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Fair risk allocation in illiquid markets

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Abstract

Let us consider a financially constrained leveraged financial firm having some divisions which have invested into some risky assets. Using coherent measures of risk the sum of the capital requirements of the divisions is larger than the capital requirement of the firm itself, there is some diversification benefit that should be allocated somehow for proper performance evaluation of the divisions. In this paper we use cooperative game theory and simulation to assess the possibility to jointly satisfy three natural fairness requirements for allocating risk capital in illiquid markets: Core Compatibility, Equal Treatment Property and Strong Monotonicity.

Core Compatibility can be viewed as the allocated risk to each coalition (subset) of divisions should be at least as much as the risk increment the coalition causes by joining the rest of the divisions. Equal Treatment Property guarantees that if two divisions have the same stand-alone risk and also they contribute the same risk to all the subsets of divisions not containing them, then the same risk capital should be allocated to them. Strong Monotonicity requires that if a division weakly reduces its stand-alone risk and also its risk contribution to all the subsets of the other divisions, then as an incentive its allocated risk capital should not increase. Analyzing the simulation results we conclude that in most of the cases it is not possible to allocate risk in illiquid markets satisfying the three fairness notions at the same time, one has to give up at least one of them.

Keywords: Market Microstructure, Coherent Measures of Risk, Market Liquidity, Funding Liquidity, Portfolio Performance Evaluation, Risk Capital Allocation, Risk Contributions, Totally Balanced Games, Simulation

JEL classification: C71, G10

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Igazságos kockázatfelosztás nem likvid piacokon

CSÓKA PÉTER

Összefoglaló

Tekintsünk egy tőkeáttételes pénzügyi vállalatot, amelynek nem likvid portfóliókat tartó divíziói vannak! Koherens kockázati mértékek használata esetén a divíziók tőkekövetelményének összege nagyobb, mint a vállalat tőkekövetelménye, diverzifikációs előny keletkezik, amelyet valahogy el kell osztani a divíziók jobb teljesítményértékelése végett. Ebben a tanulmányban kooperatív játékelmélet és szimuláció segítségével elemezzük azt, hogy mennyire lehet nem likvid piacon három természetesnek tűnő igazságossági követelmény alapján felosztani a kockázatot: ezek a magbeliség (Core Compatibility), az egyenlő kezelés (Equal Treatment Property) és az erős monotonitás (Strong Monotonicity).
A magbeliség követelménye az, hogy a divíziók tetszőleges koalíciójára (részhalmazára) legalább annyi kockázatot allokáljunk, mint amekkora kockázatnövekedést okoz az, ha a koalíció csatlakozik a többi divízióhoz. Az egyenlően kezelés tulajdonság azt garantálja, hogy ha két divízió annyira szimmetrikus, hogy ugyanaz a kockázatuk és a divíziók összes, őket nem tartalmazó koalíciójához is ugyanakkora kockázattal járulnak hozzá, akkor ugyanannyi kockázatot kell rájuk osztani. Az erős monotonitás azt követeli meg, hogy ha egy divízió gyengén csökkenti az egyéni kockázatát és a többi divízió tetszőleges koalíciójához is kevesebb kockázatot tesz hozzá, akkor (ösztönzésképpen) a ráosztott tőke nem növekedhet. A szimulációs eredményeket elemezve arra a végkövetkeztetésre jutunk, hogy a legtöbb esetben nem likvid piacokon nem lehetséges a három igazságossági követelményt egyszerre teljesíteni, valamelyikről le kell mondanunk.

Tárgyszavak: piaci mikrostruktúra, koherens kockázati mértékek, piaci likviditás, finanszírozási likviditás, portfólióteljesítmény-értékelés, tőkeallokáció, teljesen kiegyensúlyozott játékok, szimuláció

JEL kód: C71, G10

Fair risk allocation in illiquid markets*

Péter Csóka†

January 19, 2015

Abstract

Let us consider a financially constrained leveraged financial firm having some divisions which have invested into some risky assets. Using coherent measures of risk the sum of the capital requirements of the divisions is larger than the capital requirement of the firm itself, there is some diversification benefit that should be allocated somehow for proper performance evaluation of the divisions. In this paper we use cooperative game theory and simulation to assess the possibility to jointly satisfy three natural fairness requirements for allocating risk capital in illiquid markets: Core Compatibility, Equal Treatment Property and Strong Monotonicity.

Core Compatibility can be viewed as the allocated risk to each coalition (subset) of divisions should be at least as much as the risk increment the coalition causes by joining the rest of the divisions. Equal Treatment Property guarantees that if two divisions have the same stand-alone risk and also they contribute the same risk to all the subsets of divisions not containing them, then the same risk capital should be allocated to them. Strong Monotonicity requires that if a division weakly reduces its stand-alone risk and also its risk contribution to all the subsets of the other divisions, then as an incentive its allocated risk capital should not increase. Analyzing the simulation results we conclude that in most of the cases it is not possible to allocate risk in illiquid markets satisfying the three fairness notions at the same time, one has to give up at least one of them.

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1 Introduction

Let us consider a financially constrained financial firm having some divisions (subunits) which have invested into some risky assets and have some liabilities. As a cushion against possible future losses, some capital (equity) should be held by the firm, otherwise it would not be credible that it is able to pay back its liabilities. A coherent measure of risk (Artzner, Delbaen, Eber, and Heath, 1999) assigns a number to the profit and loss distribution of the value of the portfolio of the firm at a specific future point of time. When gross assets (without netting the liabilities) are taken as the portfolio of the firm, then the measure of risk reflecting the preferences of the regulator or the firm is negative, its absolute value can be seen as a "safe valuation" of the assets. The capital requirement in this case can be calculated in the following way: liabilities minus the safe value of the assets determined by the measure of risk. Since the capital should be kept by the firm in riskless assets, dividing the returns of the divisions by the respective capital requirements can serve as a performance evaluation measure. Using coherent measures of risk the sum of the capital requirements of the divisions is larger than the capital requirement of the firm itself, there is some diversification benefit hat should be allocated somehow (for more details and applications see Denault (2001), Kalkbrener (2005), Buch and Dorfleitner (2008), Homburg and Scherpereel (2008), Kim and Hardy (2009), and Csóka, Herings, and Kóczy (2009) among others).

Csóka and Herings (2014) extend the usual cooperative game theory approach (risk allocation games) to handle the problem of risk capital allocation when the divisions might have illiquid assets by combining the notions of Csóka, Herings, and Kóczy (2009) and Acerbi and Scandolo (2008). To define a cooperative game one should assign a payoff to all coalitions of players, that is to all subsets of the divisions. In a risk allocation game the payoff of a coalition is the opposite of its risk, where risk is measured by using a coherent measure of risk on the possible realizations of the value of the portfolio of the coalition. When having illiquid portfolios, the realization value of the portfolio of a coalition in a certain state depends on how easy it is to trade its assets (captured by random marginal demand curves) and on the required composition of the portfolio (called liquidity policy). To analyze how funding constraints affect fair risk allocation, in this paper we will use cash liquidity policies with short sale constraints, where a certain amount of cash should be generated and short sales are not allowed.[1] The random marginal demand curves lead to different optimal trades (sales) to satisfy the cash liquidity policy of the firm. After executing optimal trades by the coalition of divisions, the realized value of the resulting portfolio is determined by using the best bid prices of the resulting assets. Since the

[1] To cover the risk of upward moving prices, usually the proceeds of a short sale are not allowed to be used in another transaction.

coalitions of the divisions could trade at the same time, they face an externality problem: their optimal trade depends on the trades done by the other divisions outside the coalition. Csóka and Herings (2014) argue that the most reasonable way to handle this problem is that the divisions outside the coalition at hand remains inactive, so their portfolios can be considered as fixed and they define risk allocation games with liquidity this way.

Having the payoffs of the coalitions allows us to use standard game theory concepts (like the Shapley value (Shapley, 1953) or the nucleolus (Schmeidler, 1969)) as risk allocation rules to split up the risk capital of the firm to its divisions in any possible situations. In this paper we will assess the possibility to jointly satisfy three fairness requirements for allocating risk capital in illiquid markets introduced by Csóka and Pintér (2014): Core Compatibility, Equal Treatment Property and Strong Monotonicity. Core Compatibility is satisfied if the risk of the firm is allocated in such a way that no coalition of the divisions would have a lower risk by being alone. Such allocations are said to be in the core of the game. Csóka and Pintér (2014) notes that Core Compatibility can also be viewed as the allocated risk to each coalition of divisions should be at least as much as the risk increment the coalition causes by joining the rest of the divisions. Equal Treatment Property guarantees that if two divisions have the same stand-alone risk and also they contribute the same risk to all subsets of divisions not containing them, then the same risk capital should be allocated to them. Strong Monotonicity requires that if a division weakly reduces its stand-alone risk and also its risk contribution to all subsets of the other divisions, then as an incentive its allocated risk capital should not increase.

A subgame is obtained by considering a subset of the divisions of the firm and looking the resulting risk allocation game. A totally balanced game has a non-empty core in all of its subgames. Csóka and Herings (2014) show that the class of risk allocation games with liquidity coincides with the class of totally balanced games, generalizing the result by Csóka, Herings, and Kóczy (2009) for risk allocation game without liquidity. The coincidence means that firstly, any totally balanced game can be generated by a properly chosen risk allocation game with or without liquidity and secondly, it also means that any risk allocation game with or without liquidity is totally balanced, that is Core Compatibility alone can be satisfied. However, Csóka and Pintér (2014) show that on the class of totally balanced games the Shapley value is the only risk allocation rule satisfying Equal Treatment Property and Strong Monotonicity at the same time. However, it is well-known that the Shapley value does not satisfy Core Compatibility in general, hence in theory the three requirements are irreconcilable.

Looking at the impossibility problem from a practical perspective, the Shapley value in a random but realistic risk allocation game with liquidity is not always expected to satisfy Core Compatibility. Hence we can assess the possibility to allocate risk in a fair way in illiquid markets by checking the average Core Compatibility of the Shapley value in such random risk environments with liquidity considerations. In the simulation we will consider first 3, then 4 divisions and simulate 100 000 random risk allocation games with liquidity. We will see that for 3 divisions in at least 30%, for 4 divisions in at least 50 % of the cases the Shapley value (being the only risk allocation rule which satisfies Equal Treatment Property and Strong Monotonicity at the same time) does not satisfy Core Compatibility, and for more divisions we can expect that the tendency continues. So we can say that it is most of the time not possible to allocate risk in illiquid markets satisfying the three fairness notions at the same time, one has to give up at least one of them.

We are aware of two papers doing similar simulations. Homburg and Scherpereel (2008) are also checking the average Core Compatibility of the Shapley value (among other rules), but in their setting Value at Risk is used (which is not a coherent measure of risk) and there are no liquidity constraints. In their paper for 3 and 4 divisions the average Core Compatibility of the Shapley value becomes 80-90%, but using Value at Risk the resulting game is not totally balanced, and hence that result says nothing about the other two fairness requirements. Balog, Bátyi, Csóka and Pintér (2014) discuss analytically which out of the three fairness properties are met by seven different risk allocation methods. They also simulate random risk allocation games with coherent measures of risk, for normal and also for fat tailed return distributions. Without liquidity and for normal distribution our results are comparable. However, they do not take into account illiquid assets.

The structure of the paper is as follows. In Section 2 we define risk allocation games with liquidity constraint. Section 3 defines the Shapley value and discusses some of its main properties to be used as fair. Section 4 contains the simulation results and Section 5 concludes.

2 Risk allocation games with liquidity constraints

We consider a firm with n divisions, whose risk capital should be allocated. *Risk environments with liquidity considerations* are defined by Csóka and Herings (2014) and are denoted by $(N, J, S, \pi, \theta, m, L, \rho)$, where

- N is the set of divisions,
- S is the set of states of nature,
- state of nature $s \in S$ occurs with probability $\pi_s > 0$, where $\sum_{s \in S} \pi_s = 1$,
- we have cash and J is the set of risky assets,
- $\theta^i = (\theta^i_0, \theta^i_J) \in \mathbb{R} \times \mathbb{R}^J = P$ is the initial portfolio of division i,

- $\{m_j^s\}$ is the Marginal Demand Curve of asset j in state $s \in S$,
- $L \subseteq \mathbb{R}^{J+1}$ is the liquidity policy, the set of portfolios which is deemed acceptable, and
- ρ is a coherent measure of risk.

Next, we define all above mentioned elements in detail. We follow Çetin, Jarrow and Protter (2004), Jarrow and Protter (2005) and Acerbi and Scandolo (2008) in modeling the order book for asset j in state $s \in S$ by a marginal demand curve m_j^s. A function is càdlàg if it is right continuous with left limits and làdcàg if it is left continuous with right limits.

Definition 2.1. The *marginal demand curve* (MDC) for asset $j \in J$ in state $s \in S$ is given by the map $m_j^s : \mathbb{R} \setminus \{0\} \mapsto \mathbb{R}$ satisfying
(i) $m_j^s(x) \geq m_j^s(x')$ if $x < x'$;
(ii) m_j^s is càdlàg at $x < 0$ and làdcàg at $x > 0$.

For asset j the amount $m_j^s(x)$ for $x > 0$ shows the marginal revenue the firm can get by selling it, whereas $m_j^s(x)$ for $x < 0$ represents the marginal cost of buying it; $m^s(0_j^+)$ denotes the best bid and $m^s(0_j^-)$ the best ask price.

The liquidation value of a portfolio will be needed to calculate attainable portfolios.

Definition 2.2. The *liquidation mark-to-market value* of a portfolio $p \in P$ in state $s \in S$ is defined by

$$\ell^s(p) = p_0 + \sum_{j \in J} \int_0^{p_j} m_j^s(x)dx. \tag{1}$$

The liquidation mark-to-market value of a portfolio equals to the portfolio's cash plus the revenue that the firm gets by selling long positions minus the cost, which has to be paid to close short positions.

The set of portfolios *attainable* from some given portfolio $p \in P$ in state $s \in S$ by liquidating all or part of it is given by

$$A^s(p) = \{q \in P \mid q_0 = \ell^s(p_0, p_J - q_J)\}.$$

Given a portfolio p and liquidating $p_J - q_J$ results in portfolio q where the cash is p_0 plus the liquidation value of $p_J - q_J$.

The *liquidity policy* (Acerbi and Scandolo, 2008) incorporates the requirements imposed by a regulator or the contractual obligations that have to be met, and specifies that the portfolio of the firm should belong to the set $L \subset P$. In this paper we consider cash liquidity policies with short sale constraints, where the portfolio should contain at least $c \geq 0$ units of cash and short sales are not allowed: $L(c) = \{p \in P | p_0 \geq c \text{ and } p_J \geq 0\}$.

For a portfolio $p \in P$, we denote the assets hold long by $J^+(p) = \{j \in J \mid p_j > 0\}$ and the assets hold short by $J^-(p) = \{j \in J \mid p_j < 0\}$.

Definition 2.3. The *uppermost mark-to-market value* of a portfolio $p \in P$ in state $s \in S$ is defined by

$$u^s(p) = p_0 + \sum_{j \in J^+(p)} m_j^s(0^+) p_j + \sum_{j \in J^-(p)} m_j^s(0^-) p_j. \tag{2}$$

In the uppermost mark-to-market value of a portfolio long positions are evaluated by using the best bid prices and short positions are valued by using the best ask prices.

Artzner, Delbaen, Eber, and Heath (1999) have introduced coherent measures of risk. A *measure of risk* is a function $\rho : \mathbb{R}^S \to \mathbb{R}$ measuring the risk of a realization vector from the perspective of the present. In our simulations we use a particular coherent measure of risk, the k-Expected Shortfall (Acerbi and Tasche, 2002), which is the average of the worst $100k$ percent of the losses.

After completing the definition of a risk environment with liquidity considerations $(N, J, S, \pi, \theta, m, L, \rho)$ we can use tools from cooperative game theory to analyze them.

Let N denote the finite *set of players*. *A cooperative game with transferable utility* (*game*, for short) is a function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. The class of games with player set N is denoted by $\mathcal{G}^N$. For a game $v \in \mathcal{G}^N$ and a coalition $C \in 2^N$, a *subgame* v_C is obtained by restricting v to the subsets of C.

An *allocation* is a vector $x \in \mathbb{R}^N$, where x_i is the payoff of player $i \in N$. An allocation x yields payoff $x(C) = \sum_{i \in C} x_i$ to a coalition $C \in 2^N$. An allocation $x \in \mathbb{R}^N$ is called *Efficient*, if $x(N) = v(N)$ and *Coalitionally Rational* if $x(C) \geq v(C)$ for all $C \in 2^N$. The *core* (Gillies, 1959) is the set of Efficient and Coalitionally Rational allocations. The core of game v is denoted by core (v). A game is *totally balanced*, if each of its subgame has a non-empty core. Let $\mathcal{G}^N_{\text{tb}}$ denote the class of totally balanced games with player set N.

The question is how to define the cooperative game where the divisions (players) hold illiquid portfolios. Csóka and Herings (2014) argue that the most reasonable way to handle externalities in this setting is to assume that the complement of coalition C remains inactive. The portfolios which are attainable for coalition C in state $s \in S$ are given by $A^s(\theta(C))$, where $\theta(C) = \sum_{i \in C} \theta^i$. Inactivity of the complementary coalition means that those divisions stick to their initial portfolio $\theta(N \setminus C)$.

Definition 2.4. Given a risk environment with liquidity considerations $(N, J, S, \pi, \theta, m, L, \rho)$ and a coalition of divisions $C \subset N$, the *realization vector* $X(C)$ of coalition C is defined by

$$X^s(C) = \sup\{u^s(q) | q \in A^s(\theta(C)) \text{ and } q + \theta(N \setminus C) \in L^s\}, \quad s \in S.$$

When calculating $X^s(C)$, we consider the portfolios of the divisions outside the coalition as fixed, and liquidate the portfolios of the divisions in C in such a way that the resulting portfolio of the firm is attainable and satisfies the liquidity policy.

Definition 2.5. Given a risk environment with liquidity considerations $(N, J, S, \pi, \theta, m, L, \rho)$, the *risk allocation game with liquidity constraints* is the game (N, v), where the value function $v : 2^N \to \mathbb{R}$ is defined by

$$v(C) = -\rho(X(C)), \quad C \in 2^N. \tag{3}$$

Let Γ_{rl} denote the family of risk allocation games with liquidity constraints with set of players N.

Next, we introduce three fairness properties of risk allocation rules.

3 The Shapley value as a risk allocation rule

Throughout the paper we consider single-valued risk allocation rules. The function $\psi : A \to \mathbb{R}^N$, defined on $A \subseteq \mathcal{G}^N$, is called *risk allocation rule* on the class of games A if $\sum_{i \in N} \phi_i = v(N)$, that is if the value of the whole firm is allocated, where ϕ_i specifies how much the assets of division i is valued according to ϕ.

Let $v \in \mathcal{G}^N$ and $i \in N$ be a game and a player, and for all $C \subseteq N$ let $v'_i(C) = v(C \cup \{i\}) - v(C)$ denote player i's marginal contribution to coalition C in game v. Then v'_i is called player i's *marginal contribution function* in game v. Player i is a null-player in game v, if $v'_i = 0$. Moreover, players i and j are *equivalent* in game v, $i \sim^v j$, if for any $C \subseteq N$ such that $i, j \notin C$ we have that $v'_i(C) = v'_j(C)$.

Definition 3.1. For any game $v \in \mathcal{G}^N$ the *Shapley value (Shapley, 1953) of player i* , ϕ_i is given by

$$\phi_i(v) = \sum_{C \subseteq N \setminus \{i\}} v'_i(C) \frac{|C|!(|N \setminus C| - 1)!}{|N|!} \quad i \in N. \tag{4}$$

The Shapley value can be interpreted as follows. Players are entering into a room in all possible permutations $|N|!$ with equal probability. The Shapley value of a player is the expected marginal contribution she makes to the coalition preceding her. Players before her can enter $|C|!$ ways, players after her can enter $|N \setminus C| - 1)!$ ways.

Csóka and Pintér (2014) introduce four basic properties (axioms) a risk allocation rule should satisfy.

Definition 3.2. The risk allocation rule ψ on class $A \subseteq \mathcal{G}^N$ satisfies

- *Core Compatibility* if for each $v \in A$: $\psi(v) \in \text{core}(v)$,
- *Equal Treatment Property* if for each $v \in A$, $i, j \in N$: $i \sim^v j$ implies $\psi_i(v) = \psi_j(v)$,
- *Strong Monotonicity* if for any $v, w \in A$, $i \in N$: $v'_i \leq w'_i$ implies $\psi_i(v) \leq \psi_i(w)$.

The financial interpretations of the axioms are as follows.

Core Compatibility is satisfied if the risk allocation rule results in a core allocation, in which the risk of firm is allocated in such a way that no coalition of the divisions would have a lower risk by being alone. Csóka and Pintér (2014) notes that Core Compatibility can also be viewed as the allocated risk to each coalition of divisions should be at least as much as the risk increment the coalition causes by joining the rest of the divisions.

Equal Treatment Property guarantees that if two divisions have the same stand-alone risk and also they contribute the same risk to all subsets of divisions not containing them, then the same risk capital should be allocated to them.

Strong Monotonicity requires that if a division weakly reduces its stand-alone risk and also its risk contribution to all subsets of the other divisions, then as an incentive its allocated risk capital should not increase.

Csóka and Herings (2014) show that the class of risk allocation games with liquidity coincides with the class of totally balanced games, generalizing the result by Csóka, Herings, and Kóczy (2009) for risk allocation games without liquidity. The coincidence means that at first, any totally balanced game can be generated by a properly chosen risk allocation game with or without liquidity and secondly, it also means that any risk allocation game with or without liquidity is totally balanced, that is Core Compatibility alone can be satisfied. Note that the coincidence remains valid in our setting (using cash liquidity policies with short sale constraints), since they show that using any liquidity policy the generated game is totally balanced, and any totally balanced game can be generated by using any liquidity policy with perfectly liquid assets. However, Csóka and Pintér (2014) show that on the class of totally balanced games the Shapley value is the only risk allocation rule concept satisfying Equal Treatment Property and Strong Monotonicity at the same time, but it is well-known that it does not satisfy Core Compatibility in general, hence the three requirements are irreconcilable. Thus the Shapley value in a random risk allocation game with liquidity is not always expected to satisfy Core Compatibiliy. Hence we assess the possibility to allocate risk in a fair way when we have illiquid markets by checking the average Core Compatibility of the Shapley value in such random risk environments with liquidity considerations.

4 Simulation results

In the simulation we will consider first $n = 3$, then $n = 4$ divisions with the following parameters and simulate 100 000 random risk allocation games with liquidity. We have 1000 states of nature having equal probability of occurrence, $S = \{1, \dots, 1000\}$. One can look at it as considering daily market changes for 4 years. We have cash (euros) and $j = n$ risky assets, the set of possible portfolios is $P = \mathbb{R} \times \mathbb{R}^n$. The initial portfolios of the divisions in million units in case of 3 players are given by $\theta^1 = (1, 1, 0, 0)$, $\theta^2 = (2, 0, 1, 0)$ and $\theta^3 = (3, 0, 0, 1)$; in case of four players they are given by $\theta^1 = (1, 1, 0, 0, 0)$, $\theta^2 = (2, 0, 1, 0, 0)$, $\theta^3 = (3, 0, 0, 1, 0)$ and $\theta^4 = (0, 0, 0, 0, 1)$. Thus each division has some non-negative cash and 1 million units of an asset in which the other divisions have no positions. We normalize the initial price of each risky asset to be 1000 and say that the initial investment into risky assets for each division is 1000 million euros.

To define $\{m_j^s\}$, the Marginal Demand Curve (MDC) of asset j in state $s \in S$ we need the following random numbers, where $m \in \{1, \dots, 100\,000\}$ labels the risk allocation game, $s \in S$ corresponds to the state and $j \in \{1, \dots, n\}$ refers to the asset:

- Random covariance matrices $\Sigma^m \in \mathbb{R}^{n \times n}$, where Σ_{jj}^m, the daily standard deviation of r_j, the logarithmic return of asset j is uniformly distributed between 1% and 4%.
- Joint normal distribution of the returns (r_j) with mean 0 and covariance Σ^m.
- Let the market risk driver be $A_j = 1000 \times e^{r_j}$ and let the liquidity risk driver k_j be uniformly distributed between 1 and 50.
- We will use the exponential MDCs analyzed by Acerbi and Scandolo (2008), where for state $s \in S$ $m_j^s(x) = A_j^s e^{-k_j^s x}$ for $x \neq 0$.

Note that in state s the larger k_j^s is, the less liquid the market of asset j is. The best bid of asset j is calculated as $m_j^s(0^+) = A_j^s$.

The liquidity policy is the following: the portfolio of the firm should contain at least c million euros and short sales are not allowed: $L(c) = \{p \in P | p_0 \geq c \text{ and } p_J \geq 0\}$.

To define the risk allocation game with liquidity constraints consider Definition 2.4. Since both for 3 and 4 divisions the firm has in total $1 + 2 + 3 = 6$ million euros and for each coalition of the divisions the portfolio (and hence the cash) of the divisions outside of the division is given, each coalition should generate $a = c - 6$ million euros by selling its assets in an optimal way. We will consider three cases: $a = 0$, $a = 10$ and $a = 15$. Note that if $a = 0$ (and thus $c = 6$, then there is no need to liquidate any assets.

Even though short sales are not allowed at the firm level, a coalition of divisions could short sale assets owned by divisions outside the coalition, but it is not fair to allocate those proceeds to the coalition, thus we assume that a coalition can only generate cash by selling the assets of its divisions.[2] It follows from the calculations by Acerbi and Scandolo (2008) that $t_i^s(C)$, the optimal amount in millions to trade (sell) from asset i by division i in coalition $C \in N$ in state $s \in S$ is given by

$$t_i^s(C) = \frac{1}{k_i^s} \log(1 + \lambda^s(C)), \tag{5}$$

where the marginal cost of liquidation per euro liquidated, $\lambda^s(C)$ is given as

$$\lambda^s(C) = \frac{a}{\sum_{i \in C} \frac{A_i^s}{k_i^s} - a}. \tag{6}$$

The realization vector of coalition $C \in N$ in state $s \in S$ is given by

$$X^s(C) = \sum_{i \in C} A_i^s(\theta^i - t_i^s(C)) + a, \tag{7}$$

since $t_i^s(C)$ determines how much division i should trade and the remaining portfolio is valued by the uppermost value, which is the best bid, while coalition C generates a million euros.

The measure of risk ρ is the expected shortfall with $k = 1\%$ and $k = 5\%$. For each coalition $C \in 2^N$ the risk allocation game with liquidity is defined by $v(C) = -\rho(X(C))$, then the Shapley value is calculated by Equation (4). Then coalitional rationality of the Shapley value should be checked for each coalition. If coalitional rationality is not violated, then we have a core allocation and add one to a counter. If coalitional rationality is violated, then the counter is not changing. After simulating 100 000 risk allocation games with liquidity we divide the counter by 100 000 and get the average Core Compatibility of the Shapley value. Example 4.1 illustrates the simulation by showing the calculations for one realization of a risk allocation game with liquidity for 3 divisions.

Example 4.1. Let $a = 15$, $k = 1\%$ and let us see a realization of a risk allocation game for 3 divisions.

The names of the columns of Table 1 are the coalitions (C), but the first three coalitions can also be seen as the individual divisions (j). The table has three blocks: lines 1-3, 4-8 and 9-11. In the first two blocks we just have one state s, reflecting the realization of one day out of 1000. In the third block other data from other 999 simulated days are also used.

In the first block for a particular day (s) for each asset (j) we can see the realized logarithmic returns (r_j^s), the corresponding market risk drivers (best bid prices, A_j^s) and

[2] If we allowed short selling assets owned by divisions outside the coalition, then the optimal trade of each coalition would be the same as the optimal trade of the grand coalition, the firm itself, weakly decreasing the trading cost of the coalitions.

C, j	1	2	3	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
r_j^s	-1.88%	-2.72%	-2.94%				
A_j^s	981.39	973.20	971.06				
k_j^s	2.70	49.67	35.18				
$\lambda^s(C)$	0.0430	3.2662	1.1902	0.0407	0.0398	0.4659	0.0379
$t_1^s(C)$	0.0156			0.0148	0.0145		0.0138
$t_2^s(C)$		0.0292		0.0008		0.0077	0.0007
$t_3^s(C)$			0.0223		0.0011	0.0109	0.0011
$X^s(C)$	981.07	959.77	964.42	1954.28	1952.15	1941.21	2925.36
$\rho(X(C))$	-896.59	-902.86	-942.33	-1815.86	-1905.34	-1893.40	-2817.90
$v(C)$	896.59	902.86	942.33	1815.86	1905.34	1893.40	2817.90
$\phi(C)$	919.70	916.86	981.34	1836.56	1901.04	1898.20	2817.90

Table 1: One realization of a risk allocation game with liquidity

the realized liquidity risk drivers (k_j^s) rounded to two decimals. Each asset had a falling best bid price, asset 1 is the most liquid and asset 2 is the less liquid.

In the second block of Table 1 the marginal cost of liquidation per euro liquidated, $\lambda^s(C)$ is calculated by Equation (6) for a particular day (s). The resulting optimal trades by division j are given by $t_j^s(C)$ using Equation (5), the realization vector of the coalitions $X^s(C)$ are coming from Equation (7). For instance when divisions 2 and 3 trade together in coalition $\{2,3\}$ instead of alone, then the marginal cost of liquidation per euro is decreased from 3.2662 and 1.1902 to 0.4659; the trades required are also decreased from 0.0292 and 0.0223 to 0.0077 and 0.0109; and the values of the portfolios taking into the liquidity considerations are increasing from 959.77+964.42= 1924.19 to 1941.21, the coalition is saving on the trading costs.

In the last block of Table 1 first we can see the measure of risk of the coalitions $\rho(X(C))$, which is the average of the worst $k = 1\%$ of the portfolio values, that is the average of the opposite of the lowest 10 (=1000 $\times$ 1%) realizations of $X^s(C)$. The risk of the firm, -2817.90 can be interpreted as 2819.90 million euros is a safe valuation of the assets using the preferences of the regulator or the firm. Hence 3000-2819.90=180.1 million euros is the capital (equity) requirement of the firm, which, on the other side of the balance sheet means that 180.1 million euros should be invested into a safe assets by the owners of the firm. Note that we can again see diversification effects when combining individual divisions. From $v(C) = -\rho(X(C))$ the Shapley value of division j, $\phi_j(v)$ is calculated using Equation (4) and $\phi(C) = \sum_{j \in C} \phi_j(v)$ is what the coalitions get when risk is allocated by the Shapley value. To get Core Compatibility for each C the inequality $v(C) \leq \phi(C)$ should hold, but for coalition $\{1, 3\}$ it is violated, the payoff (safe valuation of the assets) to coalition $\{1, 3\}$ would be higher when they are without division 2 than when the payoff is allocated by the Shapley value to them, hence the Shapley value in this example is not in the core, it allocates capital requirements in an unfair way.

After simulating 100 000 risk allocation games with liquidity the average Core Compatibility of the Shapley value are displayed using different parameter settings for 3 divisions in Table 2 and for 4 divisions in Table 3.

	$a = 0$	$a = 10$	$a = 15$
$k = 1\%$	59.2%	62.8%	67.8%
$k = 5\%$	59.9%	64.7%	70.7%

Table 2: Average Core Compatibility of the Shapley value in case of 3 divisions

	$a = 0$	$a = 10$	$a = 15$
$k = 1\%$	39.7%	41.7%	44.2%
$k = 5\%$	40.2%	42.8%	46.5%

Table 3: Average Core Compatibility of the Shapley value in case of 4 divisions

Depending on the number of divisions each simulation lasted for about 2-5 minutes using an average computer. Repeating the simulations the numbers only changed by about 0.1-0.3%. We can confidently say that the average Core Compatibility of the Shapley value is about 60-70 % for 3 divisions and about 40-47 % for 4 divisions. Both for 3 and for 4 divisions increasing the cash to be generated in the liquidity policy (a) clearly increases Core Compatibility by about 5 to 10 percentage points, due to the extra diversification in liquidity risk on top of market risk. Increasing k, the percentage of outcomes from which the expected shortfall is calculated is also increasing Core Compatibility by about 0.5 to 3 percentage points. Note that both effects are lower for 4 divisions.

5 Conclusion

To conclude, we have observed in the simulations that for 3 divisions in at least 30%, for 4 divisions in at least 50 % of the cases the Shapley value (being the only risk allocation rule which satisfies Equal Treatment Property and Strong Monotonicity at the same time) does not satisfy Core Compatibility, and for more divisions we can expect that the tendency continues. So we can state that in most of the cases it is not possible to allocate risk in illiquid markets satisfying the three fairness notions (Equal Treatment Property and Strong

Monotonicity and Core Compatibility) at the same time, one has to give up at least one of them. Balog, Bátyi, Csóka and Pintér (2014) suggest that either the Shapley value (being not stable) or the nucleolus (being not incentive compatible) can be a good candidate.

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