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Lyapunov-type inequalities for $(m+1)$th order half-linear differential equations with anti-periodic boundary conditions

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Abstract. In this work, we will establish several new Lyapunov-type inequalities for $(m+1)$th order half-linear differential equations with anti-periodic boundary conditions, the results of this paper are new and generalize and improve some early results in the literature.

Keywords: half-linear differential equation, Lyapunov-type inequalities, anti-periodic boundary conditions.

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1 Introduction

The well-known Lyapunov inequality [6] for second-order linear differential equations states that if $u(t)$ is a nontrivial solution of the following problem

$$\begin{cases} u''(t) + r(t)u(t) = 0, & t \in (a,b), \\ u(a) = 0 = u(b), \end{cases} \tag{1.1}$$

where $r(t)$ is a continuous and nonnegative function defined in $[a,b]$, then

$$\int_a^b r(t)dt > \frac{4}{b-a}, \tag{1.2}$$

and the constant 4 cannot be replaced by a larger number.

The Lyapunov inequality has proved useful in the study of various properties of ordinary differential equations. Typical applications include bounds for eigenvalues, oscillation theory, stability criteria for periodic differential equations, and estimates for intervals of disconjugacy.

Since the appearance of Lyapunov's fundamental paper, there have been many improvements and generalizations of (1.2) in some literatures. A thorough literature review of continuous and discrete Lyapunov-type inequalities and their applications can be found in the

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survey articles by Cheng [5], Brown and Hinton [2], Tiryaki [11] and Pinasco [9]. Some other related results can be found in the articles [3,7,8,10,12–16,18] and the references cited therein.

But so far, there have been few works devoted to higher-order half-linear problems, mainly because the linear case was solved using Green's functions, which are not available now.

The study of Lyapunov-type inequalities for the differential equation under the anti-periodic boundary conditions was initiated by Wang [13]. He first obtained Lyapunov-type inequalities for $m+1$-order half-linear differential equation with anti-periodic boundary conditions, the main result is as follow.

Theorem 1.1. *Consider the following $m+1$-order half-linear differential equation*

$$(|u^{(m)}(t)|^{p-2}u^{(m)}(t))' + r(t)|u(t)|^{p-2}u(t) = 0, \qquad t \in (a,b) \text{ and } p > 1. \tag{1.3}$$

If $u(t)$ is a nonzero solution of (1.3) satisfying the anti-periodic boundary conditions

$$u^{(i)}(a) + u^{(i)}(b) = 0, \qquad i = 0,1,2,\dots,m, \tag{1.4}$$

then

$$\int_a^b |r(t)|\,dt > 2\left(\frac{2}{b-a}\right)^{m(p-1)}. \tag{1.5}$$

As a special case of Theorem 1.1, we also gave the following results.

Theorem 1.2. *Let us consider the following boundary value problem*

$$\begin{cases} u^{(2n+1)}(t) + r(t)u(t) = 0, \\ u^{(i)}(a) + u^{(i)}(b) = 0, \quad i = 0,1,\dots,2n. \end{cases} \tag{1.6}$$

If $u(t)$ is a nonzero solution of problem (1.6), then

$$\int_a^b |r(t)|dt > \frac{2^{2n+1}}{(b-a)^{2n}}. \tag{1.7}$$

Theorem 1.3. *Let us consider the following boundary value problem*

$$\begin{cases} u^{(n)}(t) + r(t)u(t) = 0, \\ u^{(i)}(a) + u^{(i)}(b) = 0, \quad i = 0,1,\dots,n-1. \end{cases} \tag{1.8}$$

If $u(t)$ is a nonzero solution of problem (1.8), then

$$\int_a^b |r(t)|dt > \frac{2^n}{(b-a)^{n-1}}. \tag{1.9}$$

Recently, there are several papers [1, 4] to discuss Lyapunov-type inequalities for half-linear system under anti-periodic boundary conditions. Very recently, Yang and Lo in [17] considered a more general higher-order anti-periodic boundary value problem, for example, they get the following result (the special case of Corollary 1).

Theorem 1.4. *Let us consider the following boundary value problem*

$$\begin{cases} u^{(n)}(t) + r(t)u(t) = 0, \\ u^{(k)}(a) + u^{(k)}(b) = 0, \quad k = 0,1,\dots,n-1. \end{cases} \tag{1.10}$$

If problem (1.10) has a nonzero solution $u(t)$, then the following inequality holds:

$$\int_a^b |r(t)|\,dt > \frac{\pi^{n-1}}{(b-a)^{n-1}} \cdot \frac{\sqrt{2}}{\sqrt{\left(1-\frac{1}{2^{2n-2}}\right)\zeta(2n-2)}}. \tag{1.11}$$

In this article, we try to generalize Lyapunov-type inequalities to more general half-linear differential equations under anti-periodic boundary conditions.

2 Main results

In this section, we give our main result Theorem 2.1 and some corollaries.

Theorem 2.1. *Consider the following $m+1$-order anti-periodic boundary value problem:*

$$(|u^{(m)}(t)|^{p-2}u^{(m)}(t))' + \sum_{j=0}^{m} r_j(t)|u^{(j)}(t)|^{p-2}u^{(j)}(t) = 0, \qquad t \in (a,b) \text{ and } p > 1, \tag{2.1}$$

$$u^{(i)}(a) + u^{(i)}(b) = 0, \qquad i = 0,1,2,\dots,m \quad \text{and} \quad u(t) \neq 0,\ \forall t \in (a,b), \tag{2.2}$$

where $m \geq 1$, $r_j(t)$, $j = 0,1,2,\dots,m$ are real continuous functions on $[a,b]$. If problem (2.1)–(2.2) has a nonzero solution $u(t)$, then the the following inequality holds:

$$\sum_{j=0}^{m-1} [(b-a)C_{m-j}]^{\frac{p-1}{2}} \int_a^b |r_j(s)|\,ds + \int_a^b |r_m(s)|\,ds > 2, \tag{2.3}$$

where

$$C_n = \frac{(2^{2n}-1)(b-a)^{2n-1}}{2^{2n-1}\pi^{2n}}\,\zeta(2n), \qquad n = 1,2,\dots,$$

and $\zeta(s) = \sum_{k=1}^{+\infty}\frac{1}{k^s}$, $\mathrm{Re}(s) > 1$ is the Riemann zeta function.

Before proving our theorem, we first give some corollaries of Theorem 2.1.

Let $r_m(t) = 0$ in (2.1), we have the following result.

Corollary 2.2. *Let us consider the following boundary value problem*

$$\begin{cases} (|u^{(m)}(t)|^{p-2}u^{(m)}(t))' + \sum_{j=0}^{m-1} r_j(t)|u^{(j)}(t)|^{p-2}u^{(j)}(t) = 0, \qquad t \in (a,b) \text{ and } p > 1, \\ u^{(i)}(a) + u^{(i)}(b) = 0, \qquad i = 0,1,\dots,m \quad \text{and} \quad u(t) \neq 0,\ \forall t \in (a,b), \end{cases} \tag{2.4}$$

where $m \geq 1$, $r_j(t)$, $j = 0,1,2,\dots,m-1$ are real continuous functions on $[a,b]$. If problem (2.4) has a nonzero solution $u(t)$, then the the following inequality holds:

$$\sum_{j=0}^{m-1} C_{m-j}^{\frac{p-1}{2}} \int_a^b |r_j(s)|\,ds > \frac{2}{(b-a)^{\frac{p-1}{2}}}. \tag{2.5}$$

For the linear case $p = 2$, we have the following result.

Corollary 2.3. *Let us consider the following boundary value problem*

$$\begin{cases} u^{(m+1)}(t) + \sum_{j=0}^{m} r_j(t)u^{(j)}(t) = 0, \\ u^{(i)}(a) + u^{(i)}(b) = 0, \quad i = 0,1,\dots,m, \end{cases} \tag{2.6}$$

where $m \geq 1$, $r_j(t)$, $j = 0,1,2,\dots,m$ are real continuous functions on $[a,b]$. If $u(t)$ is a nonzero solution of problem (2.6), then the following inequality holds:

$$\sum_{j=0}^{m-1} \sqrt{(b-a)C_{m-j}} \int_a^b |r_j(s)|\,ds + \int_a^b |r_m(s)|\,ds > 2. \tag{2.7}$$

Remark 2.4. Let $r_j(t) = 0$, $j = 1, 2, \dots, m$ in Corollary 2.3, we obtain Theorem 1.4.

Remark 2.5. If we compare Theorems 2.1 with results in [1,4], it is easy to see that they are different from each other.

Let $r_j(t) = 0$, $j = 1, 2, \dots, m$ in Theorem 2.1, For the nonlinear case, we have the following results.

Corollary 2.6. *Let us consider the following boundary value problem*

$$\begin{cases} (|u^{(m)}(t)|^{p-2}u^{(m)}(t))' + r_0(t)|u(t)|^{p-2}u(t) = 0, \quad p > 1, \\ u^{(i)}(a) + u^{(i)}(b) = 0, \quad i = 0, 1, \dots, m. \end{cases} \tag{2.8}$$

where $r_0(t)$ is a real continuous function on $[a, b]$. If $u(t)$ is a nonzero solution of problem (2.8), then the following inequality holds:

$$\int_a^b |r_0(s)|\, ds > 2\left(\frac{2}{b-a}\right)^{m(p-1)} \cdot \left[\frac{\pi^{2m}}{2(2^{2m}-1)\zeta(2m)}\right]^{\frac{p-1}{2}}. \tag{2.9}$$

Now, let us compare inequalities (2.9) and (1.5). Since for $m \geq 2$, $\zeta(2m) \leq \zeta(2) < 2$, we have

$$\frac{\pi^{2m}}{2(2^{2m}-1)\zeta(2m)} > \frac{\pi^{2m}}{2^2(2^{2m}-1)} > \frac{1}{4}\left(\frac{\pi^2}{4}\right)^m \geq \frac{1}{4}\left(\frac{\pi^2}{4}\right)^2 > 1,$$

thus

$$\left[\frac{\pi^{2m}}{2(2^{2m}-1)\zeta(2m)}\right]^{\frac{p-1}{2}} > \left[\frac{\pi^{2m}}{2^2(2^{2m}-1)}\right]^{\frac{p-1}{2}} > 1,$$

and

$$\left[\frac{\pi^{2m}}{2(2^{2m}-1)\zeta(2m)}\right]^{\frac{p-1}{2}} > \left[\frac{\pi^{2m}}{2^2(2^{2m}-1)}\right]^{\frac{p-1}{2}} \to +\infty \qquad (m \to +\infty)$$

so inequality (2.9) improves inequality (1.5) significantly.

3 Proof of Theorem 2.1

In this section, we prove our main result. For this purpose, we need the following lemmas.

Lemma 3.1 ([14])**.** *For $n \geq 1$, define the following Sobolev space:*

$$H = \left\{x \mid x^{(n)} \in L^2[a,b],\ x^{(i)}(a) + x^{(i)}(b) = 0, \quad i = 0, 1, 2, \dots, n-1\right\}.$$

For any $x \in H$, there exists a positive constant C_n such that the Sobolev inequality

$$\left(\sup_{a \leq t \leq b} |x(t)|\right)^2 \leq C_n \int_a^b |x^{(n)}(t)|^2\, dt. \tag{3.1}$$

holds, where

$$C_n = \frac{(2^{2n}-1)(b-a)^{2n-1}}{2^{2n-1}\pi^{2n}}\zeta(2n), \qquad n = 1, 2, \dots,$$

and $\zeta(s) = \sum_{k=1}^{+\infty} \frac{1}{k^s}$, $\mathrm{Re}(s) > 1$ is the Riemann zeta function, and the constants $\{C_n\}$ are sharp.

Lemma 3.2. *If $u(t)$ is a nonzero solution of (2.1) satisfying the anti-periodic boundary condition (2.2), denote $U_k = \sup_{a\le t\le b} |u^{(k)}(t)|$, then for $k = 0, 1, 2, \ldots, m-1$, we have*

$$U_k \le \sqrt{(b-a)C_{m-k}}\, U_m.$$

Proof. Applying Lemma 3.1 to $x = u^{(k)}$, $k = 0, 1, 2, \ldots, m-1$ and $n = m$ respectively, we obtain

$$\left(\sup_{a\le t\le b} |u^{(k)}(t)|\right)^2 \le C_{m-k} \int_a^b |u^{(m)}(t)|^2\, dt.$$

So,

$$\begin{aligned} U_k = \sup_{a\le t\le b} |u^{(k)}(t)| &= \sqrt{\left(\sup_{a\le t\le b} |u^{(k)}(t)|\right)^2} \le \sqrt{C_{m-k} \int_a^b |u^{(m)}(t)|^2\, dt} \\ &\le \sqrt{(b-a)C_{m-k}} \sup_{a\le t\le b} |u^{(m)}(t)| = \sqrt{(b-a)C_{m-k}}\, U_m. \end{aligned}$$

□

Proof of Theorem 2.1. Define

$$H(t,s) = \begin{cases} \frac{1}{2}, & a \le s \le t, \\ -\frac{1}{2}, & t \le s \le b. \end{cases}$$

Then, by the anti-periodic boundary condition (2.2) with $i = m$, we have

$$\begin{aligned} |u^{(m)}(t)|^{p-2}u^{(m)}(t) &= \int_a^b H(t,s)(|u^{(m)}(s)|^{p-2}u^{(m)}(s))'\, ds \\ &= -\sum_{j=0}^{m} \int_a^b H(t,s)r_j(s)|u^{(j)}(s)|^{p-2}u^{(j)}(s)\, ds, \end{aligned}$$

then

$$\begin{aligned} |u^{(m)}(t)|^{p-1} &\le \sum_{j=0}^{m} \int_a^b |H(t,s)||r_j(s)||u^{(j)}(s)|^{p-1}\, ds \\ &\le \frac{1}{2}\sum_{j=0}^{m} \int_a^b |r_j(s)||u^{(j)}(s)|^{p-1}\, ds \\ &< \frac{1}{2}\sum_{j=0}^{m} U_j^{p-1} \int_a^b |r_j(s)|\, ds \\ &= \frac{1}{2}\left(\sum_{j=0}^{m-1} U_j^{p-1} \int_a^b |r_j(s)|\, ds + U_m^{p-1} \int_a^b |r_m(s)|\, ds\right) \\ &\le \frac{1}{2}U_m^{p-1}\left[\sum_{j=0}^{m-1} ((b-a)C_{m-j})^{\frac{p-1}{2}} \int_a^b |r_j(s)|\, ds + \int_a^b |r_m(s)|\, ds\right], \end{aligned}$$

thus

$$U_m^{p-1} < \frac{1}{2}U_m^{p-1}\left[\sum_{j=0}^{m-1} ((b-a)C_{m-j})^{\frac{p-1}{2}} \int_a^b |r_j(s)|\, ds + \int_a^b |r_m(s)|\, ds\right]. \tag{3.2}$$

Now, we claim that $U_m > 0$. In fact, if it is not true, then we have $U_m = 0$ or $u^{(m)}(t) = 0$ for $t \in [a, b]$. By the anti-periodic condition (2.2), we obtain $u(t) = 0$ for $t \in [a, b]$, which contradicts to $u(t)$ is a nonzero solution of (2.1)–(2.2). Thus, $U_m > 0$, dividing both sides of the inequality (3.2) by U_m, we obtain

$$\sum_{j=0}^{m-1} ((b-a)C_{m-j})^{\frac{p-1}{2}} \int_a^b |r_j(s)|\, ds + \int_a^b |r_m(s)|\, ds > 2. \qquad \square$$

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