int t^{lq-1}\log^q t\,dt &= \frac{t^{lq}}{lq}\log^q t - \frac{1}{l}\int t^{lq-1}\log^{q-1} t\,dt\\ &= \frac{t^{lq}}{lq}\log^q t\left[1+O(\log^{-1} t)\right]\end{aligned}$$

as $t\to\infty$. This integral we use in establishing the inequality for $z \in W_0^{1,p}[T,\infty)$

$$\int_T^\infty \frac{|z'|^p}{t^{lp+1}\log^p t}\,dt \ge \frac{(l+1)^p}{\gamma_p}\int_T^\infty \frac{1+O(\log^{-1} t)}{t^{(l+1)p+1}\log^p t}|z|^p\,dt. \tag{3.6}$$

We prove (3.6) as follows. Let $r(t)>0$ be a continuous function with $\int^\infty r^{1-q}(t)\,dt=\infty$, then we have the inequality

$$\int_T^\infty r(t)|y'|^p\,dt \ge \gamma_p \int_T^\infty \frac{r^{1-q}(t)}{\left(\int_{T_0}^t r^{1-q}(s)\,ds\right)^p}|y|^p\,dt, \qquad T_0<T, \tag{3.7}$$

which follows from (3.3) with $\beta=0$. Indeed, let $s=\int_{T_0}^t r^{1-q}(\tau)d\tau$, i.e., $\frac{d}{dt}=r^{1-q}(t)\frac{d}{ds}$, then (3.7) is the same as

$$\int_S^\infty |\dot y|^p\,ds \ge \gamma_p\int_S^\infty \frac{|y|^p}{s^p}\,ds, \qquad \dot{} = \frac{d}{ds}, \qquad S=\int_{T_0}^T r^{1-q}(\tau)d\tau.$$

For $r(t)=t^{-lp-1}\log^{-p} t$ we have $r^{1-q}(t)=t^{(l+1)q-1}\log^q t$, hence

$$\int^t r^{1-q}(s)\,ds = \frac{t^{(l+1)q}}{(l+1)q}\log^q t\left(1+O(\log^{-1} t)\right)$$

as $t\to\infty$. Therefore

$$\begin{aligned}\frac{r^{1-q}(t)}{\left(\int^t r^{1-q}(s)\,ds\right)^p} &= t^{(l+1)q-1}\log^q t\left(\frac{t^{(l+1)q}}{(l+1)q}\log^q t\right)^{-p}\left(1+O(\log^{-1} t)\right)^{-p}\\ &= \frac{(l+1)^p}{\gamma_p}\frac{1}{t^{(l+1)p+1}\log^p t}\left(1+O(\log^{-1} t)\right).\end{aligned}$$

Substituting these computations into (3.7) we obtain (3.6).

Let $y \in W_0^{n,p}[T,\infty)$. Applying inequalities (3.5) and (3.6), we obtain

$$\begin{aligned}\int_T^\infty t^\alpha |y^{(n)}|^p\,dt &= \int_T^\infty t^{jp-1}|y^{(n)}|^p\,dt \geq [(j-1)!]^p \int_T^\infty t^{p-1}|y^{(n-j+1)}|^p\,dt\\ &\geq [(j-1)!]^p\gamma_p \int_T^\infty \frac{1}{t\log^p t}|y^{(n-j)}|^p\,dt\\ &\geq \frac{[(n-j)!(j-1)!]^p}{\gamma_p^{n-j-1}}\int_T^\infty \frac{1+O(\log^{-1}t)}{t^{(n-j)p+1}\log^p t}|y|^p\,dt.\end{aligned}$$

The proof is complete. □

Now we are ready to deal with the case $\alpha \in \{p-1, 2p-1, \dots, np-1\}$.

**Theorem 3.6.** *Let $\alpha = jp-1$ for some $j \in \{1,\dots,n\}$ and consider the equation*

$$\begin{aligned}&(-1)^n\left(t^{jp-1}\Phi(y^{(n)})\right)^{(n)} + \sum_{i=1}^{j-1}(-1)^{n-i}\beta_{n-i}\left(t^{(j-i)p-1}\Phi(y^{(n-i)})\right)^{(n-i)}\\ &\quad+\sum_{i=0}^{n-j-1}(-1)^{n-j-i}\beta_{n-j-i}\left(\frac{\Phi(y^{(n-j-i)})}{t^{ip+1}\log^p t}\right)^{(n-j-i)} + \beta_0\frac{\Phi(y)}{t^{(n-j)p+1}\log^p t} = 0.\end{aligned} \tag{3.8}$$

*If*

$$\begin{aligned}L :=& \frac{[(j-1)!(n-j)!]^p}{\gamma_p^{n-j-1}} + \sum_{i=1}^{j-1}\beta_{n-i}\frac{[(j-i-1)!(n-j)!]^p}{\gamma_p^{n-j-1}}\\ &+\sum_{i=0}^{n-j-1}\beta_{n-j-i}\frac{[(i+1)\cdots(n-j)]^p}{\gamma_p^{n-j-i}} + \beta_0 > 0\end{aligned} \tag{3.9}$$

*then equation (3.8) is nonoscillatory.*

*Proof.* The energy functional corresponding to (3.8) is

$$\begin{aligned}\mathcal{F}_n(y) = \int_T^\infty \Bigg[& t^{jp-1}|y^{(n)}|^p + \sum_{i=1}^{j-1}\beta_{n-i}t^{(j-i)p-1}|y^{(n-i)}|^p\\ &+\sum_{i=0}^{n-j-1}\beta_{n-j-i}\frac{|y^{(n-j-i)}|^p}{t^{ip+1}\log^p t} + \beta_0\frac{|y|^p}{t^{(n-j)p+1}\log^p t}\Bigg]\,dt\end{aligned}$$

The first term in the integral is estimated in Lemma 3.5. Concerning the terms under summation signs, for $i = 0,\dots,j-1$

$$\int_T^\infty t^{(j-i)p-1}|y^{(n-i)}|^p\,dt \geq \frac{[(j-i-1)!(n-j)!]^p}{\gamma_p^{n-j-1}}\int_T^\infty \frac{1+O(\log^{-1}t)}{t^{(n-j)p+1}\log^p t}|y|^p\,dt$$

and for $i = 0,\dots,n-j-1$

$$\int_T^\infty \frac{|y^{(n-j-i)}|^p}{t^{ip+1}\log^p t}\,dt \geq \frac{[(i+1)\dots(n-j)]^p}{\gamma_p^{n-j-i}}\int_T^\infty \frac{1+O(\log^{-1}t)}{t^{(n-j)p+1}\log^p t}|y|^p\,dt.$$

Substituting these computations into $\mathcal{F}_n(y)$, we have

$$\mathcal{F}_n(y) = \int_T^\infty \frac{|y|^p}{t^{(n-j)p+1}\log^p t}\,dt \times \left[L + \left(\int_T^\infty \frac{O(\log^{-1} t)}{t^{(n-j)p+1}\log^p t}|y|^p\,dt\right)\left(\int_T^\infty \frac{|y|^p}{t^{(n-j)p+1}\log^p t}\,dt\right)^{-1}\right].$$

Since the second term in the bracket tends to zero as $T \to \infty$, we have $\mathcal{F}_n(y; T, \infty) > 0$ for $T$ sufficiently large if (3.9) holds, which means that equation (3.8) is nonocillatory by Lemma 3.1. □

## 4 General nonoscillation criteria

We start with two nonoscillation criteria from [25] (proved in [25] via the weighted Hardy inequality) which we later compare with our results. Both criteria are contained in the following theorem.

**Theorem 4.1.** *Suppose that $c(t) \le 0$ for large $t$ and $q$ is the conjugate exponent of $p$, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. If one of the following conditions*

$$\lim_{T\to\infty} \inf_{t>T} \left(\int_T^t r^{1-q}(s)\,ds\right)^{p-1} \int_t^\infty c(s)(s-T)^{(n-1)p}\,ds > -\frac{[(n-1)!]^p}{p-1}\gamma_p \tag{4.1}$$

*or*

$$\lim_{T\to\infty} \inf_{t>T} \left(\int_T^t r^{1-q}(s)\,ds\right)^{-1} \int_T^t c(s)(s-T)^{(n-1)p}\left(\int_T^s r^{1-q}(u)\,du\right)^p ds > -\gamma_p[(n-1)!]^p, \tag{4.2}$$

*holds, then the two-term differential equation*

$$(-1)^n\big(r(t)\Phi(y^{(n)})\big)^{(n)} + c(t)\Phi(y) = 0 \tag{4.3}$$

*is nonoscillatory.*

In the next theorem we present a Hille–Nehari type nonoscillation criterion for (4.3) with $r(t) = t^\alpha$. This criterion extends the linear result given in [10]. We will need the following auxiliary statement, its proof can be found e.g. in [6].

**Lemma 4.2.** *Let $m \in \{0, \dots, n-1\}$, then we have*

$$y^{(n)} = \left\{\frac{1}{t}\left[t^{m+1}\left(\frac{y}{t^m}\right)'\right]^{(m)}\right\}^{(n-m-1)}.$$

**Theorem 4.3.** *Suppose that $\alpha \notin \{p-1, \dots, np-1\}$, $\int^\infty c_-(t)t^{(n-j)p}\,dt > -\infty$, where $c_-(t) = \min\{0, c(t)\}$ is the negative part of $c$, and*

$$\liminf_{t\to\infty} t^{jp-1-\alpha}\int_t^\infty c_-(s)s^{(n-j)p}\,ds > -\frac{\gamma_{n,p,\alpha}}{|jp-1-\alpha|} \tag{4.4}$$

*for some $j \in \{1, \dots, n\}$. Then the equation*

$$(-1)^n\big(t^\alpha\Phi(x^{(n)})\big)^{(n)} + c(t)\Phi(x) = 0, \tag{4.5}$$

*is nonoscillatory.*

*Proof.* Let $T \in \mathbb{R}$ be so large, that the limited expression in (4.4) is greater than

$$-\frac{\gamma_{n,p,\alpha}}{|jp-1-\alpha|} + \varepsilon =: K,$$

where $\varepsilon > 0$ is sufficiently small. Then for any $0 \not\equiv y \in W_0^{n,p}[T,\infty)$ we have with $z = y/t^{n-j}$ (using the inequality $\int_a^b fg \le \left(\int_a^b |f|^p\right)^{1/p}\left(\int_a^b |g|^q\right)^{1/q}$ between the fourth and fifth line and (3.3) (with $\beta = \alpha - (j-1)p$ and $x' = z'$) between the fifth and sixth line in the next computation)

$$\begin{aligned}
\int_T^\infty c(t)|y|^p\,dt &\ge \int_T^\infty c_-(t)t^{(n-j)p}\left|\left(\frac{y}{t^{n-j}}\right)\right|^p dt = p\int_T^\infty c_-(t)t^{(n-j)p}\left(\int_T^t \Phi(z)z'\,ds\right)dt \\
&= p\int_T^\infty \Phi(z)z'\frac{1}{t^{jp-1-\alpha}}t^{jp-1-\alpha}\left(\int_t^\infty c_-(s)s^{(n-j)p}\,ds\right)dt \\
&\ge p\int_T^\infty \frac{|\Phi(z)|\,|z'|}{t^{jp-1-\alpha}}t^{jp-1-\alpha}\left(\int_t^\infty c_-(s)s^{(n-j)p}\,ds\right)dt \\
&> pK\int_T^\infty \frac{|\Phi(z)|}{t^{\frac{jp-\alpha}{q}}}\cdot\frac{|z'|}{t^{-\frac{jp-\alpha}{q}+jp-1-\alpha}}\,dt = pK\int_T^\infty \frac{|\Phi(z)|}{t^{\frac{jp-\alpha}{q}}}\cdot\frac{|z'|}{t^{\frac{(j-1)p-\alpha}{q}}}\,dt \\
&\ge pK\left(\int_T^\infty \frac{|z|^p}{t^{jp-\alpha}}\,dt\right)^{\frac{1}{q}}\left(\int_T^\infty \frac{|z'|^p}{t^{(j-1)p-\alpha}}\,dt\right)^{\frac{1}{p}} \\
&\ge pK\left(\frac{p}{|jp-1-\alpha|}\right)^{\frac{p}{q}}\left(\int_T^\infty \frac{|z'|^p}{t^{(j-1)p-\alpha}}\,dt\right)^{\frac{1}{q}}\left(\int_T^\infty \frac{|z'|^p}{t^{(j-1)p-\alpha}}\,dt\right)^{\frac{1}{p}} \\
&= pK\left(\frac{p}{|jp-1-\alpha|}\right)^{p-1}\int_T^\infty \frac{|z'|^p}{t^{(j-1)p-\alpha}}\,dt.
\end{aligned}$$

In the previous computation, we have used the equality $|z(t)|^p = p\int_T^t \Phi(z(s))z'(s)\,ds$, which follows from the formula $\left(|z|^p\right)' = p\Phi(z)z'$ and from the definition of $z$ ($z(T) = 0$). We have also used the relation $|\Phi(z)|^q = |z|^p$.

Now, we apply Lemma 4.2 with $m = n-j$, i.e., $n-m-1 = j-1$, and we denote

$$v = \frac{1}{t}\left[t^{n-j+1}\left(\frac{y}{t^{n-j}}\right)'\right]^{(n-j)}, \qquad u = t^{n-j+1}\left(\frac{y}{t^{n-j}}\right)'.$$

Then, using Wirtinger inequality (3.2) (in a slightly modified form), we get for $y \in W_0^{n,p}[T,\infty)$

$$\begin{aligned}
\int_T^\infty t^\alpha |y^{(n)}|^p &= \int_T^\infty t^\alpha \left|\left\{\frac{1}{t}\left[t^{n-j+1}\left(\frac{y}{t^{n-j}}\right)'\right]^{(n-j)}\right\}^{(j-1)}\right|^p dt \\
&= \int_T^\infty t^\alpha |v^{(j-1)}|^p\,dt \ge \prod_{i=1}^{j-1}\left(\frac{|ip-1-\alpha|}{p}\right)^p \int_T^\infty t^{\alpha-(j-1)p}|v|^p\,dt \\
&= \prod_{i=1}^{j-1}\left(\frac{|ip-1-\alpha|}{p}\right)^p \int_T^\infty t^{\alpha-(j-1)p}\left|\frac{1}{t}u^{(n-j)}\right|^p dt \\
&\ge \prod_{i=1}^{j-1}\left(\frac{|ip-1-\alpha|}{p}\right)^p \prod_{i=j+1}^{n}\left(\frac{|ip-1-\alpha|}{p}\right)^p \int_T^\infty t^{\alpha-np}\left|t^{n-j+1}\left(\frac{y}{t^{n-j}}\right)'\right|^p dt \\
&= \prod_{i=1}^{j-1}\left(\frac{|ip-1-\alpha|}{p}\right)^p \prod_{i=j+1}^{n}\left(\frac{|ip-1-\alpha|}{p}\right)^p \int_T^\infty t^{\alpha-(j-1)p}|z'|^p\,dt.
\end{aligned}$$

Summarizing the previous computations

$$
\begin{aligned}
&\int_T^\infty t^\alpha |y^{(n)}|^p\,dt + \int_T^\infty c(t)|y|^p\,dt \\
&\quad \geq \left[\prod_{i=1,i\neq j}^{n}\left(\frac{|ip-1-\alpha|}{p}\right)^p + pK\left(\frac{p}{|jp-1-\alpha|}\right)^{p-1}\right]\int_T^\infty t^{\alpha-(j-1)p}\left|\left(\frac{y}{t^{n-j}}\right)'\right|^p dt \\
&\quad = p\left(\frac{p}{|jp-1-\alpha|}\right)^{p-1}\left[\frac{1}{|jp-1-\alpha|}\prod_{i=1}^{n}\left(\frac{|ip-1-\alpha|}{p}\right)^p + K\right] \\
&\qquad \times \int_T^\infty t^{\alpha-(j-1)p}\left|\left(\frac{y}{t^{n-j}}\right)'\right|^p dt \\
&\quad = p\left(\frac{p}{|jp-1-\alpha|}\right)^{p-1}\left[\frac{\gamma_{n,p,\alpha}}{|jp-1-\alpha|} + K\right]\int_T^\infty t^{\alpha-(j-1)p}\left|\left(\frac{y}{t^{n-j}}\right)'\right|^p dt.
\end{aligned}
$$

Now, according to the definition of the constant $K$ we see that the energy functional corresponding to (4.5) is positive for large $T$ and hence (4.5) is nonoscillatory. □

Next we prove a statement which relates nonoscillatory behavior of a two-term $2n$th order half-linear differential equation to nonoscillation of a certain second order half-linear equation. It also presents a simpler proof of the previous theorem with $j = n$.

**Theorem 4.4.** *Consider equation* (4.5) *with* $\alpha \notin \{p-1, \ldots, np-1\}$. *If the second order differential equation*

$$-\left(t^{\alpha-(n-1)p}\Phi(x')\right)' + \frac{c_-(t)}{\gamma_{n-1,p,\alpha}}\Phi(x) = 0 \tag{4.6}$$

*is nonoscillatory,* $\gamma_{n-1,p,\alpha} = \prod_{j=1}^{n-1}\left(\frac{|jp-1-\alpha|}{p}\right)^p$, $c_-(t) = \min\{0, c(t)\}$, *then* (4.5) *is also nonoscillatory. In particular, if* $\alpha < np-1$ *and* $\int^\infty c_-(t)\,dt > -\infty$, *equation* (4.5) *is nonoscillatory provided*

$$\liminf_{t\to\infty} t^{np-1-\alpha}\int_t^\infty c_-(s)\,ds > -\frac{\gamma_{n,p,\alpha}}{np-1-\alpha}. \tag{4.7}$$

*Proof.* Using the Wirtinger inequality (as in (3.2)) we can estimate the energy functional in (4.5) as follows

$$
\begin{aligned}
\int_T^\infty t^\alpha |y^{(n)}|^p\,dt + \int_T^\infty c(t)|y|^p\,dt &\geq \gamma_{n-1,p,\alpha}\int_T^\infty t^{\alpha-(n-1)p}|y'|^p\,dt + \int_T^\infty c_-(t)|y|^p\,dt \\
&= \gamma_{n-1,p,\alpha}\left[\int_T^\infty t^{\alpha-(n-1)p}|y'|^p\,dt + \frac{1}{\gamma_{n-1,p,\alpha}}\int_T^\infty c_-(t)|y|^p\,dt\right].
\end{aligned}
$$

The expression in brackets on the second line of the previous computation is the energy functional of (4.6) and it is positive if this equation is nonoscillatory and $T$ is sufficiently large by [16, Theorem 2.1.1]. To prove the second statement of theorem, we apply the Hille–Nehari type nonoscillation criterion to (4.6). This criterion says (see, e.g., [16, Theorem 2.1.2]) that equation (1.2) with $r_1$ satisfying $\int^\infty r_1^{1-q}(t)\,dt = \infty$ and $\int_0^\infty (r_0)_-(t) > -\infty$ (where $(r_0)_-(t) = \min\{0, r_0(t)\}$) is nonoscillatory provided

$$\liminf_{t\to\infty}\left(\int^t r_1^{1-q}(s)\,ds\right)^{p-1}\int_t^\infty (r_0)_-(s)\,ds > -\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}. \tag{4.8}$$

Hence, for $r_1(t) = t^{\alpha-(n-1)p}$, we have

$$\left(\int_0^t r_1^{1-q}(s)\,ds\right)^{p-1} = \left(\frac{t^{\alpha(1-q)+q(n-1)+1}}{\alpha(1-q)+q(n-1)+1}\right)^{p-1} = \frac{t^{np-1-\alpha}}{\left(\frac{np-1-\alpha}{p-1}\right)^{p-1}}.$$

and $\int^\infty r_1^{1-q}(t)\,dt = \infty$, since $\alpha < np-1$. Then (4.8) reads

$$\liminf_{t\to\infty} \frac{t^{np-1-\alpha}}{\left(\frac{np-1-\alpha}{p-1}\right)^{p-1}} \int_t^\infty (r_0)_-(s)\,ds > -\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}$$

which is just (4.7) with $\frac{c_-(t)}{\gamma_{n-1,p,\alpha}}$ instead of $(r_0)_-(t)$. □

**Remark 4.5.** Obviously, Theorem 4.4 applied to Euler type equation (2.3) gives Theorem 3.2.

**Remark 4.6.** Let us have a look at Theorem 4.1 with $r(t) = t^\alpha$, $\alpha \notin \{p-1, 2p-1, \dots, np-1\}$ and $c(t) \le 0$ for large $t$. Then $r^{1-q}(t) = t^{\alpha(1-q)}$ and for $\alpha < p-1$ (the case $\alpha > p-1$ is more complicated) we have

$$\int_0^t r^{1-q}(s)\,ds = \frac{t^{\alpha(1-q)+1}}{\alpha(1-q)+1}, \qquad \left(\int_0^t r^{1-q}(s)\,ds\right)^{p-1} = \frac{t^{p-1-\alpha}}{\left(\frac{p-1-\alpha}{p-1}\right)^{p-1}},$$

$$\left(\int_0^t r^{1-q}(s)\,ds\right)^{p} = \frac{t^{p-q\alpha}}{\left(\frac{p-1-\alpha}{p-1}\right)^{p}}, \qquad \left(\int_0^t r^{1-q}(s)\,ds\right)^{-1} = [1-(q-1)\alpha]t^{\alpha(q-1)-1}.$$

Hence, (4.1) takes the form

$$\begin{aligned}\liminf_{t\to\infty} t^{p-1-\alpha}\int_t^\infty c(s)(s-T)^{(n-1)p}\,ds &> -\left(\frac{p-1-\alpha}{p-1}\right)^{p-1}\cdot\left(\frac{p-1}{p}\right)^{p}\cdot\frac{[(n-1)!]^p}{p-1}\\ &= -\frac{1}{p-1-\alpha}\left(\frac{p-1-\alpha}{p}\right)^{p}[(n-1)!]^p.\end{aligned} \tag{4.9}$$

This condition is *more restrictive* than (4.4) with $j = 1$. Indeed, for $\alpha < p-1$ we have

$$\begin{aligned}-\frac{\gamma_{n,p,\alpha}}{p-1-\alpha} &= -\frac{1}{p-1-\alpha}\left(\frac{p-1-\alpha}{p}\right)^p\left(2-\frac{\alpha+1}{p}\right)^p\cdots\left(n-\frac{\alpha+1}{p}\right)^p\\ &< -\frac{1}{p-1-\alpha}\left(\frac{p-1-\alpha}{p}\right)^p[(n-1)!]^p\end{aligned}$$

since $\frac{\alpha+1}{p} < 1$. The difference in terms $\int_t^\infty c(s)s^{(n-1)p}\,ds$ and $\int_t^\infty c(s)(s-T)^{(n-1)p}\,ds$ in (4.4) (with $j = 1$) and (4.9), respectively, is not important since $\lim_{s\to\infty} s^{-(n-1)p}\cdot(s-T)^{(n-1)p} = 1$. Concerning (4.2), similarly as for (4.1) we obtain

$$\liminf_{t\to\infty} t^{\alpha(q-1)-1}\int_0^t c(s)s^{np-q\alpha}\,ds > -[(n-1)!]^p\left(\frac{p-1-\alpha}{p}\right)^p\frac{p-1}{p-1-\alpha}.$$

This condition is not covered by results presented in this paper and a subject of the present investigation is to "insert" this criterion into a general framework of even-order half-linear oscillation theory.

## 5 Remarks and comments

(i) In the previous parts of the paper, we have presented *nonoscillation* criteria for the investigated differential equations. The problem of *oscillation* of these equations is more complicated. In the linear case $p = 2$, we have the *equivalence* in Lemma 3.1, i.e., the differential equation

$$(-1)^n \left(r_n(t)y^{(n)}\right)^{(n)} + \cdots - \left(r_1(t)y'\right)' + r_0(t)y = 0$$

is oscillatory *if and only if* for every $T \in \mathbb{R}$ there exists $0 \not\equiv y \in W_0^{n,p}[T,\infty)$ such that

$$\int_T^\infty \left[r_n(t)(y^{(n)})^2 + \cdots + r_1(t)y'^2 + r_0(t)y^2\right] dt \le 0.$$

Such an equivalence is missing in the half-linear case and to find a general framework for the investigation of oscillation of (1.1) is a subject of the present investigation. In particular, we hope to prove that (2.3) is oscillatory if $\gamma < -\gamma_{n,p,\alpha}$, so the constant $-\gamma_{n,p,\alpha}$ really separates oscillation and nonoscillation in (2.3).

(ii) In the spectral theory of self-adjoint even order differential operators, an important role is played by the so-called *reciprocity principle* which claims that the two-term differential equation

$$(-1)^n \left(r(t)y^{(n)}\right)^{(n)} + c(t)y = 0 \tag{5.1}$$

with $r(t) > 0$ and $c(t) \neq 0$ for large $t$, is nonoscillatory if and only if its *reciprocal equations* (related to (5.1) by the substitution $u = ry^{(n)}$)

$$(-1)^n \left(\frac{1}{c(t)}u^{(n)}\right)^{(n)} + \frac{1}{r(t)}u = 0 \tag{5.2}$$

is also nonoscillatory, see [2]. The proof of this statement is based on the Riccati technique for Hamiltonian differential systems associated with (5.1) and (5.2) (which we miss for higher order half-linear equations as we have already mentioned in a previous part of the paper). It would be interesting to know whether a similar principle holds for the *half-linear* equation

$$(-1)^n \left(r(t)\Phi(y^{(n)})\right)^{(n)} + c(t)\Phi(y) = 0 \tag{5.3}$$

and its reciprocal equation (related to (5.3) by the substitution $u = r\Phi(y^{(n)})$)

$$(-1)^n \left(\frac{\Phi^{-1}(u^{(n)})}{\Phi^{-1}(c(t))}\right)^{(n)} + \frac{\Phi^{-1}(u)}{\Phi^{-1}(r(t))} = 0, \tag{5.4}$$

where $\Phi^{-1}(u) = |u|^{q-2}u$ is the inverse function of $\Phi$.

A positive answer to this conjecture is partially supported by considering the pair of mutually reciprocal Euler type differential equations.

**Theorem 5.1.** *The reciprocal equation to Euler differential equation* (2.3)*, which is the equation*

$$(-1)^n \left(t^{(np-\alpha)(q-1)}\Phi^{-1}(u^{(n)})\right)^{(n)} + \Phi^{-1}(\gamma)t^{-\alpha(q-1)}\Phi^{-1}(u) = 0, \tag{5.5}$$

*is again an Euler equation. Moreover, the reciprocal equation to a critical equation is again the critical equation. In particular, if $\gamma > -\gamma_{n,p,\alpha}$, then the reciprocal equation* (5.5) *is also nonoscillatory.*

*Proof.* An equation

$$(-1)^n \left(t^{\alpha_1}\Phi(y^{(n)})\right)^{(n)} + \gamma t^{\alpha_2}\Phi(y) = 0$$

is the Euler type equation if and only if $\alpha_1 - \alpha_2 = np$. Consequently, since (5.4) contains the power nonlinearity $\Phi^{-1}(u) = |u|^{q-2}u$, we compute the difference

$$(np - \alpha)(q-1) + \alpha(q-1) = (q-1)np = nq,$$

hence (5.5) is really an Euler equation. To show that the reciprocal equation to the critical equation is again a critical equation we need to show that if $\gamma = -\gamma_{n,p,\alpha}$ in (2.3), then the constant $-\Phi^{-1}(\gamma_{n,p,\alpha})$ is the critical constant for (5.5), i.e., it is

$$\Phi^{-1}(\gamma_{n,p,\alpha}) = \gamma_{n,q,\beta} = \prod_{j=1}^{n} \left(\frac{|jq-\beta-1|}{q}\right)^q$$

with $\beta = (np-\alpha)(q-1) = nq - \alpha(q-1)$. We have

$$\Phi^{-1}(\gamma_{n,p,\alpha}) = \left[\prod_{j=1}^{n} \left(\frac{|jp-1-\alpha|}{p}\right)^p\right]^{q-1}$$

On the other hand, for $j = 1, \dots, n$

$$\begin{aligned}\frac{|jq-\beta-1|}{q} &= \left|j - \frac{nq-\alpha(q-1)+1}{q}\right| = \left|j - n + \frac{\alpha}{p} - \frac{p-1}{p}\right| \\ &= \frac{|(j-n-1)p+\alpha+1|}{p} = \frac{|(n-j+1)p-\alpha-1|}{p},\end{aligned}$$

hence

$$\begin{aligned}\gamma_{n,q,\beta} = \prod_{j=1}^{n} \left(\frac{|jq-\beta-1|}{q}\right)^q &= \prod_{j=1}^{n} \left(\frac{|(n-j+1)p-\alpha-1|}{p}\right)^q \\ &= \left[\prod_{j=1}^{n} \left(\frac{|jp-1-\alpha|}{p}\right)^p\right]^{q-1},\end{aligned}$$

so really $\Phi^{-1}(\gamma_{n,p,\alpha}) = \gamma_{n,q,\beta}$. □

Note that for $n = 1$, i.e., for second order half-linear equation (1.2), the reciprocity principle holds as a simple consequence of the Rolle mean value theorem.

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