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Ground state solutions for diffusion system with superlinear nonlinearity

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Abstract. In this paper, we study the following diffusion system

$$\begin{cases} \partial_t u - \Delta_x u + b(t,x)\cdot\nabla_x u + V(x)u = g(t,x,v), \\ -\partial_t v - \Delta_x v - b(t,x)\cdot\nabla_x v + V(x)v = f(t,x,u) \end{cases}$$

where $z = (u,v)\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^2$, $b \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^N)$ and $V(x) \in C(\mathbb{R}^N,\mathbb{R})$. Under suitable assumptions on the nonlinearity, we establish the existence of ground state solutions by the generalized Nehari manifold method developed recently by Szulkin and Weth.

Keywords: diffusion systems, ground state solutions, generalized Nehari manifold, strongly indefinite functionals.

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1 Introduction and main result

We study the following diffusion system on $\mathbb{R}\times\mathbb{R}^N$

$$\begin{cases} \partial_t u - \Delta_x u + b(t,x)\cdot\nabla_x u + V(x)u = g(t,x,v), \\ -\partial_t v - \Delta_x v - b(t,x)\cdot\nabla_x v + V(x)v = f(t,x,u) \end{cases} \tag{1.1}$$

where $z = (u,v)\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^2$, $b = (b_1,\dots,b_N) \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R}^N)$ with the gauge condition $\operatorname{div} b(t,x) = 0$ ($\operatorname{div} b(t,x) := \sum_{i=1}^N \partial_{x_i} b_i(t,x)$), $V(x) \in C(\mathbb{R}^N,\mathbb{R})$, and the primitives of the nonlinearities $g(t,x,v)$, $f(t,x,u)$ are periodic in (t,x) and superquadratic in v,u at infinity. Such problem arises in control of systems governed by partial differential equations and is related to the Schrödinger equations (see [15] and [19]). In this paper, we are interested in the existence of ground state solutions of Nehari type of problem (1.1).

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For the case of a bounded domain the systems like or similar to (1.1) were studied by a number of authors. For instance, see [3–6, 8, 10, 11, 17, 18] and the references therein. When assuming $b(t,x) = 0$, $V(x) = 0$, Brézis and Nirenberg [3] considered the following system

$$\begin{cases} \partial_t u - \Delta_x u = -v^5 + f \\ -\partial_t v - \Delta_x v = u^3 + g \end{cases} \text{in } (0,T)\times\Omega.$$

Using Schauder's fixed point theorem, they obtained a solution (u,v) with $u \in L^4$ and $v \in L^6$. In [4], Clément et al. considered the problem

$$\begin{cases} \partial_t u - \Delta_x u = |v|^{q-2}v \\ -\partial_t v - \Delta_x v = |u|^{p-2}u \end{cases} \text{in } (-T,T)\times\Omega,$$

where p, q satisfy

$$\frac{N}{N+2} < \frac{1}{p} + \frac{1}{q} < 1.$$

The existence of a positive periodic solution was obtained by using a mountain pass argument and then a homoclinic solution was obtained as a limit of $2k$-periodic solution. Recently, based on a local linking theorem, Mao et al. [18] proved that the problem (1.1) has at least one nontrivial periodic solution, also see [17]. For other related elliptic system problems, we refer the readers to [5, 6, 10, 11].

The problem in the whole space $\mathbb{R}^N$ was considered recently in some works. Assuming $b(t,x) = 0$, $V(x) \neq 0$, Bartsch and Ding [2] dealt with the problem under the classic Ambrosetti–Rabinowitz condition

$$0 < \mu H(t,x,z) \leq H_z(t,x,z)z, \qquad z \neq 0 \tag{1.2}$$

for $\mu > 2$ and

$$|H_z(t,x,z)|^\nu \leq cH_z(t,x,z)z, \quad \text{for } |z| > 1 \tag{1.3}$$

for some $1 + N/(N+4) < \nu < 2$. Assumptions (1.2) and (1.3) were improved later by Schechter and Zou in [25]. Nearly, Ding et al. [9] and Wei and Yang [31] considered the case $b(t,x) \neq 0$ via variational methods. Under periodic assumption, the existence of infinitely many solutions were obtained for both superquadratic or asymptotically linear cases when the nonlinearity is symmetric. Without the symmetric assumption, Wang et al. [29] also obtained infinitely many solutions by using a reduction method. For asymptotically periodic and non-periodic case, we refer the readers to [30, 35, 36, 44, 45] and the references therein.

The main purpose of this paper is to prove the existence of ground state solutions for problem (1.1) without Ambrosetti–Rabinowitz condition. To the best of our knowledge, there is no work focusing on the existence of ground state solutions up to now. Next, we denote by $F(t,x,s)$ and $G(t,x,s)$ the primitives of $f(t,x,s)$ and $g(t,x,s)$, respectively. Our assumptions for f and g are standard, roughly speaking "superlinear" at zero and infinity and "subcritical" at infinity. More precisely, we make the following assumptions.

- (V) $V \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x_i for $i = 1, \dots, N$ and $a := \min_{x\in\mathbb{R}^N} V(x) > 0$;
- (B) $b \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^N)$ is 1-periodic in t and x_i for $i = 1, \dots, N$ and $\operatorname{div} b = 0$;
- (S_1) $f(t,x,s)$ and $g(t,x,s)$ are continuous and 1-periodic in t and x_i for $i = 1, \dots, N$, and there is a constant $C > 0$ such that

$$|f(t,x,s)| \leq C(1 + |s|^{p-1}) \quad \text{and} \quad |g(t,x,s)| \leq C(1 + |s|^{p-1}) \quad \text{for all } (t,x,s),$$

where $p \in (2, N^*)$, $N^* = \infty$ if $N = 1$ and $N^* = \frac{2(N+2)}{N}$ if $N \geq 2$;

(S_2) $f(t,x,s) = o(|s|)$ and $g(t,x,s) = o(|s|)$ as $|s| \to 0$ uniformly in (t,x);

(S_3) $\lim_{|s|\to\infty} \frac{F(t,x,s)}{|s|^2} = \infty$ and $\lim_{|s|\to\infty} \frac{G(t,x,s)}{|s|^2} = \infty$ uniformly in (t,x);

(S_4) $s \mapsto \frac{f(t,x,s)}{|s|}$ and $s \mapsto \frac{g(t,x,s)}{|s|}$ are strictly increasing on $(-\infty, 0)$ and $(0, +\infty)$.

Our main result is the following theorem.

Theorem 1.1. *Let* (V), (B) *and* (S_1)–(S_4) *be satisfied, then problem* (1.1) *has at least one ground state solutions.*

It is well known that for the study of ground state solution, Szulkin and Weth developed a powerful approach to treat the indefinite problem in [23]. More precisely, they used the generalized Nehari manifold (which was first introduced in [20] for the smooth case) to construct a natural constrained problem and obtained the ground state solution for more general strongly indefinite periodic Schrödinger equation.

Motivated by this work, in the present paper, we are devoted to study the existence of a ground state solution via the generalized Nehari manifold method for problem (1.1). Additionally, based on the linking theorem in [12] and [24], there are also many works devoted to the ground state solution for periodic Schrödinger equation, elliptic system and Hamiltonian system. For example, see [13, 16, 20, 21, 23, 26–28, 33, 34, 37–43] and the references therein.

The remainder of this paper is organized as follows. In Section 2, the variational setting and the method of the generalized Nehari manifold are briefly presented. The existence of a ground state solution is proved in Section 3.

2 Variational setting and generalized Nehari manifold method

Below by $|\cdot|_q$ we denote the usual L^q- norm, $(\cdot,\cdot)_2$ denote the usual L^2 inner product, c, c_i or C_i stand for different positive constants. Denote by $\sigma(A)$ and $\sigma_e(A)$ the spectrum and the essential spectrum of the operator A, respectively. In order to continue the discussion, we need the following notations. Set

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{J}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{S} = -\Delta_x + V$$

and

$$A_0 := \mathcal{J}_0 \mathcal{S} + \mathcal{J} b \cdot \nabla_x.$$

Then (1.1) can be read as

$$\mathcal{J}\partial_t z + A_0 z = H_z(t,x,z), \qquad z = (u,v),$$

where $H(t,x,z) = F(t,x,u) + G(t,x,v)$. It is called an unbounded Hamiltonian system [1], or an infinite-dimensional Hamiltonian system (see [2] and [8]). Indeed, it has the representation

$$\mathcal{J}\partial_t z = \operatorname{grad}_z \mathcal{H}(t,x,z)$$

with the Hamiltonian

$$\mathcal{H}(t,x,z) := -\int_{\mathbb{R}^N} (\nabla_x u \cdot \nabla_x v + b \cdot \nabla_x uv + V(x)uv - H(t,x,z))\, dx$$

in $L^2(\mathbb{R}^N, \mathbb{R}^2)$, where grad_z denotes the gradient operator in $L^2(\mathbb{R}^N, \mathbb{R}^2)$.

In order to state our main result, we introduce for $r \geq 1$ the Banach space,

$$B_r = B_r(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^2) := W^{1,r}(\mathbb{R}, L^r(\mathbb{R}^N, \mathbb{R}^2)) \cap L^r(\mathbb{R}, W^{1,r} \cap W^{1,r}(\mathbb{R}^N, \mathbb{R}^2)),$$

equipped with the norm

$$\|z\|_{B_r} = \left(\int_{\mathbb{R}\times\mathbb{R}^N} (|z|^r + |\partial_t z|^r + \sum_{j=1}^{N} |\partial^2_{x_j} z|^r) \right)^{\frac{1}{r}}.$$

Clearly, B_r is the completion of $C_0^\infty(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^2)$ with respect to the norm $\|\cdot\|_{B_r}$. If $r = 2$, B_2 is a Hilbert space.

Let $A := \mathcal{J}\partial_t + A_0$, under the conditions (V) and (B), it is easy to show that A is a self-adjoint operator acting in $L^2 := L^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^2)$ with domain $\mathcal{D}(A) = B_2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^2)$ and there exist c_1, c_2, such that

$$c_1\|z\|^2_{B_2} \leq |Az|^2_2 \leq c_2\|z\|^2_{B_2}$$

for all $z \in B_2$, (see [9, (2.1) and Lemma 2.2]).

Now, in order to establish suitable variational framework for the problem (1.1), we need the following Lemma due to [9].

Lemma 2.1 ([9, Lemma 2.1])**.** *Suppose that* (V) *and* (B) *are satisfied. Then*

(1) $\sigma(A) = \sigma_e(A)$*, i.e.,* A *has only essential spectrum;*

(2) $\sigma(A) \subset \mathbb{R} \setminus (-a, a)$ *and* $\sigma(A)$ *is symmetric with respect to the origin.*

It follows from Lemma 2.1 that L^2 possesses the orthogonal decomposition

$$L^2 = L^- \oplus L^+, \qquad z = z^- + z^+, \qquad z^\pm \in L^\pm,$$

such that A is negative definite (resp. positive definite) in L^- (resp. L^+). Let $|A|$ denote the absolute value of A and $|A|^{\frac{1}{2}}$ be the square root of $|A|$. Let $E = D(|A|^{\frac{1}{2}})$ be the Hilbert space with the inner product

$$(z, w) = \left(|A|^{\frac{1}{2}} z, |A|^{\frac{1}{2}} w \right)_2$$

and norm $\|z\| = (z, z)^{\frac{1}{2}}$. There is an induced decomposition

$$E = E^- \oplus E^+, \qquad E^\pm = E \cap L^\pm$$

which is orthogonal with respect to the inner products $(\cdot, \cdot)_2$ and $(\cdot, \cdot)$. Moreover, we have the following embedding theorem in [9].

Lemma 2.2 ([9, Lemma 2.6])**.** *E is continuously embedded in L^p for any $p \geq 2$ if $N = 1$, and for $p \in [2, N^*]$ if $N \geq 2$. E is compactly embedded in L^p_{loc} for all $p \in [1, N^*)$.*

On E we define the following energy functional of (1.1)

$$\Phi(z) = \frac{1}{2}(Az, z)_2 - \Psi(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \Psi(z), \qquad z = (u, v), \tag{2.1}$$

where $\Psi(z) = \int_{\mathbb{R}\times\mathbb{R}^N} H(t, x, z) = \int_{\mathbb{R}\times\mathbb{R}^N} (F(t, x, u) + G(t, x, v))$. Lemma 2.1 implies that Φ is strongly indefinite. Our hypotheses imply that $\Phi \in C^1(E, \mathbb{R})$, and a standard argument shows that critical points of Φ are solutions of (1.1) (see [7] and [32]).

Next, we introduce the generalized Nehari manifold method. We consider the following set introduced by Pankov [20] (see also [23] and [22]):

$$\mathcal{M} := \left\{ z \in E \setminus E^- : \Phi'(z)z = 0 \text{ and } \Phi'(z)w = 0,\ \forall w \in E^- \right\}.$$

Following Szulkin and Weth [23] (see also [22]), we will call the set $\mathcal{M}$ the generalized Nehari manifold. Obviously, $\mathcal{M}$ contains all nontrivial critical points of Φ. Let

$$c := \inf_{z \in \mathcal{M}} \Phi(z).$$

If c is attained by a solution z_0, since c is the lowest level for Φ, z_0 will be called a ground state solution of Nehari type for (1.1).

We denote by S^+ the unit sphere in E^+, that is

$$S^+ := \left\{ z \in E^+ : \|z\| = 1 \right\}.$$

For $z = z^+ + z^- \in E$, where $z^\pm \in E^\pm$, we define the subspace

$$E(z) := \mathbb{R}z \oplus E^- \equiv \mathbb{R}z^+ \oplus E^-,$$

and the convex set

$$\hat{E}(z) := \mathbb{R}^+ z \oplus E^- \equiv \mathbb{R}^+ z^+ \oplus E^-,$$

where $\mathbb{R}^+ = [0, \infty)$. It is clear that $E(z) = E(z^+)$, $\hat{E}(z) = \hat{E}(z^+)$, $E(z) = E(\alpha z)$ for $\alpha \neq 0$ and $\hat{E}(z) = \hat{E}(\beta z)$ for $\beta > 0$.

Before giving the proof of the main theorem, we need some preliminary results.

Lemma 2.3. *Assume that* (S_1)–(S_4) *are satisfied. Then for* $z \in \mathcal{M}$, *we have* $\Phi(z + w) < \Phi(z)$, *where* $w \neq 0$, $w = rz + \eta$, $\eta \in E^-$ *and* $r \geq -1$, *and* z *is the unique global maximum of* $\Phi|_{\hat{E}(z)}$.

Proof. Let $w = rz + \eta$ with $\eta = (\varphi, \psi) \in E^-$ and $r \geq -1$. Then $z + w = (1 + r)z + \eta = ((1 + r)u + \varphi, (1 + r)v + \psi)$. By (2.1) we have

$$\begin{aligned}
&\Phi(z + w) - \Phi(z) \\
&\quad = \frac{1}{2}\left(((1 + r)^2 - 1)(Az, z)_2 + 2(1 + r)(Az, \eta)_2 + (A\eta, \eta)_2\right) \\
&\qquad + \int_{\mathbb{R}\times\mathbb{R}^N} F(t, x, u) - F(t, x, (1 + r)u + \varphi) + \int_{\mathbb{R}\times\mathbb{R}^N} G(t, x, v) - G(t, x, (1 + r)v + \psi) \\
&\quad = -\frac{\|\eta\|^2}{2} + (Az, r\left(\frac{r}{2} + 1\right)z + (1 + r)\eta)_2 \\
&\qquad + \int_{\mathbb{R}\times\mathbb{R}^N} F(t, x, u) - F(t, x, (1 + r)u + \varphi) + \int_{\mathbb{R}\times\mathbb{R}^N} G(t, x, v) - G(t, x, (1 + r)v + \psi) \\
&\quad = -\frac{\|\eta\|^2}{2} + \int_{\mathbb{R}\times\mathbb{R}^N} \left[f(t, x, u) \left(r\left(\frac{r}{2} + 1\right)u + (1 + r)\varphi \right) + F(t, x, u) - F(t, x, (1 + r)u + \varphi) \right] \\
&\qquad + \int_{\mathbb{R}\times\mathbb{R}^N} \left[g(t, x, v) \left(r\left(\frac{r}{2} + 1\right)v + (1 + r)\psi \right) + G(t, x, v) - G(t, x, (1 + r)v + \psi) \right].
\end{aligned}$$

In the last step we used the fact that $z \in \mathcal{M}$ and $\xi := r(\frac{r}{2} + 1)z + (1 + r)\eta \in E(z)$, therefore

$$\begin{aligned}
0 = \langle \Phi'(z), \xi \rangle = (Au, \xi)_2 &- \int_{\mathbb{R}\times\mathbb{R}^N} f(t, x, u) \left(r\left(\frac{r}{2} + 1\right)u + (1 + r)\varphi \right) \\
&- \int_{\mathbb{R}\times\mathbb{R}^N} g(t, x, v) \left(r\left(\frac{r}{2} + 1\right)v + (1 + r)\psi \right).
\end{aligned}$$

Similar to Lemma 2.2 in [23], by (S_3) and (S_4) we have

$$f(t,x,u)\left(r\left(\frac{r}{2}+1\right)u+(1+r)\varphi\right)+F(t,x,u)-F(t,x,(1+r)u+\varphi)<0$$

and

$$g(t,x,v)\left(r\left(\frac{r}{2}+1\right)v+(1+r)\psi\right)+G(t,x,v)-G(t,x,(1+r)v+\psi)<0.$$

From the above argument, we conclude that $\Phi(z+w)<\Phi(z)$. □

Lemma 2.4. *Assume that* (S_1) *and* (S_2) *are satisfied. Then there exists* $\rho>0$ *such that* $c\geq\kappa:=\inf_{S_\rho}\Phi(z)>0$, *where* $S_\rho=\{z\in E^+:\|z\|=\rho\}$.

Proof. Observe that, given $\varepsilon>0$, there is $C_\varepsilon>0$ such that

$$|f(t,x,u)|\leq\varepsilon|u|+C_\varepsilon|u|^{p-1},\qquad |g(t,x,v)|\leq\varepsilon|v|+C_\varepsilon|v|^{p-1},\tag{2.2}$$

and

$$|F(t,x,u)|\leq\varepsilon|u|^2+C_\varepsilon|u|^p,\qquad |G(t,x,v)|\leq\varepsilon|v|^2+C_\varepsilon|v|^p\tag{2.3}$$

where $p\in[2,N^*)$. For $z=(u,v)\in E^+$ with $\|z\|$ small, by Lemma 2.2 and (2.3) we have

$$\begin{aligned}\Phi(z)&=\frac{1}{2}\|z\|^2-\int_{\mathbb{R}\times\mathbb{R}^N}F(t,x,u)+G(t,x,v)\\&\geq\frac{1}{2}\|z\|^2-\varepsilon(|u|_2^2+|v|_2^2)-C_\varepsilon(|u|_p^p+|v|_p^p)\\&\geq(\frac{1}{2}-c_2^2\varepsilon)\|z\|^2-c_p^pC_\varepsilon\|z\|^p,\end{aligned}$$

where c_2 and c_p are constants of the embedding. Hence the second inequality follows if ρ and ε are sufficiently small. Now, the first inequality follows from Lemma 2.3. □

Lemma 2.5. *Let* (S_1)–(S_4) *be satisfied. If* $\mathcal{V}\subset E^+\setminus\{0\}$ *is a compact subset. Then there exists* $R>0$ *such that* $\Phi(\cdot)\leq 0$ *on* $E(z)\setminus B_R(0)$ *for every* $z\in\mathcal{V}$.

Proof. Without loss of generality, we may assume that $\|z\|=1$ for every $z\in\mathcal{V}$. By contradiction, suppose that there exists a sequence $z_n\in\mathcal{V}$ and $w_n=(u_n,v_n)\in E(z_n)$ such that $\Phi(w_n)>0$ for all n and $\|w_n\|\to\infty$ as $n\to\infty$. Since $\mathcal{V}$ is a compact set, passing to a subsequence, we may assume that $z_n\to z\in E^+$, $\|z\|=1$. Set $y_n=\frac{w_n}{\|w_n\|}=s_nz_n+y_n^-$, then $1=\|y_n\|^2=s_n^2+\|y_n^-\|^2$ and

$$0<\frac{\Phi(w_n)}{\|w_n\|^2}=\frac{1}{2}(s_n^2-\|y_n^-\|^2)-\int_{\mathbb{R}\times\mathbb{R}^N}\frac{F(t,x,u_n)+G(t,x,v_n)}{|w_n|^2}|y_n|^2.$$

From (S_4), we have $F(t,x,u),G(t,x,v)\geq 0$ and have

$$\|y_n^-\|^2\leq s_n^2=1-\|y_n^-\|^2,$$

therefore

$$\|y_n^-\|^2\leq\frac{1}{2}\quad\text{and}\quad\frac{1}{2}\leq s_n^2\leq 1.$$

Going to a subsequence if necessary, we may assume $s_n \to s > 0$, $y_n \rightharpoonup y$ and $y_n^-(t,x) \to y^-(t,x)$ a.e. in $\mathbb{R} \times \mathbb{R}^N$. Hence $y = sz + y^- \neq 0$, therefore $|w_n| = \|w_n\||y_n| \to \infty$. By (S_3), (S_4) and Fatou's lemma, we have

$$\begin{aligned} 0 &\le \lim_{n\to\infty} \frac{\Phi(w_n)}{\|w_n\|^2} = \lim_{n\to\infty}\left(\frac{1}{2}(s_n^2 - \|y_n^-\|^2) - \int_{\mathbb{R}\times\mathbb{R}^N} \frac{F(t,x,u_n)+G(t,x,v_n)}{|w_n|^2}|y_n|^2\right) \\ &\le \frac{1}{2} - \liminf_{n\to\infty} \int_{\mathbb{R}\times\mathbb{R}^N} \frac{F(t,x,u_n)+G(t,x,v_n)}{|w_n|^2}|y_n|^2 \\ &\le \frac{1}{2} - \int_{\mathbb{R}\times\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(t,x,u_n)+G(t,x,v_n)}{|w_n|^2}|y_n|^2 \\ &= -\infty. \end{aligned}$$

This is a contradiction. □

Now we define the mappings $\hat{m}\colon E \setminus E^- \to \mathcal{M}, z \mapsto \hat{m}(z)$ and $m := \hat{m}|_{S^+}$, and have the following results.

Lemma 2.6. *Let (S_1)–(S_4) be satisfied. Then for each $z \in E \setminus E^-$, the set $\hat{E}(z) \cap \mathcal{M}$ consists of precisely one point $\hat{m}(z)$ which is the unique global maximum of $\Phi|_{\hat{E}(z)}$.*

Proof. By Lemma 2.3, it suffices to prove that $\mathcal{M} \cap \hat{E}(z)$ is not empty. Since $\hat{E}(z) = \hat{E}(z^+)$, we may assume that $z \in E^+$ and $\|z\| = 1$. By Lemma 2.4, $\Phi(sz) > 0$ for small $s > 0$ and by Lemma 2.5, $\Phi(\cdot) \le 0$ on $\hat{E}(z) \setminus B_R(0)$, hence $0 < \sup_{\hat{E}(z)} \Phi < \infty$. Suppose that $z_n \rightharpoonup z_0$ in $\hat{E}(z)$. By Lemma 2.2, up to a subsequence, we see that $z_n(t,x) \to z_0(t,x)$ a.e. in $\mathbb{R} \times \mathbb{R}^N$. It follows from (S_4) and Fatou's lemma that $\Psi(z)$ is weakly lower semicontinuous on $\hat{E}(z)$. Setting $z_n = s_n z + z_n^-$ and $z_0 = s_0 z + z_0^-$, then $z_n^- \rightharpoonup z_0^-$ and $s_n \to s_0$. Hence

$$\begin{aligned} -\Phi(z_0) &= \frac{1}{2}(-s_0^2 + \|z_0^-\|^2) + \Psi(z_0) \\ &\le \liminf_{n\to\infty}\left(\frac{1}{2}(-s_n^2 + \|z_n^-\|^2) + \Psi(z_n)\right) \\ &= \liminf_{n\to\infty} -\Phi(z_n), \end{aligned}$$

which implies that Φ is weakly upper semicontinuous on $\hat{E}(z)$. Therefore, $\sup_{\hat{E}(z)} \Phi$ is achieved at some $\tilde{z} \in \hat{E}(z) \setminus \{0\}$. This $\tilde{z}$ is a critical point of $\Phi|_{E(z)}$. So $\tilde{z} \in \mathcal{M} \cap \hat{E}(z)$. □

Lemma 2.7. *Suppose that the assumptions of Theorem 1.1 hold. Then*

(a) $\hat{m}$ is continuous;

(b) m is a homeomorphism between S^+ and $\mathcal{M}$.

Proof. The proofs were given in [22, Proposition 4.1], here we omit the details. □

Lemma 2.8. *Under the assumptions of Theorem 1.1, the functional Φ is coercive on $\mathcal{M}$.*

Proof. Suppose to the contradiction that there exist some $M \in \mathbb{R}$ and a sequence $\{z_n\} = \{(u_n, v_n)\} \subset \mathcal{M}$ such that $\Phi(z_n) \le M$ for all n and $\|z_n\| \to \infty$ as $n \to \infty$. Set $w_n = \frac{z_n}{\|z_n\|}$, after passing to a subsequence, $w_n \rightharpoonup w$ in E and $w_n(t,x) \to w(t,x)$ a.e. in $\mathbb{R} \times \mathbb{R}^N$. By (S_4) we have

$$0 \le \frac{\Phi(z_n)}{\|z_n\|^2} = \frac{1}{2}(\|w_n^+\|^2 - \|w_n^-\|^2) - \int_{\mathbb{R}\times\mathbb{R}^N} \frac{F(t,x,u_n)+G(t,x,v_n)}{\|z_n\|^2} \le \frac{1}{2}(\|w_n^+\|^2 - \|w_n^-\|^2)$$

and $\|w_n^+\|^2 \geq \|w_n^-\|^2$. Moreover, from $1 = \|w_n^+\|^2 + \|w_n^-\|^2$, we deduce that $\|w_n^+\|^2 \geq \frac{1}{2}$. After passing a subsequence, $\{w_n^+\}$ is either vanishing, i.e.,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}\times\mathbb{R}^N} \int_{B(y,1)} |w_n^+|^2 = 0,$$

or non-vanishing, i.e., there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{Z}^{N+1}$ such that

$$\lim_{n\to\infty} \int_{B(y_n,r)} |w_n^+|^2 \geq \frac{\delta}{2} > 0. \tag{2.4}$$

If $\{w_n^+\}$ is vanishing, then Lions' concentration compactness principle [14] (also [32]) implies $w_n^+ \to 0$ in L^p for $p \in (2, N^*)$. By (2.3) we know that $\Psi(\lambda w_n^+) \to 0$ as $n \to \infty$ for all $\lambda > 0$. Since $\lambda w_n^+ \in \hat{E}(z_n)$, then Lemma 2.3 implies that

$$M \geq \Phi(z_n) \geq \Phi(\lambda w_n^+) = \frac{\lambda^2}{2}\|w_n^+\|^2 - \Psi(\lambda w_n^+) \geq \frac{\lambda^2}{4} - \Psi(\lambda w_n^+) \to \frac{\lambda^2}{4}.$$

This yields a contradiction if λ is large enough. Hence, non-vanishing must hold and the invariance of Φ and $\mathcal{M}$ under translation implies that $\{y_n\}$ can be selected to be bounded. Then (2.4) implies $w^+ \neq 0$. Hence $w \neq 0$, which implies that $|z_n| \to \infty$ as $n \to \infty$. It follows from (S_3) and Fatou's lemma that

$$\frac{\Psi(z_n)}{\|z_n\|^2} = \int_{\mathbb{R}\times\mathbb{R}^N} \frac{F(t,x,u_n) + G(t,x,v_n)}{|z_n|^2} |w_n|^2 \to \infty$$

as $n \to \infty$. Therefore

$$0 \leq \frac{\Phi(z_n)}{\|z_n\|^2} = \frac{1}{2}(\|w_n^+\|^2 - \|w_n^-\|^2) - \frac{\Psi(z_n)}{\|z_n\|^2} \to -\infty$$

as $n \to \infty$, which is a contradiction. This completes the proof. □

Now, we consider the reduced functional

$$\hat{I}\colon E^+ \setminus \{0\} \to \mathbb{R}, \qquad \hat{I}(z) := \Phi(\hat{m}(z)) \quad \text{and} \quad I := \hat{I}|_{S^+}.$$

Arguing as in Proposition 2.9 in [23], $\hat{I} \in C^1(E^+ \setminus \{0\}, \mathbb{R})$ and

$$\hat{I}'(z)w := \frac{\|\hat{m}(z)^+\|}{\|z\|}\Phi'(\hat{m}(z))w, \quad \text{for all } z, w \in E^+,\ z \neq 0.$$

Therefore, by the similar argument as Corollary 2.10 in [23], we have

Lemma 2.9 ([23, Corollary 2.10]). *Suppose that the assumptions of Theorem 1.1 hold. Then*

(a) $I \in C^1(S^+, \mathbb{R})$ *and*

$$I'(z)w := \|m(z)^+\|\Phi'(m(z))w \quad \text{for all } w \in T_z(S^+),$$

where $T_z(S^+)$ is the tangent space of S^+ at z.

(b) If $\{z_n\}$ is a Palais–Smale sequence for I, then $\{m(z_n)\}$ is a Palais–Smale sequence for Φ. If $\{z_n\} \subset \mathcal{M}$ is a bounded Palais–Smale sequence for Φ, then $\{m^{-1}(z_n)\}$ is a Palais–Smale sequence for I.

(c) z is a critical point of I if and only if $m(z)$ is a nontrivial critical point of Φ. Moreover, the corresponding critical values coincide and $\inf_{S^+} I = \inf_{\mathcal{M}} \Phi$.

3 The proof of Theorem 1.1

Proof of Theorem 1.1. Using Lemma 2.9, $I \in C^1(S^+, \mathbb{R})$, $\{w_n\} \subset S^+$ is a (PS) sequence of I if and only if $\{m(w_n)\} \subset \mathcal{M}$ is a (PS) sequence of Φ, and $w \in S^+$ is a critical point of I if and only if $m(w) \in \mathcal{M}$ is a critical point of Φ, and the corresponding critical values coincide. Hence we only show that there exists a minimizer $u \in \mathcal{M}$ of $\Phi|_{\mathcal{M}}$.

By Ekeland's variational principle, there exists a sequence $\{w_n\} \subset S^+$ such that

$$I(w_n) \to \inf_{S^+} I = c \quad \text{and} \quad I'(w_n) \to 0.$$

Setting $z_n = m(w_n) \in \mathcal{M}$, then

$$\Phi(z_n) \to c \quad \text{and} \quad \Phi'(z_n) \to 0.$$

By Lemma 2.8, $\{z_n\}$ is bounded and hence $z_n \rightharpoonup z$ in E after passing to a subsequence. Therefore, $\{u_n^+\}$ is either vanishing, i.e.,

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}\times\mathbb{R}^N} \int_{B(y,1)} |z_n^+|^2 = 0,$$

or non-vanishing, i.e., there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{Z}^{N+1}$ such that

$$\lim_{n\to\infty} \int_{B(y_n,r)} |z_n^+|^2 \geq \frac{\delta}{2} > 0.$$

If $\{z_n^+\}$ is vanishing, then Lions' concentration compactness principle [14] implies $z_n^+ \to 0$ in L^p for $p \in (2, N^*)$. It follows from (2.2) that $\Psi'(z_n)z_n^+ = o(\|z_n^+\|)$, and hence

$$o(\|z_n^+\|) = \Phi'(z_n)z_n^+ = \|z_n^+\|^2 - \Psi'(z_n)z_n^+ = \|z_n^+\|^2 - o(\|z_n^+\|),$$

which implies that $\|z_n^+\|^2 = o(1)$. On the other hand, since $z_n \in \mathcal{M}$, by (S_4) we get

$$\begin{aligned} c \leq \Phi(z_n) &= \frac{1}{2}(\|z_n^+\|^2 - \|z_n^-\|^2) - \int_{\mathbb{R}\times\mathbb{R}^N} F(t,x,u_n) + G(t,x,v_n) \\ &\leq \frac{1}{2}(\|z_n^+\|^2 - \|z_n^-\|^2), \end{aligned}$$

which implies $\|z_n^+\|^2 \geq 2c > 0$. This is a contradiction. Therefore, $\{z_n\}$ is non-vanishing. Using a similar translation argument in the proof of Lemma 2.8 and the fact that Φ and $\mathcal{M}$ are $\mathbb{Z}^{N+1}$-invariant, without loss of generality, we can assume that $z = (u,v) \neq 0$. Hence $\Phi'(z) = 0$. Up to a subsequence, we assume that $z_n(t,x) \to z(t,x)$ a.e. in $\mathbb{R}\times\mathbb{R}^N$. By (S_4) and Fatou's lemma, we have

$$\begin{aligned} \Phi(z) &= \Phi(z) - \frac{1}{2}\Phi'(z)z \\ &= \int_{\mathbb{R}\times\mathbb{R}^N} \left(\frac{1}{2}f(t,x,u)u - F(t,x,u) + \frac{1}{2}g(t,x,v)v - G(t,x,v)\right) \\ &\leq \liminf_{n\to\infty} \int_{\mathbb{R}\times\mathbb{R}^N} \left(\frac{1}{2}f(t,x,u_n)u_n - F(t,x,u_n) + \frac{1}{2}g(t,x,v_n)v_n - G(t,x,v_n)\right) \\ &\leq \lim_{n\to\infty} \left(\Phi(z_n) - \frac{1}{2}\Phi'(z_n)z_n\right) \\ &= c. \end{aligned}$$

This yields that c is achieved by $z \in \mathcal{M}$. This completes the proof. □

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