On a Peculiar Attractor for Weakly Nonlinear Oscillators with a Two Period Quasiperiodic Forcing

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Abstract

We study a very peculiar nonlinear oscillator with an external two period quasiperiodic excitation, being the golden mean the ratio between the two frequencies. The two period quasi periodic forcing is characterized by an infinite frequencies number. As a consequence, we find that the motion settles down in a two period quasi periodic atttractor for a wide excitation amplitude range. The competition between the two frequencies does not produce a closed curve but fills a well defined phase space region in the Poincarè section. This attractor somehow resembles strange nonchaotic attractors because both are characterized by quasiperiodic forcing. Using a suitable perturbation method, we can understand the new attractor most important characteristics and find an approximate solution for its dynamical behavior. Numerical simulations are used to check out the analytical investigation.

***Keywords*:** Nonlinear dynamics; quasi periodic attractor; Poincarè section; perturbation method

1. Introduction

Quasi-periodic behavior and their relations with chaotic dynamical systems have been extensively studied in the last years [1-2]. Gilsinn used a series expansion in order to obtain a higher order approximation for a quasiperiodic solutions for two weakly coupled van der Pol oscillators[3].

In a very interesting paper Venkatesan and Lakshmanan [4] investigated nonlinear motion on a rotating parabola. They discovered remarkable behaviors and various bifurcations such as symmetry breaking, period doubling, and intermittency. They carefully considered a quasiperiodically driven rotating parabola system and the transition from two-frequency quasiperiodicity to chaotic behavior. As the driving parameter increases, a torus doubling bifurcation occurs with torus merging and transition from the merged torus to a strange nonchaotic attractor and at last transition from the strange nonchaotic attractor to a geometrically similar chaotic attractor.

Kuznetsov et al. [5] considered an autonomous three-dimensional system and demonstrated quasiperiodic dynamics because of presence of two coexisting oscillatory components of incommensurate frequencies. The attractor is obviously a two-dimensional torus. Moreover they considered charts of dynamical regimes on parameter planes. R. Vitolo et al. [6] studied dissipative dynamical systems and quasi-periodic behavior. They used algorithms for the computation and continuation of invariant circles and of their bifurcations. Remarkable applications are given for quasiperiodic bifurcations of saddle-node, Hopf and period-doubling type.

Hidaka et al. [7] studied the Arnold resonance web in a map generating invariant three-torus. They demonstrated that the quasi-periodic saddle-node bifurcations generate complex bifurcations and they occur when a stable and saddle invariant two-torus merge and disappear. At last just after the quasi-periodic saddle-node bifurcation, an intermittent torus can be observed.

In a very interesting review paper, Prasad et al [8] considered carefully strange nonchaotic attractor, SNAs. These amazing attractors are often observed in quasiperiodically driven nonlinear systems. Being geometrically fractal they can be considered strange attractors. Trajectories do not show exponential sensitivity to initial conditions because the largest Lyapunov exponent is zero or negative. Different experimental situations show SNAs in quasiperiodically driven mechanical systems, plasma discharges, various electronic systems and so on. Moreover they demonstrate the equivalence between a quasiperiodically driven system and the Schrödinger equation for a particle in a connected quasiperiodic potential, with a correspondence between the localized states of the quantum problem with SNAs in the connected classic dynamical system.

At last they discuss the SNAs most important features, beginning from the different bifurcations or routes for the creation of such attractors and arriving at the dynamical transitions in quasiperiodically forced systems.

In this paper we consider a weakly nonlinear oscillator with an external excitation with two frequencies Ω (ω and Ω are uncommensurable and their ratio is the golden mean, about 1.61803..)

$$\ddot{X}(t) + \omega^2 X(t) = bX^2(t) + cX^3(t) + F(t) \tag{1a}$$

where

$$F(t) = f cos\big(\Omega t + K sin(\omega t)\big) \tag{1b}$$

The quasiperiodic forcing (1b) in equation (1) is motivated by the Arnold circle or standard map [9] and contain two principal frequencies Ω (the golden mean) and ω=1, but obviously there are infinite frequencies in its Fourier spectrum. We can consider this quasiperiodic forcing to be the suitable one when there are external excitations with distorted or deformed or twisted sinusoidal waves, because only in laboratory we are able to produce pure two period quasiperiodic forcing that is a two harmonics addiction. The function F(t) in Equation (1b) is shown in Fig. 1.

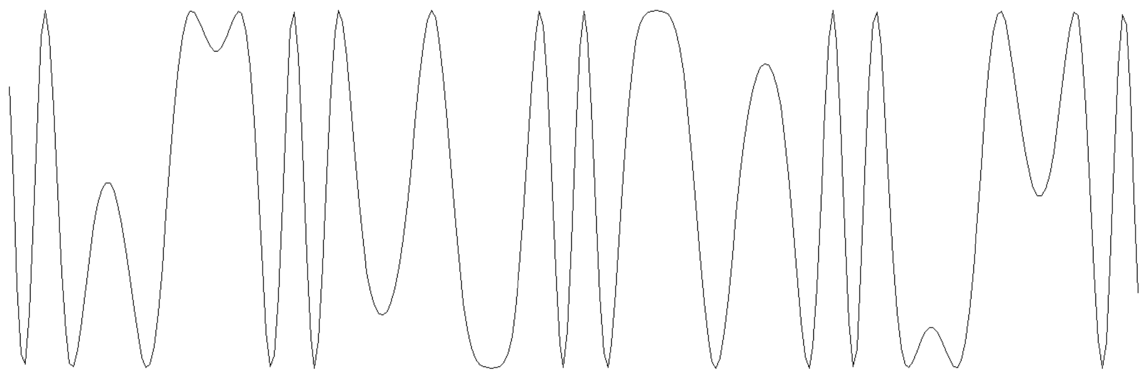

Figure 1: The quasiperiodic forcing with Ω=1.618033989.., K=5 and ω=1.

If we want to balance the effect of the nonlinearity and external excitation we need to scale the external excitation coefficient f and the coefficients b and c, as εf, εb, $\varepsilon^2 c$ where ε is a bookkeeping device, and Equations (1) yield

$$\ddot{X}(t) + \omega^2 X(t) = b\varepsilon X^2(t) + c\varepsilon^2 X^3(t) + f\varepsilon cos\big(\Omega t + Ksin(\omega t)\big) \tag{2}$$

We have chosen the golden mean because of its relation with strange nonchaotic stars [10]. In Sect. 2 we use an adequate perturbation method [11] and get a model system of nonlinear differential equations for the nonlinear oscillator. We see that the model system can be easily solved and we are able to write an approximate solutions for the equations (1) and figure out the system settles down into a particular attractor characterized by two competing frequencies ω=1 and Ω= golden mean,. We underline that the system evolution is not a pure two period quasi periodic motion, because the Poincarè section is not a closed curve. In Sect. 3 we describe the most important findings obtained from numerical simulation. We find that our prediction is correct and the oscillator drops into a remarkable attractor, characterized by a frequencies infinite number. A well defined phase space region in the Poincarè section is filled by the oscillator motion. This new attractor resembles strange nonchaotic attractors because both are characterized by quasiperiodic forcing. The approximate solution found in Sect. 2 is able to describe the solution qualitative behavior in the Poincarè section. In Sect. 4 we summarize the most important paper findings and discuss some research developments.

2. Building an approximate solution

We now introduce the slow time

$$\tau = \varepsilon^q t, \tag{3}$$

and seek a solution of Equation (2) in the following form [11]

$$X(t) = (\psi_1(\tau)\exp(-i\Omega t) + \psi_0(\tau)\varepsilon + \psi_2(\tau)\varepsilon\exp(-2i\omega t) + c.c.) + O(\varepsilon^2), \tag{4}$$

and we see that it can be considered a combination of the various harmonics with coefficients depending on τ.

Note that the introduction of the slow time (3) implies that

$$\frac{d}{\mathrm{dt}}\left(\psi_n \exp(-\mathrm{i}n\omega t)\right) = \left(-\mathrm{i}n\Omega\psi_n + \varepsilon^q \frac{d\psi_n}{dt}\right)\exp(-\mathrm{i}n\omega t). \quad (5)$$

Using Equation (4) and substituting into Equation (2) yields various equations for each harmonic *n* and for a fixed order of approximation on the perturbation parameter ε.

For *n=1*, we get

$$-2i\omega\psi_\tau\varepsilon^q = (2b)\psi_0\psi\varepsilon\varepsilon^r + (2b)\psi_2\psi_{-1}\varepsilon^2 + 3c|\psi|^2\psi\varepsilon^2 \quad (6)$$

and , after setting *q=2, r=1* for an adequate terms balance, we get for n=0, with, $\Phi = \psi_0$

$$\Phi = \frac{f}{\omega^2}cos\left(\Omega t + Ksin(\omega t)\right) + \frac{2b}{\omega^2}|\psi|^2 \quad (7)$$

for n=2

$$A_2 = \frac{-b}{3\omega^2}\ , \quad \psi_2 = A_2\psi^2 \quad (8)$$

As we can see from Equation (5), we can derive a differential equation for the evolution of the complex amplitude ψ,

$$\frac{d\psi}{dt} = (i\alpha_1)\psi\Phi + (i\beta_1)|\psi|^2\psi, \quad (9)$$

with

$$\alpha_1 = \frac{b}{\omega}, \quad (10)$$

$$\beta_1 = \frac{3c}{2\omega} - \frac{b^2}{3\omega^3}. \quad (11)$$

Substituting the polar form,

$$\psi(\tau) = \rho(\tau)\exp\left(i\theta(\tau)\right), \quad (12)$$

into Equation (9), and separating real and imaginary parts, we arrive at the following model system

$$\frac{d\rho}{dt} = 0 \quad (13)$$

$$\frac{d\theta}{dt} = \alpha_1\Phi + \beta_1\rho^2. \quad (14)$$

Taking into account Equations (4), (7), (8) and (12), the lowest order approximate solution of Equation (2) can be written as

$$X(t) = 2\rho(t)cos(\omega t - \vartheta) + \Phi + 2A_2\rho^2cos(-2\omega t + 2\theta) + o(\varepsilon^2). \quad (15)$$

and Φ is given by Equation (7). We underline this result does not depend on the quasiperiodic forcing chosen form, but the point here is that its infinite Fourier components mix up in order to produce a motion characterized by infinite frequencies. Without the nonlinear terms this result is not possible and in the linear case b=c=0 we get a closed curve in the Poincarè section. The validity of the approximate solution should be expected to be restricted on bounded intervals of the t-variable and then on time-scale $t = O\left(\frac{1}{\varepsilon}\right)$. If one wishes to construct approximate solutions on larger intervals such that $t = O\left(\frac{1}{\varepsilon}\right)$ then the higher terms will in general affect the solution and must be included. Moreover, the approximate solution (22) will be within $O(\varepsilon)$ of the true solution on bounded intervals of the t-variable, and, if the solution is periodic, for all *t*. However we can trust excessively this approximate solution, because we neglected the fundamental resonance Fourier component of the quasi periodic forcing. This component should be inserted into Equation (6) and the solution can become unstable if the quasiperiodic forcing is too strong..

In the next section we will show that the approximate solution is able to catch the most important solution characteristics.

The system (13-14) can be easily solved

$$\rho(t) = \rho_0 = const \tag{16}$$

$$\theta(t) = \beta_1 {\rho_0}^2 t + \int_0^t \Phi\,(t')dt'. \tag{17}$$

We underline that the Integral

$$I(t) = f \int_0^t cos\left(\Omega t' + Ksin(\omega t')\right)dt' \tag{18}$$

in (17) can not be written with elementary functions and the same for the approximate solution and its derivative. If we consider the solution 15 at the time t=k T, where T is the golden mean period 3.883022077… seconds and k a positive integer, we can conclude that this solution is not clearly a closed curves, because there are many frequencies coming from the quasiperiodic forcing in the equation (2) and mixing because of nonlinear terms. Roughly speaking the attractor thickness is depending on the ratio f/ω^2 and we will corroborate that in the next section by the numerical simulation.

3. Results from numerical simulation

We investigate the motion in the phase space with b=-0.1, c=0.01, f=0.11 and initial conditions X=2,0, Y=0,1. The orbit amplitude is clearly modulated, then we can investigate the Poincarè section (Fig. 2), strobing the solution every 3.883222077.. seconds(=), where Ω=φ= golden mean=1.618033 989....

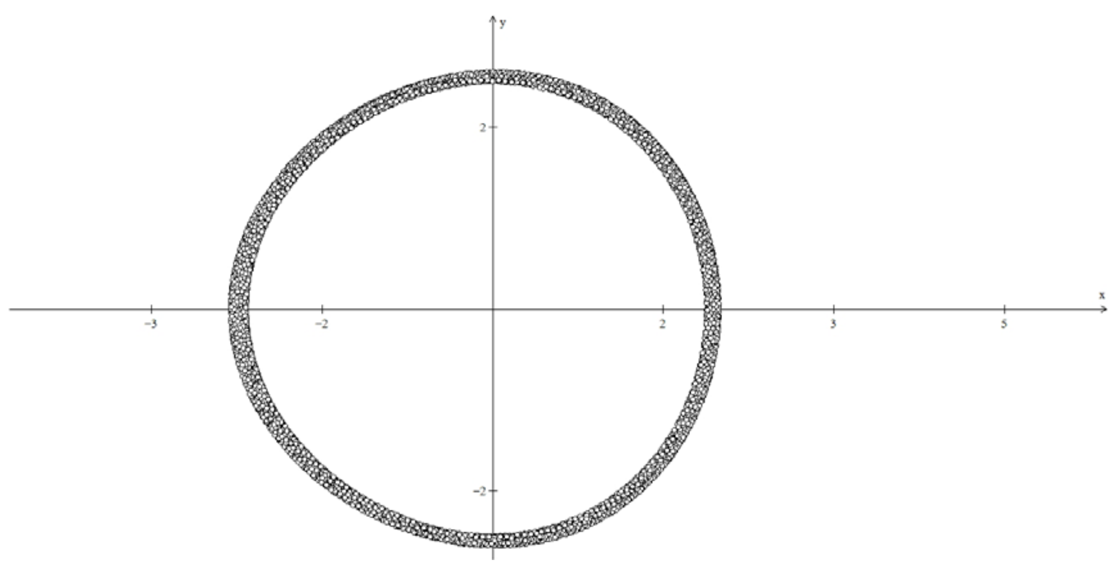

Figure 2: The attractor Poincarè section with initial conditions X=2.0 and Y=0.0, and ω=1, b=-0.1, c=0.01, f=0.11, K=5. (solution strobed every T=2π/Ω=3.883222077 seconds).

It is clear the solution does not follow a closed curve but fill in a well defined region in phase space.. The Poincarè sections are shown even with f=0.13(Fig. 3) and f=0.07 (Fig. 4) with the same initial conditions and the attractor is somehow stable for different quasiperiodic forcing amplitude.

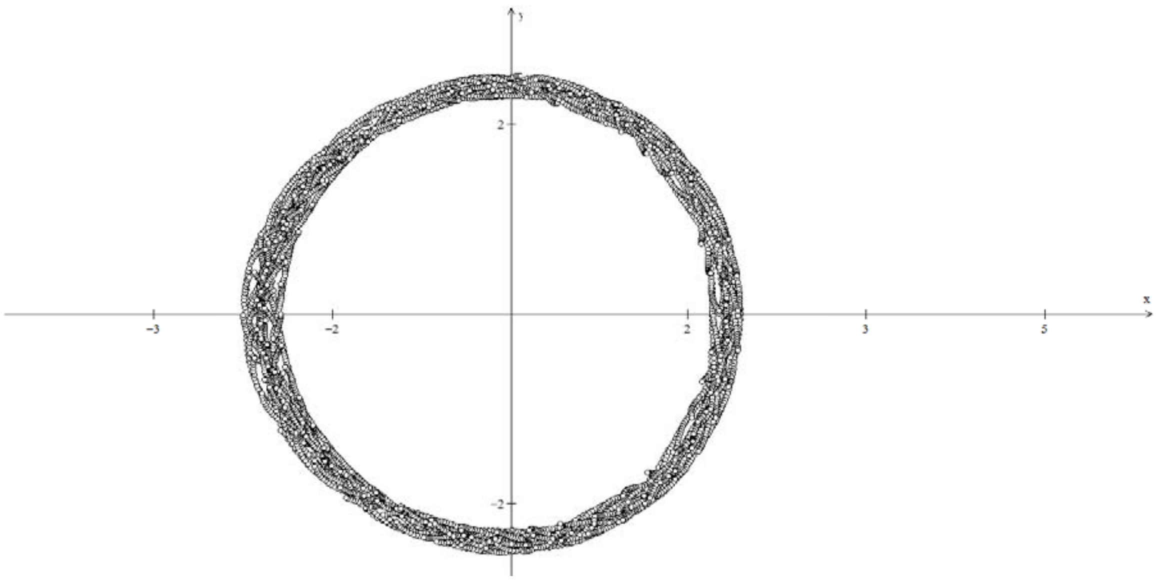

Figure 3: The attractor Poincarè section with initial conditions X=2.0 and Y=0, and ω=1, b=-0.1, c=0.01, f=0.13, K=5. (solution strobed every T=2π/Ω=3.883222077 seconds).

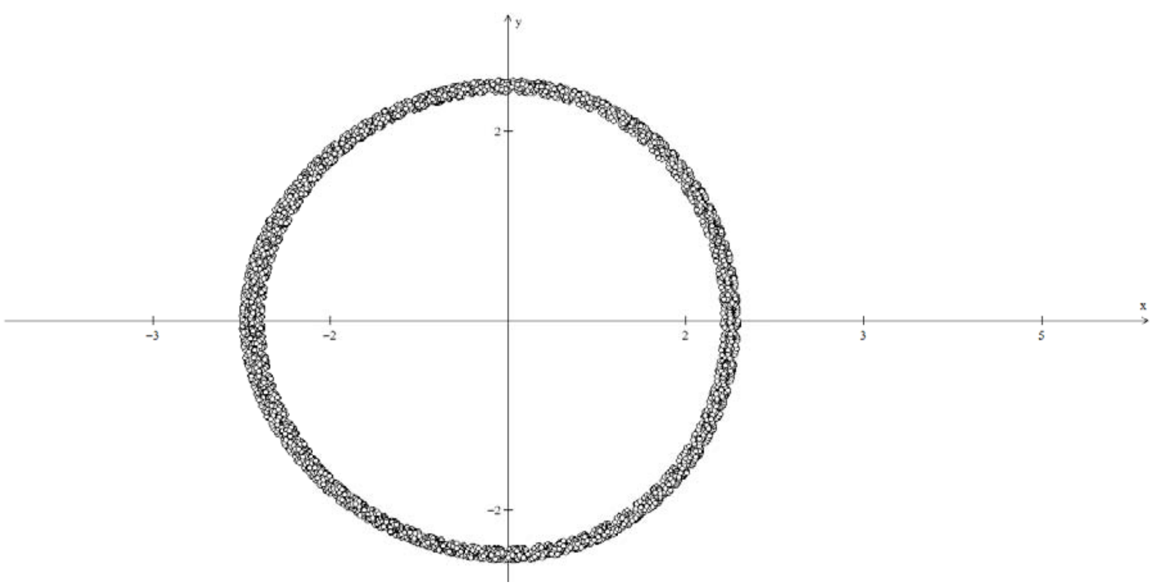

Figure 4 attractor Poincarè section with initial conditions X=2.0 and Y=0.0, and ω=1, b=-0.1, c=0.01, f=0.07, K=5. (solution strobed every T=2π/Ω=3.883222077 seconds).

The attractor shape is clearly depending on the ratio between the two frequencies and in Fig. 5 we show it for ω=0.8 with initial conditions X=2.0 and Y=0.0.

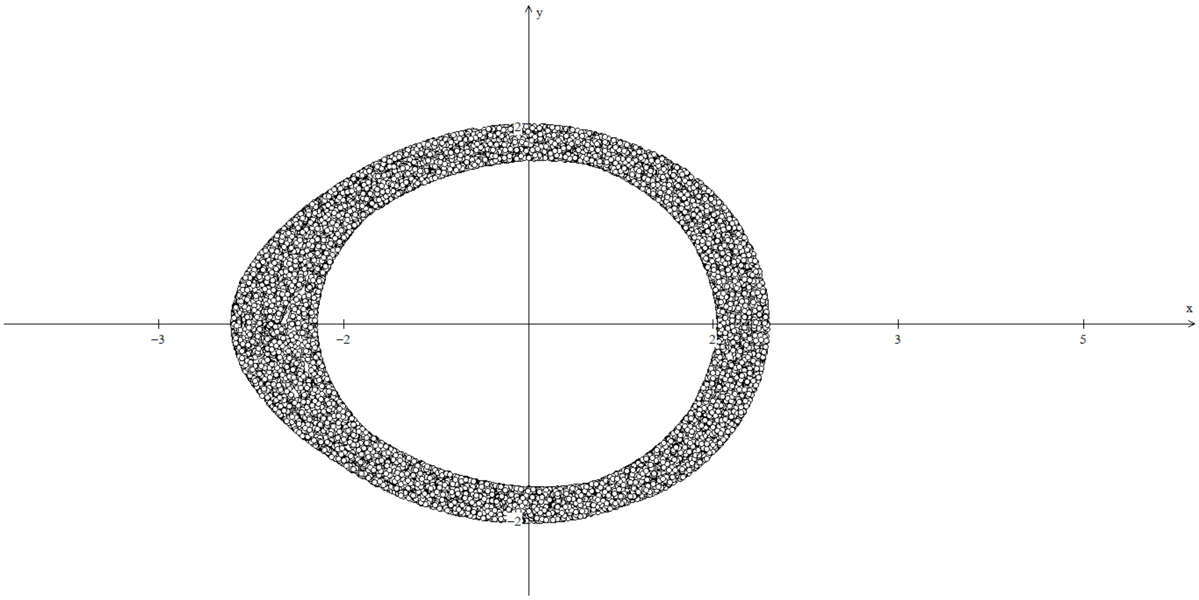

Figure 5 The attractor Poincarè section with initial conditions X=2.0 and Y=0.0, and ω=0.8, b=-0.1, c=0.01, f=0.11, K=5. (solution strobed every T=2π/Ω=3.883222077 seconds)

In Fig.6 the forcing amplitude increases (f=0.15) and the initial conditions are X=1.0 and Y=-1.0.

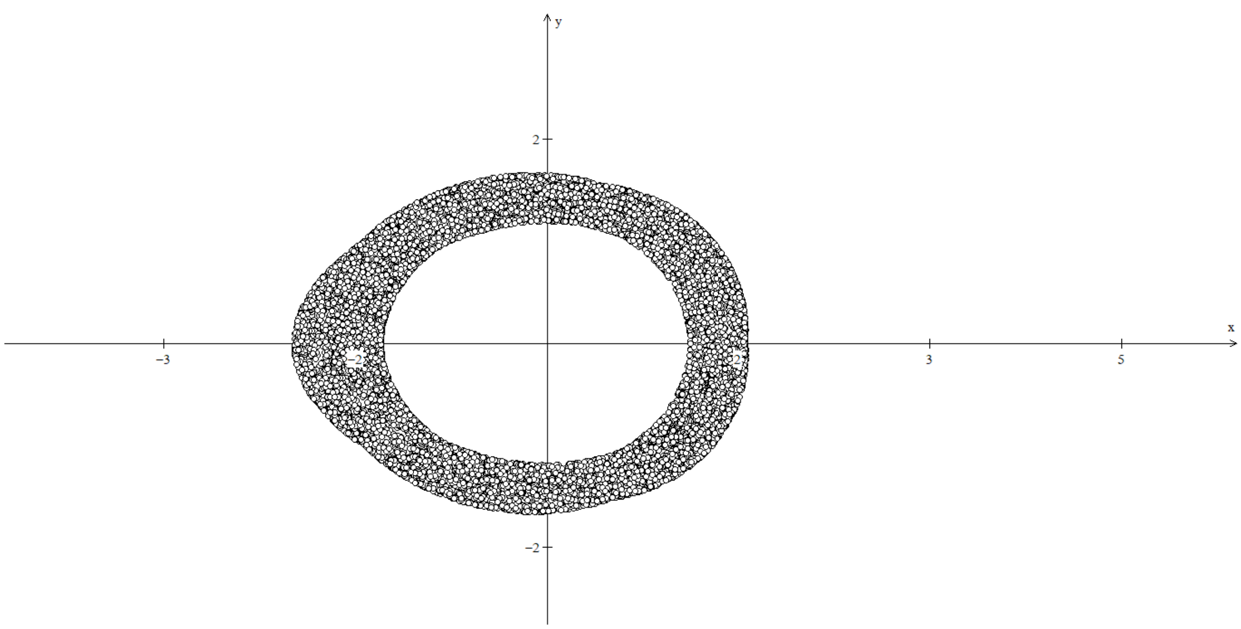

Figure 6 The attractor Poincarè section with initial conditions X=1.0 and Y=-1.0, and ω=0.8, b=-0.1, c=0.01, f=0.15, K=5. (solution strobed every T=2π/Ω=3.883222077 seconds).

From figures 7 to 10 we study the attractor dependence on the forcing amplitude and see that the attractor body enlarges for f=0.3(Fig.7) and f==0.4 (Fig. 8). Different initial conditions are shown in Fig. 9 (X=1.2 and Y=0.7) and Fig.10 (X=0.7 and Y=1.2), and for higher quasiperiodic forcing values the attractor becomes unstable and disappears. In Fig.11 we show a pictorial 3-dimensional solution representation for the Fig. 10 Poincarè section, where the z-axis stands for time.

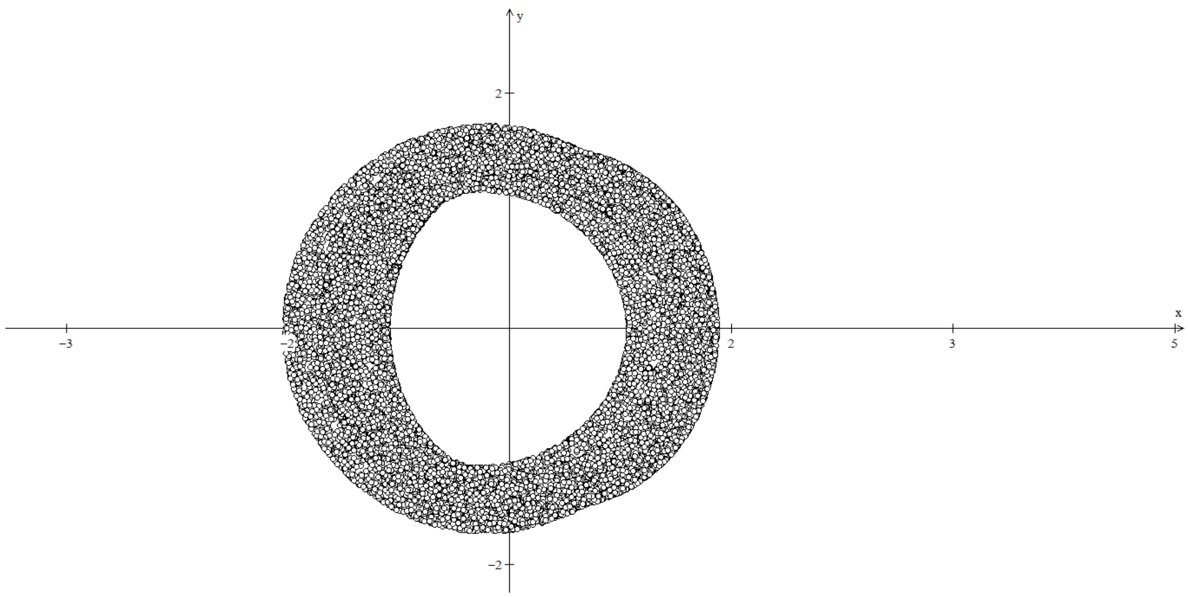

Figure 7 : The attractor Poincarè section with initial conditions X=1.1 and Y=-1.1, and ω=1, b=-0.1, c=0.01, f=0.3, K=5. (solution strobed every T=2π/Ω=3.883222077 seconds).

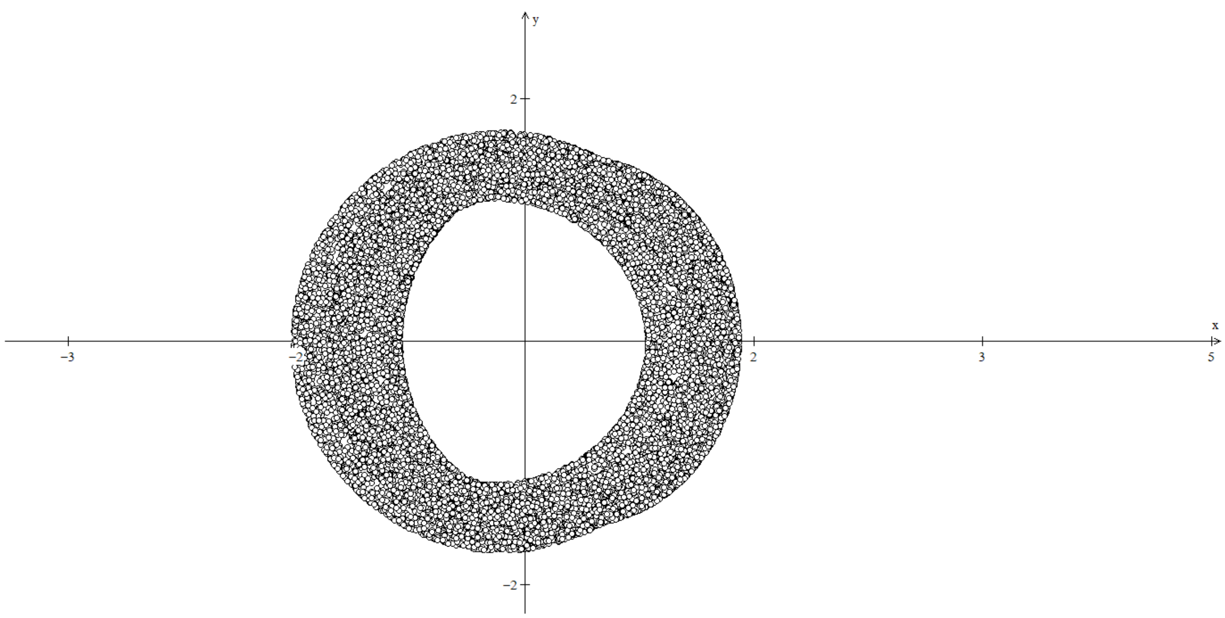

Figure 8: The attractor Poincarè section with initial conditions X=1.0 and Y=-1.0, and ω=1, b=-0.1, c=0.01, f=0.4, K=5. (solution strobed every T=2π/Ω=3.883222077 seconds).

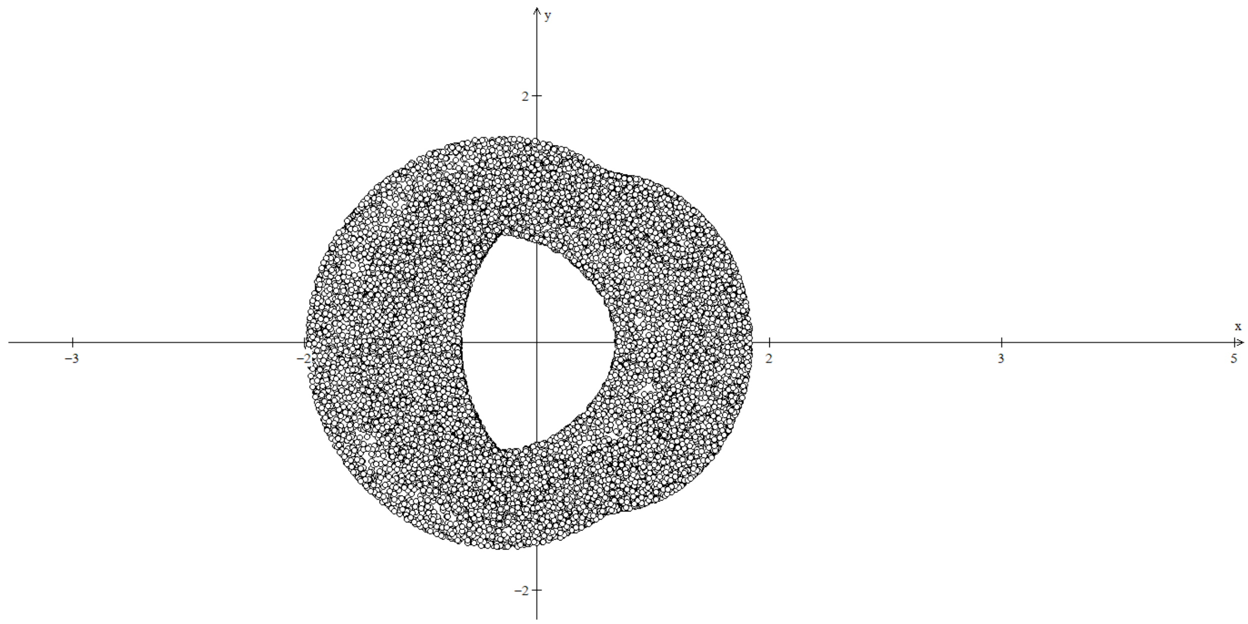

Figure 9: The attractor Poincarè section with initial conditions X=1.2 and Y=0.7, and ω=1, b=-0.1, c=0.01, f=0.4, K=5. (solution strobed every T=2π/Ω=3.883222077 seconds).

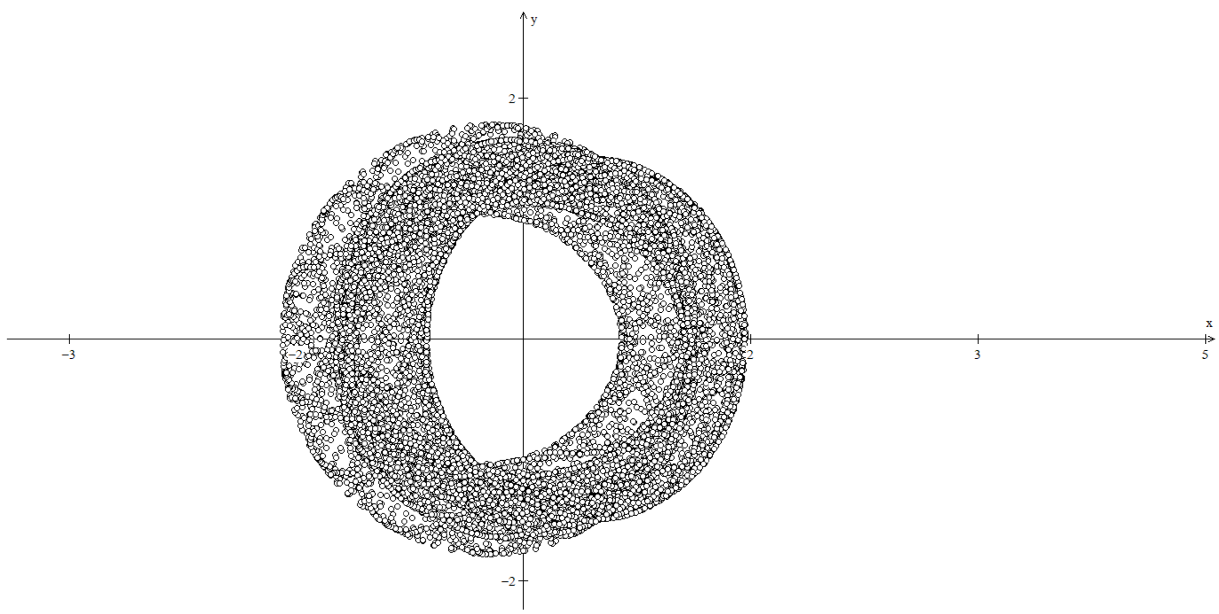

Figure 10: The attractor Poincarè section with initial conditions X=0.7 and Y=1.2, and ω=1, b=-0.1, c=0.01, f=0.4 (solution strobed every T=2π/Ω=3.883222077 seconds).

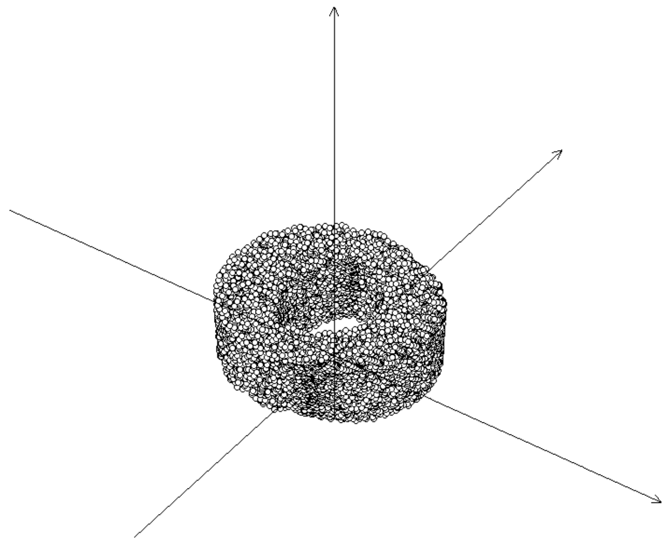

Figure 11: pictorial 3-dimensional solution representation of the Fig. 10 Poincarè section where the z-axis stands for time.

4. Conclusion

We have studied a nonlinear oscillator with an external two period quasiperiodic excitation, being the golden mean the ratio between the two frequencies. We find the motion settles down into a particular attractor, characterized by a frequencies infinite number. The competition between the two frequencies fills a well defined phase space region in the Poincarè section. This attractor resemble strange nonchaotic attractors because both are characterized by quasiperiodic forcing. Using a suitable perturbation method, we can understand the new attractor most important characteristics and find an approximate solution for its dynamical behavior. Numerical simulations are used to check out the analytical investigation.

This attractor is obviously not chaotic because two trajectories do not show exponential sensitivity to initial conditions. It is not strange because the attractor can be described by an approximate solutions.

We remember that the approximate solution is not fully adequate because we do not consider the quasiperiodic forcing Fourier component corresponding to the natural frequency.

It is not a purely two period quasiperiodic motion, that is not a two harmonics addition, because it fills not a closed curve but a well defined region in the Poincarè section Taking into account all the above reasons this attractor could be confused with a nonchaotic strange attractor.

This paper could be the starting point for some research developments -

i) The attractor characteristics can be studied more carefully including their sensitivity to the quasiperiodic forcing and to the ratio between the two frequencies:

ii) the approximate solution (Sect.2) should be improved because the quasiperiodic forcing gets a Fourier component in the fundamental resonance (ω=1) and we must consider that in our equations.

iii) This attractor could be confused with a SNA, strange nonchaotic attractor, and we need a more detailed study about their Fourier spectra.

iv) We think there are many possible physics applications out there and its our task to find them. Probably this attractor was already observed but neglected because if the quasi periodic forcing is very weak then the attractor can be confused with a closed curve in the Poincarè section. As a rule only laboratory produced quasiperiodic forcings are purely two period forcing that is the simple additions of two harmonics but in the wild quasiperiodic signals are always or usually characterized by an unlimited frequencies number.

v) It is well known the equivalence between a quasiperiodically driven system and the Schrödinger equation for a particle in a suitable quasiperiodic potential, them a new field research is investigating the correlation between the localized states of the quantum problem with this attractor in the associated dynamical system.

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