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# Bordered Complex Hadamard Matrices and Strongly Regular Graphs

Takuya IKUTA[1] and Akihiro MUNEMASA[2,*]

[1]*Faculty of Law, Kobe Gakuin University, Kobe 650-8586, Japan*
[2]*Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan*

We consider bordered complex Hadamard matrices whose core is contained in the Bose–Mesner algebra of a strongly regular graph. Examples include a complex Hadamard matrix whose core is contained in the Bose–Mesner algebra of a conference graph due to J. Wallis, F. Szöllősi, and a family of Hadamard matrices given by S. N. Singh and O. P. Dubey. In this paper, we prove that there are no other bordered complex Hadamard matrices whose core is contained in the Bose–Mesner algebra of a strongly regular graph.

KEYWORDS: association scheme, Hadamard matrix, conference graph

## 1. Introduction

A complex Hadamard matrix is a square matrix $W$ of order $n$ which satisfies $W\overline{W}^\top = nI$ and all of whose entries are complex numbers of absolute value 1. They are the natural generalization of real Hadamard matrices. Complex Hadamard matrices appear frequently in various branches of mathematics and quantum physics.

We consider the following "bordered" matrix of the form:

$$W = \begin{pmatrix} 1 & \mathbf{e} \\ \mathbf{e}^\top & W_1 \end{pmatrix}, \tag{1}$$

where $\mathbf{e}$ is the all 1's row vector of size $n$. The submatrix $W_1$ is said to be the core of $W$. In this paper, we consider complex Hadamard matrices $W$ of the form (1) whose core $W_1$ is contained in the Bose–Mesner algebra of a symmetric 2-class association scheme. In [13, 14] J. Wallis and F. Szöllősi constructed a complex Hadamard matrix $W$ whose core $W_1$ is contained in the Bose–Mesner algebra of a conference graph. And, in [11] S. N. Singh and O. P. Dubey constructed a Hadamard matrix $W$ whore core $W_1$ is contained in the Bose–Mesner algebra of strongly regular graph with $(k, \lambda, \mu) = (2r^2, r^2, r^2)$. As a natural problem, assuming $W_1$ is contained in the Bose–Mesner algebra of a strongly regular graph, we are interested in whether $W$ is a complex Hadamard matrix or not.

A similar problem has been considered in our earlier papers (see [7, 8, 13] and references therein). In [7, 8], we considered borderless complex Hadamard matrices contained in the Bose–Mesner algebra of some association schemes.

Let $X$ be a finite set with $n$ elements, and let $\mathfrak{X} = (X, \{R_i\}_{i=0}^2)$ be a symmetric 2-class association scheme with the first eigenmatrix $P = (P_{i,j})_{0 \le i,j \le 2}$:

$$\begin{pmatrix} 1 & k & \ell \\ 1 & r & -(r+1) \\ 1 & s & -(s+1) \end{pmatrix}, \tag{2}$$

where $r, s \in \mathbb{R}$, $r \ge 0$, and $s \le -1$. We let $\mathfrak{A}$ denote the Bose–Mesner algebra spanned by the adjacency matrices $A_0, A_1, A_2$ of $\mathfrak{X}$. A strongly regular graph $\Gamma$ with parameters $(k, \lambda, \mu)$ is equivalent to $\mathfrak{X}$, via the correspondence $R_1$ equal to the set of edges and $R_2$ equal to the set of non-edges. In this paper, by exchanging $R_1$ and $R_2$, we may assume that $r + s \ge -1$ without loss of generality.

Let

$$W_1 = w_0A_0 + w_1A_1 + w_2A_2 \in \mathfrak{A}. \tag{3}$$

Suppose that $w_0, w_1, w_2$ are complex numbers of absolute value 1, and $w_1 \ne w_2$. Then we have the following.

**Theorem 1.** *Suppose that $r, s \in \mathbb{R}$, $r \ge 0$, $s \le -1$, $r + s \ge -1$, and $w_1 \ne w_2$. Let $W_1$ be the matrix defined in* (3). *If*

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*Corresponding author. E-mail: munemasa@math.is.tohoku.ac.jp

*the matrix W defined by* (1) *is a complex Hadamard matrix, then one of the following holds.*
(i) $\Gamma$ *has parameter* $(k, \lambda, \mu) = (2r^2, r^2, r^2)$, *and* $(w_0, w_1, w_2) = (1, -1, 1)$.
(ii) $\Gamma$ *is a conference graph on* $2k + 1$ *vertices, and*
  (a) $(w_0, w_1, w_2) = (-1, \pm i, \mp i)$, *and*
  (b) $(w_0, w_1, w_2) = (1, \frac{-1\pm i\sqrt{k^2-1}}{k}, \frac{-1\mp i\sqrt{k^2-1}}{k})$.
*Conversely, if* (i) *or* (ii) *holds, then W is a complex Hadamard matrix.*

**Remark 2.** Strongly regular graphs having parameters (i) in Theorem 1 was considered in [11]. The list of strongly regular graphs up to 1,300 vertices are given in Brouwer's database [3]. According to that, strongly regular graphs with such a parameter exist for $r = 2, \dots, 10, 12, \dots, 16, 18$, and are unknown for $r = 11, 17$.

Complex Hadamard matrices having (a) and (b) in Theorem 1 (ii) were considered in [14] and [13, Proposition 3.4.16], respectively. The matrix in (b) of Theorem 1 (ii) is a Butson-type complex Hadamard matrix if and only if $k = 2$ [13, Remark 3.4.17]. If a conference graph on $2k + 1$ vertices exists, then $k$ must be even. A conference graph on $2k + 1$ vertices is known to exist for $k = 2, 4, \dots, 30$ except $k = 10, 16, 28$, for which the nonexistence is known. The existence is undecided for $k = 32$.

**Remark 3.** Two complex Hadamard matrices $W$ and $W'$ are said to be equivalent if there exist diagonal matrices $D, D'$ with nonzero complex diagonal entries, and permutation matrices $T, T'$, such that $DWD' = TW'T'$ holds. In Theorem 1 (i), both parts (a) and (b) give two conjugate complex Hadamard matrices of order $n = 2k + 1$. The matrices in (a) and (b) are never equivalent. This can be seen by computing the Haagerup set $H(W)$ (see [5]) defined as

$$H(W) = \left\{ \frac{W_{i_1,j_1} W_{i_2,j_2}}{W_{i_1,j_2} W_{i_2,j_1}} \,\middle|\, 1 \le i_1, i_2, j_1, j_2 \le n \right\},$$

which is an invariant for equivalence. Indeed, for a matrix $W$ in part (a) of Theorem 1 (ii), we have $H(W) \subseteq \{\pm 1, \pm i\}$, while for a matrix $W$ in part (b) of Theorem 1 (ii), we have $H(W) \cap \{\pm 1, \pm i\} = \{1\}$.

As for the two conjugate matrices in part (a) or in part (b), they are equivalent if the corresponding conference graph is self-complementary. Otherwise, it is unclear whether the two conjugate matrices are equivalent or not. There are non-self-complementary conference graphs of order 25, according to [6].

The organization of the paper is as follows. After giving preliminaries in Sect. 2, we give an overview on strongly regular graphs in Sect. 3. We also prove the "converse" part of Theorem 1 in Sect. 3. It then remains to derive (i) and (ii) of Theorem 1 under the hypotheses of that theorem. In Sect. 4, we give a quadratic equation satisfied by the real part of $w_1$ [see {(3)}], and a necessary condition that the real part lies in the interval $[-1, 1]$. In Sect. 5, we consider the special case $r + s = -1$, and derive Theorem 1 (ii). In Sect. 6, we take a closer look at the properties of the polynomials $L(X)$, $M(X)$, and $S(X)$ which are needed to express the real part of $w_1$. As a result, we obtain Theorem 1 (i) under the assumption $r + s = 0$. Finally, in Sect. 7, we rule out the case $r + s > 0$.

All the computer calculations in this paper were performed with the help of Magma [2].

## 2. Preliminaries

First we consider a more general situation than the one mentioned in the Introduction. Let $(X, \{R_i\}_{i=0}^d)$ be a symmetric $d$-class association scheme with the first eigenmatrix $P = (P_{i,j})_{0 \le i,j \le d}$. For more general and detailed theory of association schemes, see [1]. We let $\mathfrak{A}$ denote the Bose–Mesner algebra spanned by the adjacency matrices $A_0, A_1, \dots, A_d$ of $\mathfrak{X}$. Then the adjacency matrices are expressed as

$$A_j = \sum_{i=0}^{d} P_{i,j} E_i \quad (j = 0, 1, \dots, d), \tag{4}$$

where $E_0 = \frac{1}{n} J, E_1, \dots, E_d$ are the primitive idempotents of $\mathfrak{A}$.

Let

$$W_1 = \sum_{j=0}^{d} w_j A_j \in \mathfrak{A}, \tag{5}$$

where $w_0, \dots, w_d$ are complex numbers of absolute value 1. Define

$$\beta_i = \sum_{j=0}^{d} w_j P_{i,j} \quad (i = 0, 1, \dots, d). \tag{6}$$

By (4), (5) and (6) we have

$$W_1 = \sum_{i=0}^{d} \beta_i E_i. \tag{7}$$

Let $X_i$ $(0 \le i \le d)$ be indeterminates. For $j = 1, 2, \dots, d$, let $e_j$ be the polynomial defined by

$$e_j = \prod_{i=0}^{d} X_h \left( \sum_{i=0}^{d} P_{j,i}^2 + \sum_{0\le j_1<j_2\le d} P_{j,j_1}P_{j,j_2}\left(\frac{X_{j_1}}{X_{j_2}} + \frac{X_{j_2}}{X_{j_1}}\right) - (n+1)\right), \tag{8}$$

and $e_0$ be the polynomial defined by

$$e_0 = 1 + \sum_{j=0}^{d} P_{0,j}X_j. \tag{9}$$

Then we have the following.

**Lemma 4.** *The following statements are equivalent:*

(i) *The matrix $W$ defined by* (1) *is a complex Hadamard matrix,*
(ii) *$\beta_i\overline{\beta_i} = n+1$ for $i = 1,\dots,d$, and $1 + \sum_{j=0}^{d} P_{0,j}w_j = 0$,*
(iii) *$(w_i)_{0\le i\le d}$ is a common zero of $e_j$ ($j = 0,\dots,d$).*

*Proof.* By (1) we have

$$W\overline{W}^\top = \begin{pmatrix} n+1 & \mathbf{e}(I+\overline{W_1}^\top) \\ (I+W_1)\mathbf{e}^\top & J + W_1\overline{W_1}^\top \end{pmatrix}.$$

By (7) we have

$$W_1\overline{W_1}^\top = \sum_{i=0}^{d} \beta_i\overline{\beta_i}E_i. \tag{10}$$

Suppose that the matrix (1) is a complex Hadamard matrix. Since $W\overline{W}^\top = (n+1)I$, we have

$$\begin{aligned} W_1\overline{W_1}^\top &= (n+1)I - J \\ &= E_0 + (n+1)\sum_{j=1}^{d} E_j, \end{aligned} \tag{11}$$

$$(I+W_1)\mathbf{e}^\top = 0. \tag{12}$$

Therefore, by (10), (11), and (12), (i) implies (ii).

To prove the converse, it suffices to show $\beta_0\overline{\beta_0} = 1$. Since $W$ is symmetric, the diagonal entries of $W_1\overline{W_1}^\top$ are all $n$. Thus

$$\begin{aligned} n^2 &= \operatorname{Tr} W_1\overline{W_1}^\top \\ &= \sum_{j=0}^{d} \beta_j\overline{\beta_j}\operatorname{Tr} E_j \qquad \text{(by (10))} \\ &= \beta_0\overline{\beta_0} + \sum_{j=1}^{d}(n+1)\operatorname{Tr} E_j \\ &= \beta_0\overline{\beta_0} + (n+1)\operatorname{Tr}(I - E_0) \\ &= \beta_0\overline{\beta_0} + (n+1)(n-1), \end{aligned}$$

and hence $\beta_0\overline{\beta_0} = 1$.

By (6) we have

$$\beta_i\overline{\beta_i} = \sum_{j=0}^{d} P_{i,j}^2 + \sum_{0\le j_1<j_2\le d} P_{i,j_1}P_{i,j_2}\left(\frac{w_{j_1}}{w_{j_2}} + \frac{w_{j_2}}{w_{j_1}}\right)$$

for $i = 1,\dots,d$. Therefore, the equivalence of (ii) and (iii) follows. □

The following is analogous to [4, Proposition 2.2].

**Lemma 5.** *If the matrix $W$ defined by* (1) *is a complex Hadamard matrix, then we have*

$$n+1 \le \left(\sum_{i=0}^{d} |P_{j,i}|\right)^2 \quad (j = 0,1,\dots,d).$$

*Proof.* By (ii) in Lemma 4, we have

$$\begin{aligned} n+1 &= \beta_j\overline{\beta_j} \\ &= \left(\sum_{j_1=0}^{d} w_{j_1}P_{j,j_1}\right)\left(\sum_{j_2=0}^{d}\frac{P_{j,j_2}}{w_{j_2}}\right) \\ &= \sum_{i=0}^{d} P_{j,i}^2 + \sum_{0\le j_1<j_2\le d}\left(\frac{w_{j_1}}{w_{j_2}}+\frac{w_{j_2}}{w_{j_1}}\right)P_{j,j_1}P_{j,j_2}, \end{aligned}$$

for $j = 0, 1, \ldots, d$. Since $W$ is a complex Hadamard matrix, we have $|\frac{w_{j_1}}{w_{j_2}}+\frac{w_{j_2}}{w_{j_1}}| \le 2$. Then

$$\begin{aligned} n+1 &\le \sum_{i=0}^{d} |P_{j,i}|^2 + \sum_{0\le j_1<j_2\le d}\left|\frac{w_{j_1}}{w_{j_2}}+\frac{w_{j_2}}{w_{j_1}}\right||P_{j,j_1}||P_{j,j_2}| \\ &\le \sum_{i=0}^{d} |P_{j,i}|^2 + 2\sum_{0\le j_1<j_2\le d}|P_{j,j_1}||P_{j,j_2}| \\ &= \left(\sum_{i=0}^{d}|P_{j,i}|\right)^2. \end{aligned}$$

□

Let $f(X)$ be a non-constant polynomial with real coefficients. Put $f_0(X) = f(X)$ and $f_1(X) = f_0'(X)$. Define

$$f_{j+1}(X) = -\mathrm{Rem}(f_{j-1}(X), f_j(X)) \quad (j = 1, 2, \ldots)$$

where, for polynomials $a(X)$, $b(X) \neq 0$, we denote by $\mathrm{Rem}(a(X), b(X))$ the remainder when $a(X)$ is reduced modulo $b(X)$. There exists a positive integer $m$ such that $f_m(X) \neq 0$ and $f_{m+1}(X) = 0$. The sequence of the polynomials

$$f_0(X), f_1(X), f_2(X), \ldots, f_m(X)$$

is called the Strum sequence associated to $f(X)$.

Let $c_j$ be the leading coefficient of $f_j(X)$, and $d_j = \deg f_j(X)$ for $j = 0, 1, \ldots, m$. Then we have the following sequences:

$$(\mathrm{sgn}(c_j))_{j=0}^{m}, \tag{13}$$

$$(\mathrm{sgn}((-1)^{d_j}c_j))_{j=0}^{m}. \tag{14}$$

**Theorem 6** (Sturm [12]; see also [10, Corollary 10.5.4]). *With the above notation, the number of distinct real roots of $f(X)$ is given by the number of sign changes of* (14) *minus the number of sign changes of* (13).

## 3. Strongly Regular Graphs

In this section, we review basic properties of symmetric 2-class association schemes and strongly regular graphs. Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^2)$ be a symmetric 2-class association scheme with the first eigenmatrix (2). We have the following three cases in (2): (i) $r + s \ge 0$, (ii) $r + s = -1$, (iii) $r + s \le -2$. Suppose that (iii) holds. Then the eigenvalues of $R_2$ satisfy $-(r+1) - (s+1) \ge 0$. By exchanging $R_1$ and $R_2$, we may assume that $r + s \ge -1$ without loss of generality. Therefore we only consider the two cases (i) and (ii). Under this assumption, we have

$$\ell \ge 2. \tag{15}$$

Indeed, if $\ell = 1$, then $R_2$ is a matching, and hence the eigenvalues satisfy $-r-1 = -1$ and $-s-1 = 1$. This implies $r + s = -2$, contrary to our assumption.

A strongly regular graph $\Gamma$ with parameters $(k, \lambda, \mu)$ is equivalent to $\mathfrak{X}$, via the correspondence $R_1$ equal to the set of edges and $R_2$ equal to the set of non-edges. The complement of a strongly regular graph is also a strongly regular graph. Then we have

$$\mu = k + rs, \tag{16}$$

$$\lambda = r + s + \mu, \tag{17}$$

$$\ell\mu = k(k - \lambda - 1), \tag{18}$$

$$n\mu = (1 + k)\mu + \ell(k - \lambda - 1).$$

Let $m_j = \mathrm{rank}\, E_j$ for $j = 1, 2$. Then we have

$$m_1 = \frac{1}{2}\left(n - 1 - \frac{2k + (n-1)(\lambda - \mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right), \tag{19}$$

$$m_2 = \frac{1}{2}\left(n - 1 + \frac{2k + (n-1)(\lambda - \mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right).$$

A conference graph is a strongly regular graph $\Gamma$ satisfying one of the following two equivalent conditions:
(i) $k = 2r(r+1)$, $r + s = -1$,
(ii) $m_1 = m_2$.
We remark that the eigenvalues $r, s$ of a strongly regular graph $\Gamma$ are integers unless $\Gamma$ is a conference graph. If $\Gamma$ is a conference graph, then $r = \frac{-1+\sqrt{2k+1}}{2}$ and $s = \frac{-1-\sqrt{2k+1}}{2}$. In any case,

$$rs \in \mathbb{Z}. \tag{20}$$

Note that we allow disconnected strongly regular graphs. These graphs are characterized by $s = -1$, or equivalently, $\mu = 0$.

By (2), (8), and (9) we have

$$e_0 = 1 + X_0 + kX_1 + \ell X_2, \tag{21}$$

$$\begin{aligned} e_1 = &-((r+1)X_1 - rX_2)X_0^2 - (r(r+1)(X_1 - X_2)^2 + (k+\ell)X_1X_2)X_0 \\ &+ (rX_1 - (r+1)X_2)X_1X_2, \end{aligned} \tag{22}$$

$$\begin{aligned} e_2 = &-((s+1)X_1 - sX_2)X_0^2 - (s(s+1)(X_1 - X_2)^2 + (k+\ell)X_1X_2)X_0 \\ &+ (sX_1 - (s+1)X_2)X_1X_2. \end{aligned} \tag{23}$$

Let $\mathcal{I}$ be the ideal of the polynomial ring $\mathscr{R} = \mathbb{C}[X_0, X_1, X_2]$ generated by (21), (22), and (23).

**Lemma 7.** *Let $W_1$ be the matrix defined by* (3)*, and let $W$ be the matrix defined by* (1)*. Then $W$ is a complex Hadamard matrix if and only if $(w_0, w_1, w_2)$ is a common zero of the polynomials $e_k$ $(k = 0, 1, 2)$.*

*Proof.* This follows easily from Lemma 4 by setting $d = 2$. □

*Proof of the "converse" part of Theorem 1.* Assume that the matrix $W$ is one of the matrices (i), (ii) in Theorem 1. In view of Lemma 7, it suffices to show that $(w_0, w_1, w_2)$ is a common zero of the polynomials (21), (22), and (23). This can be done by direct calculation. □

**Lemma 8.** *Let $W_1$ be the matrix defined by* (3)*, and let $W$ be the matrix defined by* (1)*. If $W$ is a complex Hadamard matrix, then we have the following:*
(i) $s < -1$,
(ii) $n + 1 \le 4s^2$.

*Proof.* (i) Suppose that $s = -1$. Then, since the graph $(X, R_1)$ is the union of complete graphs, we have $k = r$. We can verify that $\mathcal{I}$ contains $X_2(\ell X_2 + 1)^2(X_2^2 + (k+\ell)X_2 + 1)$. By Lemma 7, $(w_0, w_1, w_2)$ is a common zero of the polynomials (21), (22), and (23) in $\mathcal{I}$. From this, $\ell w_2 + 1 = 0$ or $w_2^2 + (k+\ell)w_2 + 1 = 0$. By (15), we have $\ell w_2 + 1 \ne 0$. Since $|w_2^2 + 1| < 3 \le k + \ell = |(k+\ell)w_2|$, we have $w_2^2 + (k+\ell)w_2 + 1 \ne 0$. This is a contradiction.

(ii) Applying Lemma 5 for $j = 2$, we have

$$\begin{aligned} n + 1 &\le \left(\sum_{i=0}^{2} |P_{2,i}|\right)^2 \\ &= (1 + (-s) - (s+1))^2 \\ &= 4s^2. \end{aligned}$$ □

If $s \ne -1$, then we have $\mu > 0$ by (16), (17), and (18). Thus

$$\ell = \frac{-k(r+1)(s+1)}{k + rs}. \tag{24}$$

**Remark 9.** Applying Lemma 5 for $j = 1$, we have $n + 1 \le 4(r+1)^2$. This inequality is weaker than the one stated in (ii) of Lemma 8. Indeed, since we assumed that $r + s \ge -1$ in the beginning of this section, we have $(r+1)^2 \ge s^2$.

## 4. The Real Part of $w_1$

We suppose that $r, s \in \mathbb{R}$, $r \ge 0$, $s < -1$, and $r + s \ge -1$. Let $W_1$ be the matrix defined by (3), and $W$ be the matrix defined by (1). We suppose that the matrix $W$ is a complex Hadamard matrix. Let

$$w_j = a_j + b_j i$$

for $j = 0, 1, 2$, where $a_j, b_j \in \mathbb{R}$, $a_j^2 + b_j^2 = 1$, and $i^2 = -1$. Recall that $\mathcal{I}$ is the ideal of $\mathscr{R}$ generated by (21), (22), and

(23), and $(w_0, w_1, w_2)$ is a common zero of $\mathcal{I}$ by Lemma 7. Since we assume $w_1 \neq w_2 \neq 0$, in Theorem 1, we consider the ideal $\tilde{\mathcal{I}}$ of the polynomial ring $\tilde{\mathcal{R}} = \mathbb{C}[X_\infty, X_0, X_1, X_2]$ generated by (21), (22), (23), and

$$e_\infty = 1 + X_\infty(r-s)(X_1 - X_2)X_2.$$

Note that we have included the factor $r - s$ for a technical reason. In computer implementation, we regard $r$ and $s$ as indeterminates as well, but $r$ and $s$ are assumed to take distinct values.

Define the polynomials $L(X)$, $M(X)$, and $S(X)$ as follows:

$$L(X) = X^3 + \frac{4rs - r - s + 3}{2}X^2 + \frac{-4rs(r+s-1)+1}{2}X + \frac{rs(r^2 + 2(3s+1)r + s^2 + 2s + 2)}{2}, \tag{25}$$

$$M(X) = L(X) - 4(X + rs)^2, \tag{26}$$

$$S(X) = s_4X^4 + s_3X^3 + s_2X^2 + s_1X + s_0, \tag{27}$$

where

$$\begin{aligned}
s_4 &= (r+s+1)^2,\\
s_3 &= 4sr^3 + 8s(s+1)r^2 + (4s^3 + 8s^2 + 8s + 2)r + 2s + 2,\\
s_2 &= 2s(2s-1)r^4 + 2s(s+1)(4s-3)r^3 + 2s(2s^3 + s^2 + 6s + 4)r^2\\
&\quad - 2s(s+1)(s^2 + 2s - 6)r + 1,\\
s_1 &= -2rs(2sr^4 + 6s(s+1)r^3 + (6s^3 - 4s^2 - 8s - 1)r^2\\
&\quad + 2(s+1)(s^3 + 2s^2 - 6s - 1)r - s^2 - 2s - 2),\\
s_0 &= r^2s^2(r^4 + 4(s+1)r^3 + (22s^2 + 28s + 8)r^2\\
&\quad + 4(s+1)(s^2 + 6s + 2)r + (s^2 + 2s + 2)^2).
\end{aligned}$$

**Lemma 10.** *We have*

$$(L(k) - M(k))^2a_1^2 + 2(L(k)^2 - M(k)^2)a_1 + (L(k) + M(k))^2 - S(k) = 0, \tag{28}$$

$$2(k+rs)^2rsa_0 - 2k^2(k+rs)^2a_1 + h_0 = 0, \tag{29}$$

$$2(k+rs)^3a_1 - 2(k+rs)r(r+1)s(s+1)a_2 + \ell_0 = 0, \tag{30}$$

*where*

$$\begin{aligned}
h_0 &= -k^5 + (r+s-2rs+1)k^4 + 3rs(r+s+1)k^3\\
&\quad - rs((r+s)^2 + 2(r+s) - 1)k^2 + 4r^2s^2k + 2r^3s^3,\\
\ell_0 &= k(k-r-s-1)(k^2 + 2rsk - rs(r+s+1)).
\end{aligned}$$

*Proof.* With the help of Magma, we can verify that $\tilde{\mathcal{I}}$ contains $f_1(X_0, X_1, X_2)$ and $f_2(X_0, X_1, X_2)$, where

$$\begin{aligned}
f_1(X_0, X_1, X_2) &= X_0^2 + (r+s+1)(X_1 - X_2)X_0 - X_1X_2,\\
f_2(X_0, X_1, X_2) &= X_1^3X_2^2 - X_0X_1(X_1^2 + X_2^2) + X_2(X_0^2 + X_1^2 - X_1X_2)\\
&\quad + (r+s+1)X_2(X_1 - X_2)(X_0 - X_1X_2)\\
&\quad + rsX_1(X_1 + X_2)(X_1 - X_2)^2.
\end{aligned}$$

By Lemma 7, we have $f_1(w_0, w_1, w_2) = 0$ and $f_2(w_0, w_1, w_2) = 0$.

Consider the polynomial ring

$$\mathcal{P} = \mathbb{C}[Y_\infty, \alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2].$$

Let $\chi$ be the homomorphism from $\tilde{\mathcal{R}}$ to $\mathcal{P}$ defined by $\chi(X_\infty) = Y_\infty$ and $\chi(X_j) = \alpha_j + \beta_j i$ for $j = 0, 1, 2$. Let $\mathcal{J}$ denote the ideal of the polynomial ring $\mathcal{P}$ generated by $\chi(\tilde{\mathcal{I}})$, $\chi(f_1(X_0, X_1, X_2))$, $\chi(f_2(X_0, X_1, X_2))$ and $\alpha_j^2 + \beta_j^2 - 1$ for $j = 0, 1, 2$. With the help of Magma, we can verify that $\mathcal{J}$ contains

$$\begin{aligned}
&(L(k) - M(k))^2\alpha_1^2 + 2(L(k)^2 - M(k)^2)\alpha_1 + (L(k) + M(k))^2 - S(k),\\
&2(k+rs)^2rs\alpha_0 - 2k^2(k+rs)^2\alpha_1 + h_0,\\
&2(k+rs)^3\alpha_1 - 2(k+rs)r(r+1)s(s+1)\alpha_2 + \ell_0.
\end{aligned}$$

Therefore we have the assertion. □

**Lemma 11.** *If the real part of $w_1$ is in the interval $[-1, 1]$, then the following holds:*
(i) $S(k) \geq 0$,
(ii) $M(k) \leq \frac{\sqrt{S(k)}}{2} \leq L(k)$ *or* $M(k) \leq \frac{-\sqrt{S(k)}}{2} \leq L(k)$.

*Proof.* By (i) in Lemma 8 and (26) we have $L(k) - M(k) \neq 0$. Assume that $a_1 \in [-1, 1]$. Then by (28), using the notation of (25), (26), and (27), we have

$$a_1 = \frac{-L(k) - M(k) \pm \sqrt{S(k)}}{L(k) - M(k)}.$$

Since $a_1 \in \mathbb{R}$, we have (i). Since $a_1 \in [-1, 1]$, we have (ii). □

## 5. Properties of the Polynomials $L(X)$, $M(X)$, and $S(X)$ for the Case $r + s = -1$

In this section, we suppose that $r + s = -1$, $2r(r + 1) \in \mathbb{Z}$, and $(r, s) \neq (0, -1)$. We consider properties of the polynomials (25), (26), and (27). By (25), (26), and (27) we have

$$L(X) = X^3 - 2(r^2 + r - 1)X^2 - \frac{8r^2 + 8r - 1}{2}X + \frac{r(r + 1)(4r^2 + 4r - 1)}{2}, \tag{31}$$

$$M(X) = L(X) - 4(X - r(r + 1))^2, \tag{32}$$

$$S(X) = -(X - r(r + 1))S_1(X), \tag{33}$$

where

$$S_1(X) = 4r(r + 1)(X - s_+)(X - s_-), \tag{34}$$

$$s_\pm = 2r(r + 1) + \frac{1 \pm \sqrt{16r^2(r + 1)^2 + 1}}{8r(r + 1)}. \tag{35}$$

**Lemma 12.** *We have $r(r + 1) < s_\pm$.*

*Proof.* Since $s_- < s_+$ by (35), we show that $r(r + 1) < s_-$. To do this, we have only to show that $\sqrt{16r^2(r + 1)^2 + 1} < 8r^2(r + 1)^2 + 1$ by (35). Since $(8r^2(r + 1)^2 + 1)^2 - (16r^2(r + 1)^2 + 1) = 64r^4(r + 1)^4 > 0$, we have the assertion. □

**Lemma 13.** *Suppose that $r(r + 1) < x$. Then $S(x) \geq 0$ if and only if $s_- \leq x \leq s_+$.*

*Proof.* This follows easily from (33), (34), and Lemma 12. □

**Lemma 14.** *We have $\mathbb{Z} \cap \{x \mid s_- \leq x \leq s_+\} = \{2r(r + 1)\}$.*

*Proof.* Note that $r(r + 1) \in \mathbb{Z}$ by (20). It is easy to show that $s_- < 2r(r + 1) < s_+$ by (35). Thus, it is enough to show that $2r(r + 1) - 1 < s_-$ and $s_+ < 2r(r + 1) + 1$, or equivalently, $\sqrt{16r^2(r + 1)^2 + 1} < 8r(r + 1) \pm 1$. We have only to show that $\sqrt{16r^2(r + 1)^2 + 1} < 8r(r + 1) - 1$. Since

$$(8r(r + 1) - 1)^2 - (16r^2(r + 1)^2 + 1) = 16r(r + 1)(3r(r + 1) - 1) > 0,$$

the result holds. □

**Lemma 15.** *Suppose that $z \in \mathbb{Z}$ and $r(r + 1) < z$. Then $M(z) \leq \frac{\sqrt{S(z)}}{2} \leq L(z)$ or $M(z) \leq \frac{-\sqrt{S(z)}}{2} \leq L(z)$ holds if and only if $z = 2r(r + 1)$.*

*Proof.* First suppose that $M(z) \leq \frac{\sqrt{S(z)}}{2} \leq L(z)$ or $M(z) \leq \frac{-\sqrt{S(z)}}{2} \leq L(z)$ holds. Since $S(z) \geq 0$, by Lemma 13 we have $s_- \leq z \leq s_+$. By Lemma 14, we have $z = 2r(r + 1)$. Secondly suppose that $z = 2r(r + 1)$. Since

$$L(2r(r + 1)) = \frac{r(r + 1)(2r + 1)^2}{2},$$

$$M(2r(r + 1)) = \frac{-r(r + 1)(4r(r + 1) - 1)}{2} < 0,$$

$$\sqrt{S(2r(r + 1))} = r(r + 1)$$

by (31), (32), and (33), we have $M(2r(r + 1)) \leq \frac{\sqrt{S(2r(r+1))}}{2} \leq L(2r(r + 1))$. □

**Lemma 16.** *Let $W_1$ be the matrix defined by* (3)*, and $W$ be the matrix defined by* (1)*. Suppose that $W$ is a complex Hadamard matrix. If $r + s = -1$, then we have* (ii) *in Theorem 1.*

*Proof.* By Lemma 8, we have $(r, s) \neq (0, -1)$. Thus, we may use results of this section. In particular, by Lemmas 15 and 11, we have $k = 2r(r + 1)$. By Sect. 3, $\Gamma$ is a conference graph on $(2r + 1)^2$ vertices.

By (28) we have $2r^3(r + 1)^3 a_1((2r + 1)a_1 + 1) = 0$. Hence $a_1 = 0$ or $a_1 = -1/(2r + 1)$. If $a_1 = 0$ then by (29), (30) we have $a_0 = -1$, $a_2 = 0$, respectively. By $w_1 \neq w_2$ we have $(b_0, b_1, b_2) = (0, \pm 1, \mp 1)$. Therefore we have (a) of (ii) in Theorem 1. If $a_1 = -1/(2r + 1)$ then by (29), (30) we have $a_0 = 1$, $a_2 = -1/(2r + 1)$, respectively. By $w_1 \neq w_2$ we have $(b_0, b_1, b_2) = (0, \frac{\pm\sqrt{4r^2(r+1)^2-1}}{2r(r+1)}, \frac{\mp\sqrt{4r^2(r+1)^2-1}}{2r(r+1)})$. Therefore we have (b) of (ii) in Theorem 1. □

## 6. Properties of the Polynomials $L(X)$, $M(X)$, and $S(X)$ for the Case $r + s \geq 0$

In this section, we suppose that $r, s \in \mathbb{Z}$ and $r + s \geq 0$. We further assume $r \geq 2$ and $s \leq -2$. We consider properties of the polynomials (25), (26), and (27). We put

$$h = \sqrt{4r(r+1)s(s+1)+1}, \tag{36}$$

$$\alpha_\pm = \frac{r+s-1}{2} \pm \frac{\sqrt{(s-1)^2 - 6rs + r(r-2)}}{2}, \tag{37}$$

$$\beta_\pm = -rs - \frac{1}{2} \pm \frac{h}{2}, \tag{38}$$

$$\gamma_\pm = \frac{r+s+3}{2} \pm \frac{\sqrt{r^2 + 2(5s+3)r + (s+3)^2}}{2}, \tag{39}$$

$$\delta = -rs + \sqrt{r(r+1)s(s+1)}. \tag{40}$$

Then $\alpha_\pm$, $\beta_\pm$, $\delta \in \mathbb{R}$. By (25), (26), and (27) we have

$$L(X)^2 - \frac{S(X)}{4} = (X-\alpha_-)(X-\alpha_+)(X-\beta_-)^2(X-\beta_+)^2, \tag{41}$$

$$M(X)^2 - \frac{S(X)}{4} = (X-\gamma_-)(X-\gamma_+)(X-(\beta_-+1))^2(X-(\beta_++1))^2. \tag{42}$$

**Lemma 17.** *We have the following:*
(i) $\alpha_\pm, \beta_- + 1 < -rs$,
(ii) $-rs < \beta_+ < \delta < \beta_+ + 1$,
(iii) *if* $\gamma_\pm \in \mathbb{R}$ *then* $\gamma_\pm < -rs$.

*Proof.* (i) The inequality $\beta_- + 1 < -rs$ follows easily from (38). Since $\alpha_- < \alpha_+$, it remains to show that $\alpha_+ < -rs$. Then by (37) we have only to show that

$$\sqrt{(s-1)^2 - 6rs + r(r-2)} < -2rs - (r+s-1).$$

Since $-2rs - (r+s-1) = -(2s+1)r - s + 1 > 0$ and

$$\begin{aligned}&(-(2s+1)r - s + 1)^2 - ((s-1)^2 - 6rs + r(r-2))\\ &\quad = 4r(r+1)s(s+1) > 0,\end{aligned}$$

we have $\alpha_+ < -rs$.

(ii) First, the inequality $-rs < \beta_+$ follows easily from the definition (38). Since

$$h < 1 + 2\sqrt{r(r+1)s(s+1)},$$

we have $\beta_+ < \delta$ from the definitions (38) and (40), while $\delta < \beta_+ + 1$ follows trivially from these.

(iii) Since $\gamma_- < \gamma_+$, it is enough to show that $\gamma_+ < -rs$. By (39) we have only to show that

$$\sqrt{r^2 + 2(5s+3)r + (s+3)^2} < -2rs - (r+s+3).$$

Since $-2rs - (r+s+3) = -(2s+1)r - (s+3) > 0$ and

$$\begin{aligned}&(-(2s+1)r - (s+3))^2 - (r^2 + 2(5s+3)r + (s+3)^2)\\ &\quad = 4r(r+1)s(s+1) > 0,\end{aligned}$$

we have $\gamma_+ < -rs$. □

**Lemma 18.** *We have* $L(-rs) = M(-rs) < 0$.

*Proof.* By (25) and (26) we have

$$L(-rs) = M(-rs) = \frac{r(r+1)s(s+1)((2r+1)(s+1) - r)}{2} < 0.$$

□

**Lemma 19.** *We have the following:*
(i) $L(X)$ *has exactly one real root* $\zeta$ *in* $(-rs, \infty)$, *and* $\beta_+ \leq \zeta < \delta$,
(ii) $L(x) < 0$ *for* $-rs < x < \zeta$, *and* $L(x) \geq 0$ *for* $\zeta \leq x$.

*Proof.* Since $L'(X) = 3(X-\theta_-)(X-\theta_+)$, where

$$\theta_\pm = \frac{(-4s+1)r+s-3}{6} \pm \frac{\sqrt{\iota}}{6},$$
$$\iota = (16s(s+1)+1)r^2 + 2(8s^2+s-3)r + s^2 - 6s + 3 > 0,$$

$L(X)$ has the local maximum at $X=\theta_-$ and the local minimum at $X=\theta_+$.

(i) We show that (a) $\theta_- < -rs < \theta_+$, (b) $\theta_+ < \beta_+$, $L(\beta_+) \le 0$, and $L(\delta) > 0$. Then (a) together with Lemma 18 implies that the first half of (i) holds, and (b) implies that the latter half of (i) holds.

First we show that (a) holds. Since

$$\begin{aligned}6(-rs-\theta_-) &> -6rs - ((-4s+1)r+s-3)\\ &= -(2s+1)r - s + 3 > 0,\end{aligned}$$

we have $\theta_- < -rs$. To show that $-rs < \theta_+$, we have only to show that $-(2s+1)r - s + 3 < \sqrt{\iota}$. Since

$$\iota - (-(2s+1)r - s + 3)^2 = 12r(r+1)s(s+1) - 6 > 0,$$

we have $-rs < \theta_+$. Hence $\theta_- < -rs < \theta_+$.

Secondly we show that (b) holds. To show that $\theta_+ < \beta_+$, by (38) and the definition of $\theta_+$ we have only to show that $\sqrt{\iota} < -(2s+1)r - s + 3h$. Since

$$\begin{aligned}&(-(2s+1)r - s + 3h)^2 - \iota\\ &= -6((2s+1)r+s)h + 24s(s+1)r^2 + 6(4s^2+6s+1)r + 6(s+1)\\ &> 0,\end{aligned}$$

we have $\theta_+ < \beta_+$. We have

$$L(\beta_+) = \frac{\kappa_1 h + \kappa_2}{4},$$

where

$$\begin{aligned}\kappa_1 &= (2s+1)r + s < 0,\\ \kappa_2 &= 4s(s+1)r^2 + (2s(2s+1)-1)r - s.\end{aligned}$$

Since

$$\kappa_1^2 h^2 - \kappa_2^2 = 4r(r+1)s(s+1)(r+s)(r+s+2) \ge 0,$$

by our assumption, we have

$$L(\beta_+) \le 0. \tag{43}$$

We have

$$L(\delta) = \frac{\sqrt{r(r+1)s(s+1)}}{2} + 2r(r+1)s(s+1) > 0. \tag{44}$$

(ii) This follows easily from (i), (43), and (44). □

**Lemma 20.** *We have the following:*

(i) *$M(X)$ has exactly one real root $\eta$ in $(-rs, \infty)$, and $\delta < \eta \le \beta_+ + 1$,*

(ii) *$M(x) \le 0$ for $-rs < x \le \eta$, and $M(x) > 0$ for $\eta < x$.*

*Proof.* Since $M'(X) = 3(X-\nu_-)(X-\nu_+)$, where

$$\nu_\pm = \frac{(-4s+1)r+s+5}{6} \pm \frac{\sqrt{\rho}}{6},$$
$$\rho = (16s(s+1)+1)r^2 + 2(8s^2+17s+5)r + s^2 + 10s + 19 > 0,$$

$M(X)$ has the local maximum at $X=\nu_-$ and the local minimum at $X=\nu_+$.

(i) It is enough to show that (a) $\nu_- < -rs < \nu_+$, (b) $\nu_+ < \delta$, $M(\delta) < 0$, and $M(\beta_+ + 1) \ge 0$. Then (a) together with Lemma 18 implies that the first half of (i) holds, and (b) implies that the latter half of (i) holds.

First we show that (a) holds. Since

$$\begin{aligned}6(-rs-\nu_-) &= -(2s+1)r - s - 5 + \sqrt{\rho},\\ &> -(2s+1)r - s - 5 > 0,\end{aligned}$$

we have $\nu_- < -rs$. To show that $-rs < \nu_+$, we have only to show that $-(2s+1)r - s - 5 < \sqrt{\rho}$. Since

$$\rho - (-(2s+1)r - s + 3)^2 = 12r(r+1)s(s+1) - 6 > 0,$$

we have $-rs < \nu_+$. Hence $\nu_- < -rs < \nu_+$.

Secondly we show that (b) holds. To show that $\nu_+ < \delta$, by (40) and the definition of $\nu_+$ we have only to show that $(2s+1)r+s+5+\sqrt{\rho} < 6\sqrt{r(r+1)s(s+1)}$. Since

$$\begin{aligned}&(\sqrt{r(r+1)s(s+1)})^2-((2s+1)r+s+5+\sqrt{\rho})^2\\&=-((4s+2)r+2s+10)\sqrt{\rho}\\&\quad+(16s^2+16s-2)r^2+(16s^2-20s-20)r-2s^2-20s-44>0,\end{aligned}$$

we have $\nu_+ < \delta$. It is easy to show that

$$M(\delta)=\frac{\sqrt{r(r+1)s(s+1)}}{2}-2r(r+1)s(s+1)<0. \tag{45}$$

We have

$$M(\beta_+ + 1)=\frac{\sigma_1 h+\sigma_2}{4},$$

where

$$\begin{aligned}\sigma_1&=-((2s+1)r+s+2)>0,\\ \sigma_2&=-4s(s+1)r^2-(4s^2+6s+1)r-s-2.\end{aligned}$$

Since

$$\sigma_1^2h^2-\sigma_2^2=4r(r+1)s(s+1)(r+s)(r+s+2)\geq 0,$$

by our assumption, we have

$$M(\beta_+ + 1)\geq 0. \tag{46}$$

(ii) This follows easily from (i), (45), and (46). □

**Lemma 21.** *For $-rs \leq x$ we have the following:*

(i) $L(x)^2 \geq \frac{S(x)}{4}$, *and equality holds if and only if $x=\beta_+$,*

(ii) $M(x)^2 \geq \frac{S(x)}{4}$, *and equality holds if and only if $x=\beta_+ + 1$.*

*Proof.* (i) Since $\alpha_\pm < -rs$ by (i) in Lemma 17, by (41) we have the claimed inequality. Since $\alpha_\pm, \beta_- < -rs < \beta_+$ by (i) and (ii) in Lemma 17, equality holds if and only if $x=\beta_+$.

(ii) First suppose that $\gamma_\pm \in \mathbb{R}$. Since $\gamma_\pm < -rs$ by (iii) in Lemma 17, by (42) we have the claimed inequality. Since $\beta_- + 1, \gamma_\pm < -rs$ by (i) and (iii) in Lemma 17, equality holds if and only if $x=\beta_+ + 1$.

Secondly suppose that $\gamma_\pm \notin \mathbb{R}$. Then $\overline{\gamma_+}=\gamma_-$ by (39), so we also have the claimed inequality. Since $\beta_- + 1 < -rs$ by (i) in Lemma 17, equality holds if and only if $x=\beta_+ + 1$. □

For the remainder of this subsection, we suppose that $r+s=0$. By (25), (26), and (27) we have

$$L(X)=\frac{(X-\tau_-)(X-\tau_+)(X-\beta_+)}{2}, \tag{47}$$

$$M(X)=\frac{(2X^2-5X+2r^2+1)(X-(\beta_+ + 1))}{2}, \tag{48}$$

$$S(X)=(X-\beta_+)^2(X-(\beta_+ + 1))^2\geq 0, \tag{49}$$

where

$$\begin{aligned}\tau_\pm&=\frac{-1\pm\sqrt{16r^2+1}}{4},\\ \beta_+&=2r^2-1\in\mathbb{Z}.\end{aligned} \tag{50}$$

By (37) and (38) we have

$$\begin{aligned}\alpha_\pm&=\frac{-1\pm\sqrt{8r^2+1}}{2},\\ \beta_-&=0.\end{aligned}$$

**Lemma 22.** *Suppose that $r+s=0$. Let $\zeta$ and $\eta$ be as defined in* Lemmas 19 and 20. *Then we have $\zeta=\beta_+$ and $\eta=\beta_+ + 1$.*

*Proof.* We have $\tau_\pm < r^2$. Then by (i) in Lemma 19 and (47) we have $\zeta=\beta_+$. Since the discriminant of $2x^2-5x+2r^2+1$ is $-16r^2+17<0$, by (i) in Lemma 20 and (48) we have $\eta=\beta_+ + 1$. □

**Lemma 23.** *Suppose that $r + s = 0$, $z \in \mathbb{Z}$ and $r^2 < z$. Then the following are equivalent:*
(i) $S(z) \geq 0$ *and* $M(z) \leq \frac{\sqrt{S(z)}}{2} \leq L(z)$,
(ii) $S(z) \geq 0$ *and* $M(z) \leq \frac{-\sqrt{S(z)}}{2} \leq L(z)$,
(iii) $S(z) = 0$,
(iv) $z \in \{\beta_+, \beta_+ + 1\}$.

*Proof.* Since $z \in \mathbb{Z}$ and $\beta_+, \beta_+ + 1 \in \mathbb{Z}$ by (50), the condition (iv) is equivalent to $\beta_+ \leq z \leq \beta_+ + 1$.

First suppose that (i) holds. Since $L(z) \geq 0$, by (ii) in Lemma 19 and Lemma 22 we have

$$\beta_+ \leq z. \tag{51}$$

Suppose that $M(z) \leq 0$. By (ii) in Lemma 20 and Lemma 22 we have $z \leq \beta_+ + 1$. By (51) we have $z = \beta_+, \beta_+ + 1$. Suppose that $M(z) > 0$. Then $M(z)^2 \leq \frac{S(z)}{4}$. By (ii) in Lemma 21 we have $z = \beta_+ + 1$. Thus we have (iv).

Secondly suppose that (ii) holds. Since $M(z) \leq 0$, by (ii) in Lemma 20 and Lemma 22 we have

$$z \leq \beta_+ + 1. \tag{52}$$

Suppose that $L(z) \geq 0$. Then by (ii) Lemma 19 and Lemma 22 we have $\beta_+ \leq z$. By (52) we have $z = \beta_+, \beta_+ + 1$. Suppose that $L(z) < 0$. Then $L(z)^2 \leq \frac{S(z)}{4}$. By (i) in Lemma 21 we have $z = \beta_+$. Thus we have (iv).

The equivalence of (iii) and (iv) follows immediately from (49).

Finally suppose that (iv) holds. Since $S(z) = 0$ by (iii), it suffices to show $L(z) \geq 0$ and $M(z) \leq 0$. By (ii) in Lemma 19 and Lemma 22 we have $L(z) \geq 0$. By (ii) in Lemma 20 and Lemma 22 we have $M(z) \leq 0$. □

**Lemma 24.** *Let $W_1$ be the matrix defined by* (3)*, and $W$ be the matrix defined by* (1)*. Suppose that $W$ is a complex Hadamard matrix. If $r + s = 0$, then we have* (i) *in Theorem 1.*

*Proof.* By Lemma 8, we have $r^2 = -rs < k$. Also, by Lemma 8, we have $s \leq -2$ and $r \geq 2$. Thus, we may use results of this section. In particular, by Lemmas 23, 11, and (50) we have $k = 2r^2$ or $k = 2r^2 - 1$. First suppose $k = 2r^2$. By (28) we have $a_1 = -1$. Then by (29), (30) we have $a_0 = 1$, $a_2 = 1$, respectively. Therefore we have (i) in Theorem 1. Secondly suppose $k = 2r^2 - 1$. By (19) we have $m_1 = \frac{(2r-1)(2r^2-1)}{2r}$. This is a contradiction since $m_1$ must be an integer. □

## 7. The Case $r + s > 0$

In this section, we suppose that $r, s \in \mathbb{Z}$ and $r + s > 0$. Then by Lemma 8 we have $r \geq 3$ and $s \leq -2$. We consider properties of the polynomials (25), (26), and (27). Let

$$\begin{aligned}\kappa(x) = {} & (2s+1)^2x^3 - (2s+1)(8s^3 - 2s^2 - s + 2)x^2 \\ & - (16s^5 + 8s^2 + 2s - 1)x + 4s^2 + s.\end{aligned}$$

**Lemma 25.** *Assume that $-s + 1 \leq x < -2s + 1$. Then we have $\kappa(x) < 0$.*

*Proof.* Since

$$\begin{aligned}\kappa(-s) &= -4s^2(s-1)(2s(s+1)+1) > 0, \\ \kappa(-s+1) &= s^2(8s^3 + 4s^2 - 8s + 1) < 0, \\ \kappa(-2s+1) &= -32s^4(s^2 - 1) < 0,\end{aligned}$$

we have the assertion. □

Let

$$\psi(x) = (s+1)(x+1)((2s+1)x - 1), \tag{53}$$

$$\phi(x) = 2\psi(x) - (2s+1)(x + 2s - 1). \tag{54}$$

**Lemma 26.** *We have $\psi(r) > 0$ and $\phi(r) > 0$.*

*Proof.* The inequality $\psi(r) > 0$ follows immediately by (53). Since $r > -s$ and

$$\begin{aligned}\phi(0) &= -2s(2s+1) - 1 < 0, \\ \phi(-s) &= (s-1)(4s^2(s+2) - 2(s+1)(s-2) - 3) > 0,\end{aligned}$$

we have $\phi(r) > 0$. □

Let $h$ be defined as (36).

**Lemma 27.** *Assume that $k = -rs + \frac{h+\epsilon}{2}$ and $h \in \mathbb{Z}$, where $\epsilon \in \{\pm 1\}$. Then we have*

$$n \geq -(2s+1)r + 2 + \frac{2\psi(r)}{h+1}.$$

*Proof.* First we show that

$$h \geq -2(s+1)r + 3. \tag{55}$$

To do this, since $h > 0$ and

$$h^2 - (-2(s+1)r+1)^2 = 4r(s+1)(s-r+1) > 0,$$

we have $h > -2(s+1)r + 1$. Since $h$ is odd, we have (55).

Secondly we show the assertion. Since

$$\begin{aligned} k &\geq -rs + \frac{h-1}{2} \\ &\geq -(2s+1)r + 1, \qquad \text{(by (55))} \end{aligned} \tag{56}$$

we have

$$\begin{aligned} n &= 1 + k + \ell \\ &= 1 + k - \frac{k(r+1)(s+1)}{k+rs} && \text{(by (24))} \\ &\geq 1 - (2s+1)r + 1 + \frac{(s+1)(r+1)((2s+1)r-1)}{k+rs} && \text{(by (56))} \\ &\geq -(2s+1)r + 2 + \frac{2\psi(r)}{h+1} && \text{(by (53))}. \end{aligned}$$

□

Let $u = r + s$. Then $u \in \mathbb{Z}$ and

$$1 \leq u \leq r - 2. \tag{57}$$

**Lemma 28.** *The polynomial $S''(X)$ has two distinct real roots:*

$$\tau_\pm = \frac{c_1 \pm \sqrt{c_2}}{6(u+1)^2}, \tag{58}$$

*where*

$$\begin{aligned} c_1 &= 3(u+1)^2(2r(r-u)-1) + 3(r(r+1) + (r-u)(r-u-1)), \\ c_2 &= 12r(r+1)u(u+2)(u^2+2u-2)(r-u)(r-u-1) + 3(u+1)^2. \end{aligned}$$

*Proof.* Observe $c_2 > 0$ follows from (57). Since

$$\begin{aligned} S''(X) = 12&(u+1)^2X^2 \\ &- 12(2(u^2+2u+2)r^2 - 2u(u^2+2u+2)r - (u+1))X \\ &+ 8(u^2+2u+6)r^4 - 16u(u^2+2u+6)r^3 \\ &+ 4(2u^4+5u^3+15u^2-4u-6)r^2 \\ &- 4u(u+1)(u^2+2u-6)r + 2 \end{aligned}$$

by (27), we have (58). □

**Lemma 29.** *Let $\tau_\pm$ be the real number defined by (58). Then $\tau_\pm < \beta_+$.*

*Proof.* Since $\tau_- < \tau_+$, it is enough to show that $\tau_+ < \beta_+$. By (38), we have

$$\beta_+ = r(r-u) - \frac{1}{2} + \frac{h}{2}.$$

Since

$$h^2 - (2r(r-u-1)+1)^2 = 4r(2r-u-1)(r-u-1) > 0,$$

we have

$$\beta_+ - r(2(r-u)-1) = \frac{1}{2}(h - (2r(r-u-1)+1)) > 0. \tag{59}$$

Since

$$\begin{aligned}&(6(u+1)^2 r(2(r-u)-1)-c_1)^2-c_2\\&\quad=6(u+1)^2(2r(r-u-1)(u(u+2)(2r(r-u-2)+u)+3)+1)>0,\end{aligned}$$

we have

$$r(2(r-u)-1)-\tau_+=\frac{6(u+1)^2 r(2(r-u)-1)-c_1-\sqrt{c_2}}{6(u+1)^2}>0. \tag{60}$$

By (59) and (60), we obtain $\tau_+<\beta_+$. □

Define

$$\begin{aligned}g_1&=4u(u+2)(u(u+2)-2)r(r+1)(r-u)(r-u-1)+(u+1)^2,\\g_2&=2r(r+1)(r-u)(r-u-1)\\&\quad\times(8u(u+2)r(r+1)(r-u)(r-u-1)+7u(u+2)-1)-1,\\g_3&=16u(u+2)r(r+1)(r-u)(r-u-1)-1.\end{aligned}$$

**Lemma 30.** *We have $g_1>0$, $g_2>0$, and $g_3>0$.*

*Proof.* These follow immediately from (57). □

**Lemma 31.** *The polynomial $S(X)$ has exactly two real roots, say, $\xi_1$, $\xi_2$, and $\beta_+<\xi_1<\delta<\xi_2<\beta_++1$. Moreover, both $\xi_1$ and $\xi_2$ are simple.*

*Proof.* Set $f_0(X)=S(X)$ and $f_1(X)=f_0'(X)$. Set

$$f_j(X)=-\mathrm{Rem}(f_{j-2}(X),f_{j-1}(X))$$

for $j=2,3,4$. Let $c_j$ be the leading coefficient of $f_j(X)$, and $d_j=\deg f_j(X)$. We have $(d_0,d_1,d_2,d_3,d_4)=(4,3,2,1,0)$. Then we have the following:

$$\begin{aligned}c_0&=(u+1)^2>0,\\c_1&=4(u+1)^2>0,\\c_2&=\frac{g_1}{4(u+1)^2},\\c_3&=\frac{-32u^2(u+1)^2(u+2)^2r(r+1)(r-u)(r-u-1)g_2}{g_1^2},\\c_4&=\frac{-r^2(r+1)^2(r-u)^2(r-u-1)^2g_1^2g_3}{4(u+1)^2g_2^2}.\end{aligned}$$

By Lemma 30 we have $c_2>0$, $c_3<0$, and $c_4<0$. Therefore we have Table 1. Applying Theorem 6 for $S(X)$ using Table 1, we see that $S(X)$ has exactly two real roots.

We show that $S(\beta_+)>0$, $S(\beta_++1)>0$, and $S(\delta)<0$. We have

$$S(\beta_+)=\frac{h_1h+h_2}{2},$$

where

$$\begin{aligned}h_1&=-(2r(r-u)-u)((4(r-u)r-2(2u+1))(r-u)r-u),\\h_2&=u^2+2r(r-u)\\&\quad\times(8r^2(r-u)^2(r(r-u)-(2u+1))+u^2(9r(r-u)-u+1)\\&\qquad+2r(r-u)(3u+1))>0\end{aligned}$$

since $r-u\geq 2$. Since

$$h_2^2-h_1^2h^2=4r^2(r+1)^2u^2(u+2)^2(r-u)^2(r-u-1)^2>0,$$

we have $S(\beta_+)>0$. We have

Table 1. Sturm's sequence.

| $j$ | 0 | 1 | 2 | 3 | 4 | ♯ sign changes |
|---|---|---|---|---|---|---|
| $\mathrm{sgn}(c_j)$ | + | + | + | − | − | 1 |
| $\mathrm{sgn}((-1)^{d_j}c_j)$ | + | − | + | + | − | 3 |

$$S(\beta_+ + 1) = \frac{h_3 h + h_4}{2},$$

where

$$h_3 = -(2r(r-u) - (u+2))((4r(r-u) - 2(2u+3))(r-u)r + u + 2),$$
$$h_4 = u^2 + 2(r+1)(r-u-1)$$
$$\times (r(r-u)(8((r-u)r - (u+2))(r-u)r + u^2 + 6u + 10) - 2) > 0$$

since $r - u \geq 2$. Since

$$h_4^2 - h_3^2 h^2 = 4r^2(r+1)^2 u^2 (u+2)^2 (r-u)^2 (r-u-1)^2 > 0,$$

we have $S(\beta_+ + 1) > 0$. We have

$$S(\delta) = r(r+1)(r-u)(r-u-1)(h_5\sqrt{r(r+1)(r-u)(r-u-1)} + h_6),$$

where

$$h_5 = -4(2r(r-u) - (u+1)) < 0,$$
$$h_6 = 8r(r+1)(r-u)(r-u-1) + 1.$$

Since

$$h_5^2 r(r+1)(r-u)(r-u-1) - h_6^2 = r(r+1)u(u+2)(r-u)(r-u-1) - 1 > 0,$$

we have

$$S(\delta) < 0. \tag{61}$$

The polynomial $S(X)$ has exactly two real roots, say, $\xi_1$, $\xi_2$, and $\beta_+ < \xi_1 < \delta < \xi_2 < \beta_+ + 1$.

We show that the roots $\xi_1, \xi_2$ are simple. Since $\deg S(X) = 4$ and the number of imaginary roots of $S(X)$ is even, the sum of multiplicities of $\xi_1$ and $\xi_2$ is 2 or 4. If both $\xi_1$ and $\xi_2$ are double roots, then $S(x) > 0$ for $\xi_1 < x < \xi_2$. This contradicts (61). By Lemmas 28, 29 we have $S''(x) \neq 0$ for $\beta_+ \leq x \leq \beta_+ + 1$. Thus neither $\xi_1$ nor $\xi_2$ is triple. □

**Lemma 32.** *We have* $L(\beta_+) \leq 0$ *and* $M(\beta_+ + 1) \geq 0$.

*Proof.* We have

$$L(\beta_+) = \frac{\tau_1 h + \tau_2}{4},$$

where

$$\tau_1 = -2r(r-u) + u < 0,$$
$$\tau_2 = 4r^4 - 8r^3 u + (4u^2 - 4u - 2)r^2 + 2u(2u+1)r - u.$$

Since

$$\tau_1^2 h^2 - \tau_2^2 = 4r(r+1)u(u+2)(r-u)(r-u-1) \geq 0,$$

we have $L(\beta_+) \leq 0$. Also, we have

$$M(\beta_+ + 1) = \frac{\tau_3 h + \tau_4}{4},$$

where

$$\tau_3 = 2r(r-u) - u - 2 > 0,$$
$$\tau_4 = -4r^4 + 8ur^3 - (4u^2 - 4u - 6)r^2 - 2u(2u+3)r - u - 2.$$

Since

$$\tau_3^2 h^2 - \tau_4^2 = 4r(r+1)u(u+2)(r-u)(r-u-1) \geq 0,$$

we have $M(\beta_+ + 1) \geq 0$. □

**Lemma 33.** *We have* $\xi_1 < \zeta < \eta < \xi_2$.

*Proof.* Suppose that $\zeta \leq \xi_1$. By (i) in Lemma 19 we have $L(\zeta) = 0$. By (i) in Lemma 21 we have $L(\zeta)^2 \geq \frac{S(\zeta)}{4}$, and by Lemma 31 we have $\frac{S(\zeta)}{4} \geq 0$. Hence $L(\zeta)^2 = \frac{S(\zeta)}{4} = 0$. This contradicts (41) and Lemma 32.

Suppose that $\xi_2 \leq \eta$. By (i) in Lemma 20 we have $M(\eta) = 0$. By (ii) in Lemma 21 we have $M(\eta)^2 \geq \frac{S(\eta)}{4}$, and by Lemma 31 we have $\frac{S(\eta)}{4} \geq 0$. Hence $M(\eta)^2 = \frac{S(\eta)}{4} = 0$. This contradicts (42) and Lemma 32.

We have

$$M(x) \le L(x) \tag{62}$$

for $x \in \mathbb{R}$ by (26). The inequality $\zeta < \eta$ follows from (62), (i) in Lemma 19, and (i) in Lemma 20. □

Let

$$A = (-rs, \xi_1], \tag{63}$$
$$B = [\xi_2, \infty). \tag{64}$$

**Lemma 34.** *We have the following:*
(i) $S(x) \ge 0$ *for* $x \in \mathbb{R}$ *holds if and only if* $x \in A \cup B$.
(ii) *For* $x \in A \cup B$,
(a) $M(x) \le \frac{\sqrt{S(x)}}{2} \le L(x)$ *holds if and only if* $x = \beta_+ + 1$,
(b) $M(x) \le \frac{-\sqrt{S(x)}}{2} \le L(x)$ *holds if and only if* $x = \beta_+$.

*Proof.* (i) This follows from Lemma 31 since the leading coefficient of $S(X)$ is positive.

(ii) (a) Suppose that $M(x) \le \frac{\sqrt{S(x)}}{2} \le L(x)$ holds. Since $L(x) \ge 0$, by (ii) in Lemma 19 we have $\zeta \le x$. Since $x \in A \cup B$, by Lemma 33 we have $x \in B$. Hence $\eta < x$. Then by (ii) in Lemma 20 we have $M(x) \ge 0$. Hence $M(x)^2 \le \frac{S(x)}{4}$. By (ii) in Lemma 21 we have $x = \beta_+ + 1$.

Conversely, suppose that $x = \beta_+ + 1$. By (ii) in Lemma 21 and Lemma 32 we have $M(\beta_+ + 1) = \frac{\sqrt{S(\beta_+ + 1)}}{2}$. Since $-rs < \beta_+ + 1$ by (ii) in Lemma 17, by (26) we have $M(\beta_+ + 1) < L(\beta_+ + 1)$. Therefore $M(\beta_+ + 1) = \frac{\sqrt{S(\beta_+ + 1)}}{2} < L(\beta_+ + 1)$.

(ii) (b) Suppose that $M(x) \le \frac{-\sqrt{S(x)}}{2} \le L(x)$ holds. Since $M(x) \le 0$, by (ii) in Lemma 20 we have $x \le \eta$. Since $x \in A \cup B$, by Lemma 33 we have $x \in A$. Hence $x < \zeta$. Then by (ii) in Lemma 19 we have $L(x) < 0$. Thus $L(x)^2 \le \frac{S(x)}{4}$. By (i) in Lemma 21 we have $x = \beta_+$.

Conversely, suppose that $x = \beta_+$. By (i) in Lemma 21 and Lemma 32 we have $\frac{-\sqrt{S(\beta_+)}}{2} = L(\beta_+)$. Since $-rs < \beta_+$ by (ii) in Lemma 17, by (26) we have $M(\beta_+ + 1) < L(\beta_+ + 1)$. Therefore $M(\beta_+) < \frac{-\sqrt{S(\beta_+)}}{2} = L(\beta_+)$. □

For the remainder of this section, we assume that $W$ defined by (1) is a complex Hadamard matrix for the case $r + s > 0$. By (i) in Lemma 34 and (i) in Lemma 11 we have $k \in A \cup B$ by (63) and (64). By (ii) (a) and (b) in Lemma 34 and (ii) in Lemma 11 we have $k \in \{\beta_+, \beta_+ + 1\}$, that is, $k = -rs + \frac{h+\epsilon}{2}$, where $\epsilon \in \{\pm 1\}$. Then by (38) we have $h \in \mathbb{Z}$. By (ii) in Lemma 8 and Lemma 27 we have

$$4s^2 - 1 \ge -(2s+1)r + 2 + \frac{2\psi(r)}{h+1}. \tag{65}$$

Since

$$\begin{aligned} 0 &< \frac{2\psi(r)}{h+1} && \text{(by Lemma 26)} \\ &\le (2s+1)(r+2s-1) - 2 && \text{(by (65))} \\ &< (2s+1)(r+2s-1), \end{aligned}$$

we have $r < -2s + 1$. Then by Lemma 25 we have $\kappa(r) < 0$.

By (65) we have

$$\begin{aligned} (2s+1)(r+2s-1)h &> 2\psi(r) - (2s+1)(r+2s-1) \\ &= \phi(r) && \text{(by (54))} \\ &> 0 && \text{(by Lemma 26).} \end{aligned}$$

Since

$$\begin{aligned} 0 &< ((2s+1)(r+2s-1)h)^2 - \phi(r)^2 \\ &= -4(s+1)(r+1)\kappa(r), \end{aligned}$$

we have $\kappa(r) > 0$. This is a contradiction. Therefore there does not exist such a complex Hadamard matrix.

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