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Some special types of determinants in graded skew PBW extensions

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Dedicated to professor Oswaldo Lezama for his brilliant academic career at the Universidad Nacional de Colombia - Sede Bogotá

Abstract. In this paper, we prove that the Nakayama automorphism of a graded skew PBW extension over a finitely presented Koszul Auslander-regular algebra has trivial homological determinant. For $A = \sigma(R)\langle x_1, x_2\rangle$ a graded skew PBW extension over a connected algebra R, we compute its P-determinant and the inverse of σ. In the particular case of quasi-commutative skew PBW extensions over Koszul Artin-Schelter regular algebras, we show explicitly the connection between the Nakayama automorphism of the ring of coefficients and the extension. Finally, we give conditions to guarantee that A is Calabi-Yau. We provide illustrative examples of the theory concerning algebras of interest in noncommutative algebraic geometry and noncommutative differential geometry.

Keywords: Calabi-Yau algebra, skew PBW extension, double Ore extension, homological determinant, P-determinant, Nakayama automorphism.

MSC2010: 16S37, 16W50, 16W70, 16S36, 13N10.

Algunos tipos especiales de determinantes en extensiones PBW torcidas graduadas

Resumen. En este artículo, demostramos que el automorfismo de Nakayama de una extensión PBW torcida graduada sobre un álgebra de Koszul finitamente presentada y Auslander-regular tiene determinante homológico trivial. Para $A = \sigma(R)\langle x_1, x_2\rangle$ una extensión PBW torcida graduada sobre un álgebra conexa R, calculamos su P-determinante y el inverso de σ. En el caso particular de extensiones PBW torcidas cuasi-conmutativas sobre álgebras de

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Koszul Artin-Schelter regulares, mostramos explícitamente la relación entre el automorfismo de Nakayama del anillo de coeficientes y la extensión. Finalmente, damos condiciones para garantizar que A sea Calabi-Yau. Proporcionamos ejemplos ilustrativos de la teoría con álgebras de interés en geometría algebraica no conmutativa y geometría diferencial no conmutativa.

Palabras clave: Álgebra Calabi-Yau, extensión PBW torcida, extensión de Ore doble, determinante homológico, P-determinante, automorfismo de Nakayama.

1. Introduction

In [16] the *homological determinant* was defined and used to study *the Artin-Schelter regular* property of some algebras. Shen et al., [34] and Wu et al., [41] gave equivalent definitions of the homological determinant and established connections between homological determinant and the usual determinant. On the other hand, *Nakayama automorphism* plays an important role in noncommutative algebraic geometry (see Reyes et al., [27]) and its computation is not easy in the general case. Some authors have computed and studied this automorphism for special types of algebras, see for example [22, 23, 24, 27, 33, 34, 39, 45]. The remarkable fact is the relationship between both notions, see Reyes et al., [27], Shen et al., [34] and Zhun et al., [45], for more details. *Double Ore extensions*, denoted by $A = R_P[x_1, x_2; \sigma, \delta, \tau]$, were defined by Zhang and Zhang [43] where they proved that a connected graded double Ore extension of an Artin-Schelter regular algebra is Artin-Schelter regular. Later, in [44], they constructed 26 families of Artin-Schelter regular algebras of global dimension four by using double Ore extensions. For R a Koszul Artin-Schelter regular algebra, Zhun et al., [45] calculated the Nakayama automorphism of a trimmed double Ore extension $R_P[x_1, x_2; \sigma]$, and the Nakayama automorphism of an iterated skew polynomial extension of a Koszul Artin-Schelter regular algebra by using the homological determinant. Another properties of double Ore extensions have been explored by Carvalho et al., [6] and Zhun et al., [45].

Now, for $A = R_P[x_1, x_2; \sigma, \delta, \tau]$ a right double Ore extension, Zhang and Zhang [43] introduced the $\mathbb{K}$-linear map $\det_P\sigma : R \to R$ of σ called the *P-determinant.* This P-determinant was used to prove the Artin-Schelter regular property of double Ore extensions. Zhun et al., [45] used the P-determinant to calculate the Nakayama automorphism of certain algebras. For the case where R is a Koszul Artin-Schelter regular algebra and $\sigma : R \to \mathrm{M}_2(R)$ is an algebra homomorphism, Zhu et al., [45] defined the *homological determinant* of σ, denoted by hdet_M, and presented some connections between the Nakayama automorphism, the homological determinant, and the P-determinant. These authors used the homological determinant of σ to know when a trimmed double Ore extension of a Koszul Artin-Schelter regular algebra turns out to be Calabi-Yau.

The noncommutative rings of polynomial type of interest in this paper are the *skew PBW extensions*, denoted by $A = \sigma(R)\langle x_1, \dots, x_n\rangle$, which were defined by Gallego and Lezama [8] with the aim of generalizing another families of noncommutative rings appearing in several branches of mathematics (c.f. Bell and Goodearl [5] and Ore [25]). These objects have been recently studied (e.g., [1, 2, 11, 12, 13, 17, 18, 19, 21, 40, 42]), and, as a matter of fact, a book containing some of the works developed for these objects has been published recently by Springer [7]. As a particular class of these objects, Suárez [35] defined *graded skew PBW extensions* over an $\mathbb{N}$-graded algebra R and showed that if R is a finitely presented Koszul algebra, then every graded skew PBW extension over R is Koszul. The Artin-Schelter regular property and the skew Calabi-Yau condition for graded skew PBW extensions were studied by Suárez et al., [36]. There, the authors proved that every graded quasi-commutative skew PBW extension of an Artin-Schelter regular algebra is also Artin-Schelter regular; every graded quasi-commutative skew PBW extension over a connected skew Calabi-Yau algebra is skew Calabi-Yau; and every graded skew PBW extension over a connected Auslander regular algebras is Artin-Schelter regular and skew Calabi-Yau. With respect to the Nakayama automorphism, Suárez et al., [39] described it for $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ by using the Nakayama automorphism of an Artin-Schelter regular algebra R, and also they calculated explicitly the Nakayama automorphism of some skew PBW extensions. About doble Ore extensions, Gómez and Suárez [9] gave necessary and sufficient conditions for a graded (trimmed) double Ore extension to be a graded (quasi-commutative) skew PBW extension. They proved that graded skew PBW extensions $A = \sigma(R)\langle x_1, x_2\rangle$ over Artin-Schelter regular algebras R are also Artin-Schelter regular, and graded skew PBW extensions $A = \sigma(R)\langle x_1, x_2\rangle$ over connected skew Calabi-Yau algebras R of dimension d are skew Calabi-Yau of dimension $d + 2$.

With all above results in mind, next, we present the structure of the article by mentioning the original results presented here. In Section 2, we present definitions and basic properties of ring theory and skew PBW extensions. In Section 3, we present some new results about these extensions. Briefly, Theorem 3.1 asserts that the Nakayama automorphism of a graded skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ over a finitely presented Koszul Auslander-regular algebra R has trivial homological determinant. A particular case of this fact is presented in Corollary 3.2, where we show that the Nakayama automorphism of graded skew PBW extensions over the commutative polynomial ring $R = \mathbb{K}[t_1, \ldots, t_m]$ is equal to 1. Now, since a graded skew PBW extension over a connected algebra R is a connected graded double Ore extension $R_P[x_1, x_2; \sigma, \delta, \tau]$, in Theorem 3.4 we calculate the P-determinant of σ and the inverse of σ. In Theorem 3.5, for a quasi-commutative Calabi-Yau skew PBW extension $A = \sigma(R)\langle x_1, x_2\rangle = R_P[x_1, x_2; \sigma]$ over a Koszul Artin-Schelter regular algebra, we calculate the Nakayama automorphism of R and the homological determinant of σ in terms of the definition of skew PBW extension. In this sense, we also show the correlation between the Nakayama automorphism, the homological determinant and the homological determinant of σ. Finally in Corollary 3.6, we calculate the Nakayama automorphism of a graded quasi-commutative skew PBW extension $A = \sigma(R)\langle x_1, x_2\rangle$ over a Koszul Artin-Schelter regular algebra R with Nakayama automorphism, and we establish a sufficient and necessary condition to guarantee that A is Calabi-Yau. In this way, this paper continues the research about Koszul, Calabi-Yau, Artin-Schelter and related ring theoretic notions for skew PBW extensions (c.f. [29, 36, 37, 38, 39]).

2. Preliminaries

Throughout the paper, the word ring means an associative ring with identity not necessarily commutative. If B is a ring, $\mathrm{M}_n(B)$ denotes the set of matrices of size $n \times n$ with entries in B. $\mathbb{K}$ denotes a field; all algebras are $\mathbb{K}$-algebras. The symbol $\mathbb{N}$ will be used to denote the set of natural numbers including zero, and the tensor product $\otimes$ means $\otimes_{\mathbb{K}}$. An algebra B is $\mathbb{Z}$-*graded*, if there exists a family of subspaces $\{B_p\}_{p\in\mathbb{Z}}$ of B such that $B = \bigoplus_{p\in\mathbb{Z}} B_p$ and $B_pB_q \subseteq B_{p+q}$, for all $p, q \in \mathbb{Z}$. A graded algebra B is called *positively graded* (or $\mathbb{N}$-*graded*), if $B_p = 0$, for all $p < 0$. An $\mathbb{N}$-graded algebra is called *connected*, if $B_0 = \mathbb{K}$. A non-zero element $x \in B_p$ is called a *homogeneous element* of B of degree p. If $B = \bigoplus_{p\in\mathbb{Z}} B_p$ is a graded algebra,we set $B(l) = \bigoplus_{p\in\mathbb{Z}} B(l)_p$, where $B(l)_p = B_{p+l}$, for $l \in \mathbb{Z}$. An algebra B is *finitely generated* as $\mathbb{K}$-algebra, if there exists a finite set of elements $t_1, \dots, t_m \in B$ such that the set $\{t_{i_1}t_{i_2}\cdots t_{i_p} \mid 1 \leq i_j \leq m, p \geq 1\} \cup \{1\}$ spans B as a $\mathbb{K}$-space. For example the free associative algebra (tensor algebra) L in m generators $t_1, \dots, t_m$ is denoted by $L := \mathbb{K}\langle t_1, \dots, t_m\rangle$. The *degree* (deg) of a word $t_{i_1}t_{i_2}\dots t_{i_p}$ is p; the degree of $f \in L$ is the maximum of the degrees of the words in f. We include among the words a symbol 1, which we think of as the empty word and it has degree 0. Note that L is positively graded with graduation given by $L := \bigoplus_{j\geq 0} L_p$, where $L_0 = \mathbb{K}$ and L_p is spanned by all words of degree p in the alphabet $\{t_1, \dots, t_m\}$, for $p > 0$. A connected graded algebra $B = L/\langle R\rangle$ is called a *quadratic algebra*, if R is a subspace of L_2. An algebra is *finitely presented*, if it is a quotient $\mathbb{K}\langle t_1, \dots, t_m\rangle/I$, where $I = \langle r_1, \dots, r_s\rangle$ is a finitely generated two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m\rangle$. We call $\mathbb{K}\langle t_1, \dots, t_m\rangle/\langle r_1, \dots, r_s\rangle$ a *presentation* of B with generators $t_1, \dots, t_m$ and relations $r_1, \dots, r_s$. We always assume that $r_1, \dots, r_s$ is a minimal set of relations for B and every generator t_i has degree 1. For $B = \mathbb{K}\langle t_1, \dots, t_m\rangle/\langle r_1, \dots, r_s\rangle$ a finitely presented algebra, if the relations $r_1, \dots, r_s$ are all homogeneous, then B is a *connected graded algebra*.

Definition 2.1. A quadratic algebra B is called *Koszul*, if the trivial left B-module ${}_B\mathbb{K}$ admits a projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to {}_B\mathbb{K} \to 0,$$

such that P_n is generated in degree n, for all $n \geq 0$.

For more details about Koszul algebras, we refer the reader to Polischuk [26]. Backelin and Fröberg [4] showed several equivalent definitions of Koszul algebras.

Now, we recall the definition of Artin-Schelter regular algebra introduced by Artin and Schelter [3].

Definition 2.2. A connected graded algebra B is said to be *Artin-Schelter regular* of dimension d, if B has finite global dimension d, finite Gelfand-Kirillov dimension, and $\mathrm{Ext}^i_B(\mathbb{K}, B) = 0$, if $i \neq d$, and $\mathrm{Ext}^d_B(\mathbb{K}, B) \cong \mathbb{K}(l)$, for some integer l.

Let M be a B-bimodule, and let ν, τ be algebra automorphisms of B. Then ${}^{\tau}M^{\nu}$ denotes the induced B-bimodule such that ${}^{\tau}M^{\nu} = M$ as a $\mathbb{K}$-space, and where $a \star m \star b = \tau(a)m\nu(b)$, for all $a, b \in B$ and all $m \in {}^{\tau}M^{\nu} = M$. If τ or ν is the identity map of B, then we write M^{ν} or ${}^{\tau}M$, respectively. The *enveloping algebra* of an algebra B is the tensor product $B^e = B \otimes B^{op}$, where B^{op} is the opposite algebra of B. Bimodules over

B are essentially the same as modules over the enveloping algebra of B. An algebra B is said to be *homologically smooth*, if as an B^e-module, B has a projective resolution with finite length and such that every term in the projective resolution is finitely generated.

Definition 2.3. A graded algebra B is called *skew Calabi-Yau* of dimension d, if

(i) B is homologically smooth, and

(ii) there exists an algebra automorphism ν of B such that

$$\mathrm{Ext}^i_{B^e}(B, B^e) \cong \begin{cases} 0, & i \neq d; \\ B^{\nu}(l), & i = d, \end{cases}$$

as B^e-modules, for some integer l.

The automorphism ν is called the *Nakayama automorphism* of B. If, in addition, B^e is isomorphic to B as B^e-modules, or equivalently, ν is inner, then B is called *Calabi-Yau* of dimension d.

Next, we recall the definition of our objects of interest introduced by Gallego and Lezama [8].

Definition 2.4. Let R and A be rings. We say that A is a *skew PBW extension over R*, if the following conditions hold:

(i) R (the subring of coefficients) is a subring of A sharing the same identity element.

(ii) There exist finitely many elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis the set of standard monomials $\mathrm{Mon}(A) := \{x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$. Moreover, $x_1^0 \cdots x_n^0 := 1 \in \mathrm{Mon}(A)$.

(iii) For every $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For $1 \leq i, j \leq n$, there exists $d_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n. \tag{1}$$

If A is a skew PBW extension over R, then we denote it by $A := \sigma(R)\langle x_1, \dots, x_n \rangle$. For $X = x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathrm{Mon}(A)$, $\deg(X) = \alpha_1 + \cdots + \alpha_n$.

The notation and the name of the skew PBW extensions are due to the following proposition.

Proposition 2.5 ([8], Proposition 3)**.** *Let $A = \sigma(R)\langle x_1, \dots, x_n \rangle$ be a skew PBW extension of R. For each $1 \leq i \leq n$, there exists an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, where $r \in R$.*

From now on, σ_i and δ_i are the injective endomorphisms and σ_i-derivations, respectively, as in Proposition 2.5. A skew PBW extension A is called *bijective*, if σ_i is bijective and $d_{i,j}$ is invertible, for any $1 \leq i < j \leq n$. A is called *quasi-commutative*, if the conditions (iii) and (iv) in Definition 2.4 are replaced by the following:

(iii') for each $1 \leq i \leq n$ and every $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$;

(iv') for any $1 \leq i, j \leq n$, there exists $d_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = d_{i,j} x_i x_j$.

Different examples of skew PBW extensions can be found in Lezama et al., [20] and Suárez [35]. Ring and theoretic properties of these objects have been studied by the authors in [28, 30, 31].

The next result is key to define *graded skew PBW extensions* (see Definition 2.4).

Proposition 2.6 ([35], Proposition 2.7(ii)). *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension over an $\mathbb{N}$-graded algebra $R = \bigoplus_{m\geq 0} R_m$. If the following conditions hold,*

(i) *σ_i is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded σ_i-derivation, for all $1 \leq i \leq n$, and*

(ii) *$x_j x_i - d_{i,j} x_i x_j \in R_2 + R_1 x_1 + \cdots + R_1 x_n$, as in (1) and $d_{i,j} \in R_0$,*

then A is an $\mathbb{N}$-graded algebra with graduation given by $A = \bigoplus_{p\geq 0} A_p$, where, for $p \geq 0$, A_p is the $\mathbb{K}$-space generated by the set

$$\Big\{ r_t x^\alpha \mid t + \deg(x^\alpha) = p,\ r_t \in R_t \text{ and } x^\alpha \in \mathrm{Mon}(A) \Big\}.$$

Definition 2.7 ([35], Definition 2.6). Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension over an $\mathbb{N}$-graded algebra $R = \bigoplus_{m\geq 0} R_m$. If A satisfies the conditions (i) and (ii) established in Proposition 2.6, then we say that A is a *graded skew PBW extension over R*.

Remark 2.8. If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a quasi-commutative skew PBW extension over a ring R, then A is isomorphic to an iterated Ore extension of endomorphism type ([20], Theorem 2.3). Nevertheless, skew PBW extensions of endomorphism type are more general than iterated Ore extensions. With the aim of illustrating the differences between these structures, we consider the situations with two and three indeterminates.

If we take the iterated Ore extension of endomorphism type $R[x; \sigma_x][y; \sigma_y]$, by definition (see Ore [25] or Goodearl and Warfield [10]), for any element $r \in R$, we have the following relations: $xr = \sigma_x(r)x$, $yr = \sigma_y(r)y$, and $yx = \sigma_y(x)y$. On the other hand, if we have $\sigma(R)\langle x, y\rangle$ a skew PBW extension of endomorphism type over R, then for any $r \in R$, by Definition 2.4, we have the relations $xr = \sigma_1(r)x$, $yr = \sigma_2(r)y$, and $yx = d_{1,2}xy + r_0 + r_1x + r_2y$, for some elements $d_{1,2}, r_0, r_1$ and r_2 belong to R. When we compare the defining relations of both algebraic structures, it is clear which one of them is more general.

Now, if we have the iterated Ore extension $R[x; \sigma_x][y; \sigma_y][z; \sigma_z]$, then for any $r \in R$, $xr = \sigma_x(r)x$, $yr = \sigma_y(r)y$, $zr = \sigma_z(r)z$, $yx = \sigma_y(x)y$, $zx = \sigma_z(x)z$, $zy = \sigma_z(y)z$. On the other hand, for the skew PBW extension of automorphism type $\sigma(R)\langle x, y, z\rangle$, we have the relations given by $xr = \sigma_1(r)x$, $yr = \sigma_2(r)y$, $zr = \sigma_3(r)z$, $yx = d_{1,2}xy + r_0 + r_1x + r_2y + r_3z$, $zx = d_{1,3}xz + r_0' + r_1'x + r_2'y + r_3'z$, and $zy = d_{2,3}yz + r_0'' + r_1''x + r_2''y + r_3''z$, for some elements $d_{1,2}, d_{1,3}, d_{2,3}, r_0, r_0', r_0'', r_1, r_1', r_1'', r_2, r_2', r_2'', r_3, r_3', r_3''$ of R. Of course, as the number of indeterminates increases, the differences between both algebraic structures are more remarkable.

As we can expect from these facts, graded iterated Ore extensions are strictly contained in graded skew PBW extensions, see Suárez [35], Remark 2.11. Examples of graded skew PBW extensions can be found in Suárez et al., [9, 35, 36].

The following proposition contains several results about skew PBW extensions and it will be useful in the next section.

Proposition 2.9. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a graded skew PBW extension.*

(i) *If R is graded Noetherian, then A is graded Noetherian ([36], Proposition 2.7).*

(ii) *If A is quasi-commutative, then R is Koszul if and only if A is Koszul ([35], Proposition 3.3).*

(iii) *If R is Koszul finitely presented, then A is Koszul ([35], Theorem 5.5).*

(iv) *If R is Artin-Schelter regular and A is quasi-commutative, then A is Artin-Schelter regular ([36], Theorem 3.6).*

(v) *If R is a finitely presented connected Auslander-regular algebra, then A is Artin-Schelter regular ([36], Proposition 3.5).*

(vi) *If A is quasi-commutative, R is connected and skew Calabi-Yau of dimension d, then A is skew Calabi-Yau of dimension $d + n$ ([36], Theorem 4.5 (i)).*

(vii) *If R is finitely presented connected Auslander-regular, then A is skew Calabi-Yau ([36], Theorem 4.5 (ii)).*

Zhang and Zhang in [43] introduced a special class of algebras called *double Ore extensions.*

Definition 2.10 ([43], Definition 1.3)**.** Let R be an algebra and A be another algebra containing R as a subring.

(1) A is a *right double Ore extension* of R, if the following conditions hold:

  (i) A is generated by R and two new variables x_1 and x_2.

  (ii) The variables $\{x_1, x_2\}$ satisfy the relation

$$x_2x_1 = p_{12}x_1x_2 + p_{11}x_1^2 + \tau_1x_1 + \tau_2x_2 + \tau_0, \tag{2}$$

  where $p_{12}, p_{11} \in \mathbb{K}$ and $\tau_1, \tau_2, \tau_0 \in R$.

  (iii) As a left R-module, $A = \sum_{\alpha_1,\alpha_2\geq 0} Rx_1^{\alpha_1}x_2^{\alpha_2}$ and it is a left free R-module with basis $\{x_1^{\alpha_1}x_2^{\alpha_2} \mid \alpha_1 \geq 0, \alpha_2 \geq 0\}$.

  (iv)

$$x_1R + x_2R \subseteq Rx_1 + Rx_2 + R. \tag{3}$$

(2) A is a *left double Ore extension* of R, if the following conditions hold:

(ib) A is generated by R and two new variables x_1 and x_2.

(iib) The variables $\{x_1, x_2\}$ satisfy the relation

$$x_1x_2 = p'_{12}x_2x_1 + p'_{11}x_1^2 + x_1\tau'_1 + x_2\tau'_2 + \tau'_0, \tag{4}$$

where $p'_{12}, p'_{11} \in \mathbb{K}$ and $\tau'_1, \tau'_2, \tau'_0 \in R$.

(iiib) As a right R-module, $A = \sum_{\alpha_1,\alpha_2\geq 0} Rx_1^{\alpha_1}x_2^{\alpha_2}$ and it is a right free R-module with basis $\{x_1^{\alpha_1}x_2^{\alpha_2} \mid \alpha_1 \geq 0, \alpha_2 \geq 0\}$.

(ivb) $x_1R + x_2R \subseteq Rx_1 + Rx_2 + R$.

(iii) A is a *double Ore extension*, if it is left and right double Ore extension of R with the same generating set $\{x_1, x_2\}$.

(iv) A is a *graded right (left) double Ore extension*, if all relations of A are homogeneous with assignment $\deg(x_1) = \deg(x_2) = 1$.

Let P denote the set of scalar parameters $\{p_{12}, p_{11}\}$ and let τ denote the set $\{\tau_1, \tau_2, \tau_0\}$. We call P the *parameter* and τ the *tail* (see [43], p. 2671). By considering the datum P, Zhu et al., [45] defined a matrix $\mathbb{P} \in \mathrm{M}_2(\mathbb{K})$ in the following way:

$$\mathbb{P} := \begin{pmatrix} p_{12} & 0 \\ -(1+p_{12}^{-1})p_{11} & p_{12}^{-1} \end{pmatrix}.$$

By using the ideas given in [43], p. 2671, we rewrite the condition (3) above as follows:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} r := \begin{pmatrix} x_1r \\ x_2r \end{pmatrix} = \begin{pmatrix} \sigma_{11}(r) & \sigma_{12}(r) \\ \sigma_{21}(r) & \sigma_{22}(r) \end{pmatrix} + \begin{pmatrix} \delta_1(r) \\ \delta_2(r) \end{pmatrix}, \text{ for every } r \in R. \tag{5}$$

Writing

$$\sigma(r) = \begin{pmatrix} \sigma_{11}(r) & \sigma_{12}(r) \\ \sigma_{21}(r) & \sigma_{22}(r) \end{pmatrix} \quad \text{and} \quad \delta(r) = \begin{pmatrix} \delta_1(r) \\ \delta_2(r) \end{pmatrix},$$

it is clear that σ is a $\mathbb{K}$-linear map from R to $\mathrm{M}_2(R)$, δ is a $\mathbb{K}$-linear map from R to the column R-module $R^{\oplus 2} := \left(\begin{smallmatrix} R \\ R \end{smallmatrix}\right)$, and σ and δ are uniquely determined. In this way, equation (5) can also be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} r = \sigma(r) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \delta(r),$$

which is a generalization of the multiplication in an Ore extension.

Let A be a right double Ore extension of R. Having in mind Definition 2.10, all symbols of $\{P, \sigma, \delta, \tau\}$ are now defined. When no confusion arises, a right double Ore extension or a double Ore extension A is also denoted by $A = R_P[x_1, x_2; \sigma, \delta, \tau]$. Let $A = R_P[x_1, x_2; \sigma, \delta, \tau]$ be a right double Ore extension. If δ is a zero map and τ consists of zero elements, then the right double Ore extension is denoted by $R_P[x_1, x_2; \sigma]$ and is called a *trimmed right double Ore extension* (see [43, Convention 1.6]).

Lemma 2.11 ([43], Lemma 1.7). *If $A = R_P[x_1, x_2; \sigma, \delta, \tau]$ is a right double Ore extension of R with $\{\sigma, \delta\}$ defined as in (5), then the following holds:*

(i) *$\sigma : R \to \mathrm{M}_2(R)$ is an algebra homomorphism.*

(ii) *$\delta : R \to R^{\oplus 2}$ is a σ-derivation.*

(iii) *If $\sigma : R \to \mathrm{M}_2(R)$ is an algebra homomorphism and $\delta : R \to R^{\oplus 2}$ is a σ-derivation, then (5) holds for all elements $r \in R$ if and only if it holds for a set of generators.*

By Lemma 2.11, when $A = R_P[x_1, x_2; \sigma, \delta, \tau]$ is a right double Ore extension, then σ is an algebra homomorphism and δ a σ-derivation. We now call σ a homomorphism, δ a derivation. Note that homomorphism σ is not surjective, so it cannot be invertible in the usual sense. Zhang and Zhang [43] defined invertibility of σ in right double Ore extensions.

Definition 2.12 ([43], Definition 1.8). Let $\sigma : R \to \mathrm{M}_2(R)$ be an algebra homomorphism. We say that σ is *invertible*, if there exists an algebra homomorphism

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} : \ R \to \mathrm{M}_2(R), \tag{6}$$

satisfying the following conditions:

$$\sum_{k=1}^{2} \phi_{jk}(\sigma_{ik}(r)) = \begin{cases} r, & i = j, \\ 0, & i \neq j, \end{cases} \qquad \text{and} \qquad \sum_{k=1}^{2} \sigma_{kj}(\phi_{ki}(r)) = \begin{cases} r, & i = j, \\ 0, & i \neq j, \end{cases}$$

for all $r \in R$. Equivalently,

$$\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \bullet \begin{pmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \bullet \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} = \begin{pmatrix} Id_R & 0 \\ 0 & Id_R \end{pmatrix},$$

where $\bullet$ is the multiplication of the matrix algebra $\mathrm{M}_2(\mathrm{End}_{\mathbb{K}}(R))$. The multiplication of $\mathrm{End}_{\mathbb{K}}(A)$ is the composition of $\mathbb{K}$-linear maps. The map ϕ is called the inverse of σ.

Let $A = R_P[x_1, x_2; \sigma, \delta, \tau]$ be a right double Ore extension. Zhang and Zhang [43] introduced the $\mathbb{K}$-linear map $\det_P\sigma : R \to R$ of $\sigma := \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$, which is called the *$P$-determinant*.

Definition 2.13. If $A = R_P[x_1, x_2; \sigma, \delta, \tau]$ is a right double Ore extension, then the *P-determinant* of σ is defined to be

$$\det{}_P\sigma : r \mapsto -p_{11}\sigma_{12}(\sigma_{11}(r)) + \sigma_{22}(\sigma_{11}(r)) - p_{12}\sigma_{12}(\sigma_{21}(r)), \quad \text{for } r \in R. \tag{7}$$

Remark 2.14. If $P = \{1, 0\}$, then $\det_P\sigma = \sigma_{22}\sigma_{11} - \sigma_{12}\sigma_{21}$. If $p_{12} \neq 0$, then $\det_P\sigma = -p_{12}^{-1}p_{11}\sigma_{11}\sigma_{22} - p_{12}^{-1}\sigma_{21}\sigma_{12} + \sigma_{11}\sigma_{22}$ (see [43], p. 2677). Moreover, $\det_P\sigma$ is an algebra endomorphism of R (see Zhang [43], Proposition 2.1).

Let V be a finite-dimensional vector space and $T(V)$ be the tensor algebra with the usual grading. A connected graded algebra $B = T(V)/\langle R\rangle$ is *quadratic*, if R is a subspace of $V^{\otimes 2}$. The homogeneous dual of B is then defined as $B^! = T(V^*)/\langle R^\perp\rangle$, where V^* is the dual space of V and $R^\perp = \{\lambda \in V^* \otimes V^* \mid \lambda(r) = 0, \text{ for all } r \in R\}$. If $B = T(V)/\langle R\rangle$ is a Koszul algebra, then its *Yoneda algebra* is defined as $E(B) := \bigoplus_{i\in\mathbb{N}} \operatorname{Ext}^i_B(\mathbb{K}, \mathbb{K})$ which is isomorphic to $B^!$. For a graded automorphism ν of B, we define a map $\nu^* : V^* \to V^*$ by $\nu^*(f)(x) = f(\nu(x))$, for each $f \in V^*$ and $x \in V$. Note that ν induces a graded automorphism of $B^!$ because ν is assumed to preserve the relation space R. We still use the notation ν^* for this algebra automorphism (see Zhun [45], p. 559).

Proposition 2.15 ([41], Proposition 1.11). *Let B be a Koszul Artin-Schelter regular algebra with global dimension d. If ν is a graded automorphism of B and ν^* is its corresponding dual graded automorphism of the dual algebra $B^!$, then there exists a unique c in the multiplicative group $\mathbb{K} \setminus \{0\}$ such that $\nu^*(u) = cu$, for all $u \in \operatorname{Ext}^d_A(\mathbb{K}, \mathbb{K})$.*

In [16] the homological determinant of a graded automorphism of an Artin-Schelter Gorenstein algebra was defined to study the noncommutative invariant theory. Since every Artin-Schelter Gorenstein algebra is Artin-Schelter regular, then for an Artin-Schelter Gorenstein algebra B, its homological determinant, denoted hdet, it is a homomorphism from the graded automorphism group of B to the multiplicative group $\mathbb{K} \setminus \{0\}$. The precise definition and its application can be found in [16, 27]. We use Proposition 2.15 to define the homological determinant.

Definition 2.16 ([41], Definition 1.5). Let B be a Koszul Artin-Schelter regular algebra and let ν be a graded automorphism of B. The *homological determinant* $\operatorname{hdet}(\nu)$ of σ is defined as $\operatorname{hdet}(\nu) := c$, where c is as in Proposition 2.15.

Zhu et al., [45] defined the homological determinant of σ, denoted hdet_M, in the following way:

Definition 2.17 ([45], Definition 2.5). Let B be a Koszul Artin-Schelter regular algebra. Let $\sigma : B \to \mathrm{M}_2(B)$ be an algebra homomorphism and let $\sigma^* : B^! \to \mathrm{M}_2(B^!)$ be the dual homomorphism of algebras. The *homological determinant* of σ, denoted by $\operatorname{hdet}_M(\sigma)$, is defined to be

$$\operatorname{hdet}_M(\sigma) := \begin{pmatrix} W & X \\ Y & Z \end{pmatrix},$$

for $W, X, Y, Z \in \mathbb{K}$ satisfying $\sigma^*(x_0) = \begin{pmatrix} Wx_o & Xx_o \\ Yx_o & Zx_o \end{pmatrix}$, where x_0 is a base element of the highest nonzero component $B^!_d$, which is 1-dimensional $\mathbb{K}$-space, of $B^!$.

Some properties, applications and examples of the homological determinant of σ can be found in Zhun et al., [45].

3. Main results

In this section, we present the new results for graded skew PBW extensions concerning ring-theoretic properties.

Theorem 3.1. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a graded skew PBW extension over a finitely presented Koszul Auslander-regular algebra R, then the homological determinant of the Nakayama automorphism of A is 1.*

Proof. Since R is finitely presented Koszul, Proposition 2.9(iii) asserts that A is Koszul. Moreover, since R is Koszul, it is known that R is quadratic, and therefore connected. By assumption that R is Auslander-regular, and by Proposition 2.9(v), we have that A is Artin-Schelter regular. Note that by Proposition 2.9(vii), A is skew Calabi-Yau and there is the Nakayama automorphism ν of A. Now, since R is Auslander-regular, necessarily it is Noetherian. Thus, by Proposition 2.9(i), A is Noetherian. Finally, by Reyes et al., [27], Theorem 6.3, $\text{hdet}(\nu) = 1$. ☑

Corollary 3.2. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a graded skew PBW extension over the commutative polynomial ring $R = \mathbb{K}[t_1, \dots, t_m]$, then the homological determinant of the Nakayama automorphism of A is 1.*

Example 3.3. *Some algebras which are graded skew PBW extensions over commutative polynomial rings are the following:*

1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\Bbbk$ with basis $x_1, \dots, x_n$ and $U(\mathfrak{g})$ its enveloping algebra. The homogenized enveloping algebra of $\mathfrak{g}$ is $\mathcal{A}(\mathfrak{g}) := T(\mathfrak{g} \oplus \Bbbk z)/\langle R\rangle$, where $T(\mathfrak{g} \oplus \Bbbk z)$ is the tensor algebra, z is a new indeterminate, and R is spanned by the union of sets $\{z \otimes x - x \otimes z \mid x \in \mathfrak{g}\}$ and $\{x \otimes y - y \otimes x - [x, y] \otimes z \mid x, y \in \mathfrak{g}\}$. The algebra $A(\mathfrak{g})$ is a skew PBW extension over $\Bbbk[z]$ (c.f. Suárez et al., [37], Section 2.2).*

2. *The Jordan Algebra introduced by Jordan [15] is the free $\Bbbk$-algebra $\mathcal{J}$ defined by $\mathcal{J} := \Bbbk\{x, y\}/\langle yx - xy - y^2\rangle$. It is immediate that this algebra is not a skew polynomial ring of automorphism type but an easy computation shows that $\mathcal{J} \cong \sigma(\Bbbk[y])\langle x\rangle$.*

3. *Diffusion algebras arose in physics as a possible way to understand a large class of 1-dimensional stochastic process, see [14]. A diffusion algebra $\mathcal{A}$ with non-zero parameters a_{ij} in the complex numbers $\mathbb{C}$, is an algebra over $\mathbb{C}$ generated by variables $x_1, \dots, x_n$ subject to relations $a_{ij}x_ix_j - b_{ij}x_jx_i = r_jx_i - r_ix_j$, whenever $i < j$, $b_{ij}, r_i \in \mathbb{C}$, for all $i < j$, such that the indeterminates x's form a $\mathbb{C}$-basis of the algebra $\mathcal{A}$. In the applications to physics the parameters a_{ij} are strictly positive reals and the parameters b_{ij} are positive reals as they are unnormalised measures of probability. One can see that these algebras are not skew polynomial rings over $\mathbb{C}[x_1, \dots, x_n]$ but are skew PBW extensions over this ring (see [32], Section 5.3).*

Theorem 3.4. *Let $A = \sigma(R)\langle x_1, x_2\rangle$ be a graded skew PBW extension over a connected algebra R.*

(i) *A is a connected graded double Ore extension $R_P[x_1, x_2; \sigma, \delta, \tau]$, where*

$$\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix},$$

with σ_1, σ_2, δ_1 and δ_2 as in Proposition 2.5 and $d_{1,2}$ as in (1).

(ii) $\det_P\sigma = \sigma_2\sigma_1,\ P = \{c_{1,2}, 0\}$ *and*

$$\mathbb{P} = \begin{pmatrix} c_{1,2} & 0 \\ 0 & c_{1,2}^{-1} \end{pmatrix}.$$

(iii) σ *is invertible with inverse*

$$\phi = \begin{pmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_2^{-1} \end{pmatrix}$$

such that for all $r \in R$, *the equation*

$$r(x_1\ x_2) = (x_1\ x_2)\phi(r) + \delta'(r)$$

holds for

$$\delta' = \begin{pmatrix} -\delta_1(\sigma_1^{-1}) \\ -\delta_2(\sigma_2^{-1}) \end{pmatrix}.$$

(iv)

$$\frac{A}{\bigoplus_{n\geq 1} R_n} \cong \frac{\mathbb{K}\langle x_1, x_2\rangle}{\langle x_2x_1 - c_{1,2}x_1x_2\rangle}$$

Proof. Suppose that $A = \sigma(R)\langle x_1, x_2\rangle$ is a graded skew PBW extension of a connected algebra R. It is clear that A is a connected algebra. (i) Definition 2.4 (iii) establishes that for all $r \in R \setminus \{0\}$, there are elements $c_{1,r}$, $c_{2,r}$, $c_{1,0}$, $c_{2,0} \in R$, with $c_{1,r}$, $c_{2,r} \neq 0$, such that

$$x_1 r = c_{1,r}x_1 + c_{1,0}, \qquad x_2 r = c_{2,r}x_2 + c_{2,0}. \tag{8}$$

In this way, for all $r \in R$, $\sigma_1(r) = c_{1,r}$, $\sigma_2(r) = c_{2,r}$, $\delta_1(r) = c_{1,0}$, $\delta_2(r) = c_{2,0}$, where σ_1, σ_2, δ_1 and δ_2 are as in Proposition 2.5. From [9], Theorem 3.2, we can assert that A is a connected graded double Ore extension $R_P[x_1, x_2; \sigma, \delta, \tau]$. (ii) Comparing relations (5) and (8), we have that

$$\begin{pmatrix} x_1 r \\ x_2 r \end{pmatrix} = \begin{pmatrix} \sigma_1(r) & 0 \\ 0 & \sigma_2(r) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \delta_1(r) \\ \delta_2(r) \end{pmatrix}, \text{ for all } r \in R.$$

Therefore,

$$\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}.$$

Hence, $\det_P\sigma = \sigma_2\sigma_1$. Now, from Definition 2.4 (iv), we know that there exists an element $c_{1,2} \in R \setminus \{0\}$ such that

$$x_2x_1 = c_{1,2}x_1x_2 + r_0 + r_1x_1 + r_2x_2, \tag{9}$$

with r_0, r_1, $r_2 \in R$. By assumption, A is graded, so $\deg(r_0) = 2$, $\deg(r_1) = \deg(r_2) = 1$ and $\deg(c_{1,2}) = 0$. Since R is connected, $R_0 = \mathbb{K}$, whence $c_{1,2} \in \mathbb{K} \setminus \{0\}$. Comparing (2) and (9), we have that $p_{12} = c_{1,2},\ p_{11} = 0,\ \tau_1 = r_1,\ \tau_2 = r_2,\ \tau_0 = r_0$. Thus $P = \{c_{1,2}, 0\}$ and

$$\mathbb{P} = \begin{pmatrix} c_{1,2} & 0 \\ 0 & c_{1,2}^{-1} \end{pmatrix}.$$

(iii) By (i) and Zhun et al., [45], Lemma 1.8(1), we have that σ is invertible with its inverse given by

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix},$$

such that the equation $r(x_1x_2) = (x_1x_2)\phi(r) + \delta'(r)$ holds for some

$$\delta' = \begin{pmatrix} \delta'_1 \\ \delta'_2 \end{pmatrix} : R \to \mathrm{M}_{1\times 2}(R).$$

Then $rx_1 = x_1\phi_{11}(r) + x_2\phi_{21}(r) + \delta'_1(r)$ and $rx_2 = x_1\phi_{12}(r) + x_2\phi_{22}(r) + \delta'_2(r)$. Now, since $A = \sigma(R)\langle x_1, x_2\rangle$ is a skew PBW extension, then

$$\begin{aligned} rx_1 &= x_1\phi_{11}(r) + x_2\phi_{21}(r) + \delta'_1(r) \\ &= \sigma_1(\phi_{11}(r))x_1 + \delta_1(\phi_{11}(r)) + \sigma_2(\phi_{21}(r))x_2 + \delta_2(\phi_{21}(r)) + \delta'_1(r), \end{aligned} \tag{10}$$

and

$$\begin{aligned} rx_2 &= x_1\phi_{12}(r) + x_2\phi_{22}(r) + \delta'_2(r) \\ &= \sigma_1(\phi_{12}(r))x_1 + \delta_1(\phi_{12}(r)) + \sigma_2(\phi_{22}(r))x_2 + \delta_2(\phi_{22}(r)) + \delta'_2(r). \end{aligned} \tag{11}$$

By Definition 2.4(ii), in Equation 10 we have that

$$r = \sigma_1(\phi_{11}(r)) \tag{12}$$

and

$$\sigma_2(\phi_{21}) = \delta_1(\phi_{11}) + \delta_2(\phi_{21}) + \delta'_1 = 0. \tag{13}$$

Since A is a graded skew PBW extension, then A is bijective and therefore σ_1 and σ_2 are bijective. So, by Equation (12), $\sigma_1^{-1} = \phi_{11}$ and by Equation (13) we have that $\phi_{21} = 0$ and $-\delta_1(\phi_{11}) = -\delta_1(\sigma_1^{-1}) = \delta'_1$. Analogous to the above reasoning we have $\phi_{12} = 0$ and $-\delta_2(\sigma_2^{-1}) = \delta'_2$. (iv) It follows from (i) and [43], Proposition 1.14(a). ☑

Theorem 3.5. *Let R be a Koszul Artin-Schelter regular algebra with Nakayama automorphism ν and let $A = \sigma(R)\langle x_1, x_2\rangle$ be a graded quasi-commutative Calabi-Yau skew PBW extension. Then*

$$\nu = \sigma_2\sigma_1, \ \mathrm{hdet}_M(\sigma) = \begin{pmatrix} c_{1,2} & 0 \\ 0 & c_{1,2}^{-1} \end{pmatrix}, \quad \mathrm{hdet}(\sigma_1) = c_{1,2} \quad \textit{and} \quad \mathrm{hdet}(\sigma_2) = c_{1,2}^{-1},$$

where σ_1, σ_2 are as in Proposition 2.5 and $d_{1,2}$ is as in (1).

Proof. Since A is a connected graded quasi-commutative skew PBW extension of R then by [9], Corollary 3.3, we have that A is a connected graded trimmed right double Ore extension of R, of the form $A = R_P[x_1, x_2; \sigma]$. By Theorem 3.4, $\det_P\sigma = \sigma_2\sigma_1$ and

$$\mathbb{P} = \begin{pmatrix} c_{1,2} & 0 \\ 0 & c_{1,2}^{-1} \end{pmatrix}.$$

By the assumption, R is Artin-Schelter regular and A is quasi-commutative, so Proposition 2.9(iv) implies that A is Artin-Schelter regular. Since R is Koszul then by Proposition

2.9(iii) we have that A is Koszul. Since A is Calabi-Yau, then by Zhun [45], Theorem 3.12,

$$\det{}_P\sigma = \nu \quad \text{and} \quad \operatorname{hdet}_M(\sigma) = \mathbb{P},$$

i.e.,

$$\nu = \sigma_2\sigma_1 \text{ and } \operatorname{hdet}_M(\sigma) = \begin{pmatrix} c_{1,2} & 0 \\ 0 & c_{1,2}^{-1} \end{pmatrix}.$$

According to the calculations presented in [45], Example 2.7,

$$\operatorname{hdet}_M(\sigma) = \begin{pmatrix} \operatorname{hdet}(\sigma_1) & 0 \\ 0 & \operatorname{hdet}(\sigma_2) \end{pmatrix},$$

which implies that $\operatorname{hdet}(\sigma_1) = c_{1,2}$, and $\operatorname{hdet}(\sigma_2) = c_{1,2}^{-1}$. ☑

Corollary 3.6. *If $A = \sigma(R)\langle x_1, x_2\rangle$ is a graded quasi-commutative skew PBW extension over a Koszul Artin-Schelter regular algebra R with Nakayama automorphism ν, then*

(i) *The Nakayama automorphism μ of A is given by*

$$\mu(r) = \sigma_1^{-1}\sigma_2^{-1}\nu(r), \ \textit{for } r \in R, \ \textit{and } \mu\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \operatorname{hdet}_M(\sigma)\begin{pmatrix} c_{1,2}^{-1}x_1 \\ c_{1,2}x_2 \end{pmatrix}. \tag{14}$$

(ii) *A is Calabi-Yau if and only if $\nu = \sigma_2\sigma_1$ and*

$$\operatorname{hdet}_M(\sigma) = \begin{pmatrix} c_{1,2} & 0 \\ 0 & c_{1,2}^{-1} \end{pmatrix}.$$

Proof. From the proof of Theorem 3.5, we know that A is a Koszul Artin-Schelter regular connected graded trimmed right double Ore extension of R given by $A = R_P[x_1, x_2; \sigma]$.

(i) By [45], Proposition 3.11, the Nakayama automorphism μ of A is given by

$$\mu(r) = (\det{}_P\sigma)^{-1}\nu(r), \ \text{for } r \in R, \text{and} \mu\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \operatorname{hdet}_M(\sigma)\mathbb{P}^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{15}$$

Now, from Theorem 3.4, we have that

$$\det{}_P\sigma = \sigma_2\sigma_1 \text{ and } \mathbb{P} = \begin{pmatrix} c_{1,2} & 0 \\ 0 & c_{1,2}^{-1} \end{pmatrix}. \tag{16}$$

By replacing (16) in (15), we obtain (14).

(ii) If A is Calabi-Yau, then by Theorem 3.5, $\nu = \sigma_2\sigma_1$ and

$$\operatorname{hdet}_M(\sigma) = \begin{pmatrix} c_{1,2} & 0 \\ 0 & c_{1,2}^{-1} \end{pmatrix}.$$

Conversely, by replacing $\nu = \sigma_2\sigma_1$ and

$$\operatorname{hdet}_M(\sigma) = \begin{pmatrix} c_{1,2} & 0 \\ 0 & c_{1,2}^{-1} \end{pmatrix}$$

in (14), we have that μ is the identity map of A. Definition 2.3 asserts that A is Calabi-Yau. ☑

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