ALMOST PERIODIC PROCESSES AND THE EXISTENCE OF ALMOST PERIODIC SOLUTIONS

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1. INTRODUCTION

There are many results on the existence of almost periodic solutions for almost periodic systems (cf. [6] and its references). In the methodology, there are three types. One is a separation condition, the others are stability conditions and the existence of some Liapunov function.

To discuss the evolution operator, the concept of processes introduced by Dafermos [1,2] is a usefull tool. In fact, Dafermos [2] has given the existence of almost periodic solutions for almost periodic evolution equations under the separation condition by using general theories for almost periodic processes.

In this paper, we also give existence theorems of almost periodic solutions for some almost periodic evolution equations. Our method is based on the stability property. In Section 2, we shall give some definitions of stabilities that are discussed in this paper. In Section 3, we introduce some equivalent concept, which is called Property (A), to an asymptotically almost periodic integral of almost periodic processes (Theorem 1). Furthermore, we shall show that the existence of an asymptotically almost periodic integral of an almost periodic process implies the existence of an almost periodic integral

*Partly supported in part by Grant-in-Aid for Scientific Research (C), No.09640156, Japanese Ministry of Education, Science, Sports and Culture.

†Partly supported in part by Grant-in-Aid for Scientific Research (C), No.09640235, Japanese Ministry of Education, Science, Sports and Culture.

(Theorem 2) and that uniform asymptotic stability of an almost periodic integral has Property (A) (Theorem 3). In Section 4, we shall give some equivalence relation with respect to the separation condition and some stability property (Theorem 5). In Section 5, we shall give an example as an illustration.

2. PROCESSES AND DEFINITIONS OF STABILITIES

In this section, we shall give the concepts of processes and stability properties for processes. Suppose that $\mathcal{X}$ is a separable metric space with metric d and let $w : R^+ \times R \times \mathcal{X} \mapsto \mathcal{X}$, $R^+ := [0, \infty)$ and $R := (-\infty, \infty)$, be a function satisfying the following properties for all $t, \tau \in R^+, s \in R$ and $x \in \mathcal{X}$:

(p1) $w(0, s, x) = x$.

(p2) $w(t + \tau, s, x) = w(t, \tau + s, w(\tau, s, x))$.

(p3) the mapping $w : R^+ \times R \times \mathcal{X} \mapsto \mathcal{X}$ is continuous.

We call the mapping w a process on $\mathcal{X}$. Denote by W the set of all processes on $\mathcal{X}$. For $\tau \in R$ and $w \in W$, we define the translation $\sigma(\tau)w$ of w by

$$(\sigma(\tau)w)(t, s, x) = w(t, \tau + s, x), \qquad (t, s, x) \in R^+ \times R \times \mathcal{X},$$

and set $\gamma_\sigma(w) = \bigcup_{t \in R} \sigma(t)w$. Clearly $\gamma_\sigma(w) \subset W$. We denote by $H_\sigma(w)$ all functions $v : R^+ \times R \times \mathcal{X} \mapsto \mathcal{X}$ such that for some sequence $\{\tau_n\} \subset R$, $\{\sigma(\tau_n)w\}$ converges to v pointwise on $R^+ \times R \times \mathcal{X}$, that is, $\lim_{n\to\infty}(\sigma(\tau_n)w)(t, s, x) = v(t, s, x)$ for any $(t, s, x) \in R^+ \times R \times \mathcal{X}$. The set $H_\sigma(w)$ is considered as a topological space and it is called the hull of w.

Consider a process w on $\mathcal{X}$ satisfying

(p4) $\qquad H_\sigma(w) \subset W$.

Clearly, $H_\sigma(w)$ is invariant with respect to the tanslation $\sigma(\tau), \tau \in R$. Now we suppose that $H_\sigma(w)$ is sequentially compact. Let $\Omega_\sigma(w)$ be the ω-limit set of w with respect to the translation semigroup $\sigma(t)$.

A continuous function $\mu : R^+ \mapsto \mathcal{X}$ is called *an integral* on R^+ of the process w, if $w(t, s, \mu(s)) = \mu(t + s)$ for all $t, s \in R^+$ (cf. [3, p.80]). In the following, we suppose that there exists an integral μ on R^+ of the process w such that the set $O^+(\mu) = \{\mu(t) : t \in R^+\}$ is relatively compact in $\mathcal{X}$. For any $t \in R^+$, we consider a function $\pi(t) : \mathcal{X} \times H_\sigma(w) \mapsto \mathcal{X} \times H_\sigma(w)$ defined by

$$\pi(t)(x, v) = (v(t, 0, x), \sigma(t)v)$$

for $(x, v) \in \mathcal{X} \times H_\sigma(w)$. $\pi(t)$ is called *the skew product flow of the process* w, if the following property holds true:

(p5) $\pi(t)(x, v)$ is continuous in $(t, x, v) \in R^+ \times \mathcal{X} \times H_\sigma(w)$.

From (p5) we see that $\pi(\delta)(\mu(s), \sigma(s)w) = (w(\delta, s, \mu(s)), \sigma(s+\delta)w) = (\mu(s+\delta), \sigma(s+\delta)w)$ tends to $(\mu(s), \sigma(s)w)$ as $\delta \to 0^+$, uniformly for $s \in R^+$; consequently, the integral μ on R^+ must be uniformly continuous on R^+. From Ascoli-Arzéla's theorem and the sequential compactness of $H_\sigma(w)$, it follows that for any sequence $\{\tau'_n\} \subset R^+$, there exist a subsequence $\{\tau_n\}$ of $\{\tau'_n\}$, a $v \in H_\sigma(w)$ and a function $\nu : R^+ \mapsto \mathcal{X}$ such that $\lim_{n\to\infty} \sigma(\tau_n)w = v$ and $\lim_{n\to\infty} \mu(t + \tau_n) = \nu(t)$ uniformly on any compact interval in R^+. In this case, we write as

$$(\mu^{\tau_n}, \sigma(\tau_n)w) \to (\nu, v) \text{ compactly on } R^+,$$

for simplicity. Denote by $H(\mu, w)$ the set of all (ν, v) such that $(\mu^{\tau_n}, \sigma(\tau_n)w) \to (\nu, v)$ compactly on R^+ for some sequence $\{\tau_n\} \subset R^+$. Clearly, ν is an integral on R^+ of v for any $(\nu, v) \in H(\mu, w)$. Likewise, for any sequence $\{\tau'_n\} \subset R^+$ with $\tau'_n \to \infty$ as $n \to \infty$, there exist a subsequence $\{\tau_n\}$ of $\{\tau'_n\}$, a $v \in \Omega_\sigma(w)$ and a function $\nu : R \to \mathcal{X}$ such that $\lim_{n\to\infty} \sigma(\tau_n)w = v$ and $\lim_{n\to\infty} \mu(t + \tau_n) = \nu(t)$ uniformly on any compact interval in R. In this case, we write as

$$(\mu^{\tau_n}, \sigma(\tau_n)w) \to (\nu, v) \text{ compactly on } R,$$

for simplicity. Denote by $\Omega(\mu, w)$ the set of all (ν, v) such that $(\mu^{\tau_n}, \sigma(\tau_n)w) \to (\nu, v)$ compactly on R for some sequence $\{\tau_n\} \subset R^+$ with $\tau_n \to \infty$ as $n \to \infty$. When $(\nu, v) \in \Omega(\mu, w)$, we often write $\nu \in \Omega_v(\mu)$. In this case, ν is an integral on R of v, that is, ν satisfies the relation $v(t, s, \nu(s)) = \nu(t + s)$ for all $t \in R^+$ and $s \in R$.

For any $x_0 \in \mathcal{X}$ and $\varepsilon > 0$, we set $V_\varepsilon(x_0) = \{x \in \mathcal{X} : d(x, x_0) < \varepsilon\}$. We shall give the definition of stabilities for the integral μ of the process w.

Definition 1 *The integral $\mu : R^+ \mapsto \mathcal{X}$ of the process w is said to be:*

(i) *uniformly stable (US) (resp. uniformly stable in $\Omega_\sigma(w)$) if for any $\varepsilon > 0$, there exists a $\delta := \delta(\varepsilon) > 0$ such that $w(t, s, V_\delta(\mu(s))) \subset V_\varepsilon(\mu(t+s))$ for $(t, s) \in R^+ \times R^+$ (resp. $v(t, s, V_\delta(\nu(s))) \subset V_\varepsilon(\nu(t+s))$ for $(\nu, v) \in \Omega(\mu, w)$ and $(t, s) \in R^+ \times R^+$);*

(ii) *uniformly asymptotically stable (UAS) (resp. uniformly asymptotically stable in $\Omega_\sigma(w)$), if it is US (resp. US in $\Omega_\sigma(w)$) and there exists a $\delta_0 > 0$ with the property that for any $\varepsilon > 0$, there is a $t_0 > 0$ such that $w(t, s, V_{\delta_0}(\mu(s))) \subset V_\varepsilon(\mu(t+s))$ for $t \geq t_0,\ \ s \in R^+$ (resp. $v(t, s, V_{\delta_0}(\nu(s))) \subset V_\varepsilon(\nu(t+s))$ for $(\nu, v) \in \Omega(\mu, w)$ and $t \geq t_0,\ \ s \in R^+$).*

3. ALMOST PERIODIC INTEGRALS FOR ALMOST PERIODIC PROCESSES

In this section, we shall discuss an existence theorem for an almost periodic integral of almost periodic processes.

A process $w : R^+ \times R \times \mathcal{X} \mapsto \mathcal{X}$ is said to be almost periodic if $w(t, s, x)$ is almost periodic in s uniformly with respect to t, x in bounded sets. Let w be an almost periodic process on $\mathcal{X}$. Bochner's theorem implies that $\Omega_\sigma(w) = H_\sigma(w)$ is a minimal set. Also, for any $v \in H_\sigma(w)$, there exists a sequence $\{\tau_n\} \subset R^+$ such that $w(t, s+\tau_n, x) \to v(t, s, x)$ as $n \to \infty$, uniformly in $s \in R$ and (t, x) in bounded sets of $R^+ \times \mathcal{X}$. Consider a metric ρ on $H_\sigma(w)$ defined by

$$\rho(u, v) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\rho_n(u, v)}{1 + \rho_n(u, v)}$$

with $\rho_n(u, v) = \sup\{|u(t, s, x) - v(t, s, x)| : 0 \leq t \leq n, s \in R, d(x, x_0) \leq n\}$, where x_0 is a fixed element in $\mathcal{X}$. Then $v \in H_\sigma(w)$ means that $\rho(\sigma(\tau_n)w, v) \to 0$ as $n \to \infty$, for some sequence $\{\tau_n\} \subset R^+$.

Definition 2 *An integral $\mu(t)$ on R^+ is said to be asymptotically almost periodic if it is a sum of a continuous almost periodic function $\phi(t)$ and a continuous function $\psi(t)$ defined on R^+ which tends to zero as $t \to \infty$, that is*

$$\mu(t) = \phi(t) + \psi(t).$$

Let $\mu(t)$ be an integral on R^+ such that the set $O^+(\mu)$ is relatively compact in $\mathcal{X}$. As noted in [5], $\mu(t)$ is asymptotically almost periodic if and only if it satisfies the following property:

(L) For any sequence $\{t'_n\}$ such that $t'_n \to \infty$ as $n \to \infty$ there exists a subsequence $\{t_n\}$ of $\{t'_n\}$ for which $\mu(t+t_n)$ converges uniformly on R^+.

Now, for an integral μ on R^+ of the almost periodic process w we consider the following property:

(A) For any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\nu(t) \in V_\varepsilon(\mu(t+\tau))$, for all $t \geq 0$, whenever $(\nu, v) \in \Omega(\mu, w)$, $\nu(0) \in V_{\delta(\varepsilon)}(\mu(\tau))$, and $\rho(\sigma(\tau)w, v) < \delta(\varepsilon)$ for some $\tau \geq 0$.

Theorem 1 *Assume that $\mu(t)$ is an integral on R^+ of the almost periodic process w such that the set $O^+(\mu) = \{\mu(t) : t \in R^+\}$ is relatively compact in $\mathcal{X}$. Then $\mu(t)$ is asymptotically almost periodic if and only if it has Property* (A).

Proof Assume that $\mu(t)$ has Property (A), and let $\{t'_n\}$ be any sequence such that $t'_n \to \infty$ as $n \to \infty$. Then there exist a subsequence $\{t_n\}$ of $\{t'_n\}$ and a $(\nu, v) \in \Omega(\mu, w)$ such that $(\mu^{t_n}, \sigma(t_n)w) \to (\nu, v)$ compactly on R. For any $\varepsilon > 0$, there exists an $n_0(\varepsilon) > 0$ such that if $n \geq n_0(\varepsilon)$, then $\nu(0) \in V_{\delta(\varepsilon)}(\mu(t_n))$ and $\rho(\sigma(t_n)w, v) < \delta(\varepsilon)$, where $\delta(\varepsilon)$ is the one for Property (A), which implies

$$\nu(t) \in V_\varepsilon(\mu(t+t_n)) \quad \text{for} \quad t \geq 0.$$

Thus $\mu(t)$ satisfies Property (L) and hence it is asymptotically almost periodic.

Next, suppose that $\mu(t)$ is asymptotically almost periodic, but does not have Property (A). Then there exists an $\varepsilon > 0$ and sequences $(\nu^n, v^n) \in \Omega(\mu, w), \tau_n \geq 0$ and $t_n > 0$ such that

$$\nu^n(t_n) \in \partial V_\varepsilon(\mu(t_n + \tau_n)), \tag{1}$$

$$\nu^n(0) \in V_{1/n}(\mu(\tau_n)) \tag{2}$$

and

$$\rho(v^n, \sigma(\tau_n)w) < \frac{1}{n}, \tag{3}$$

where ∂V_ε is the boundary of V_ε. We can assume that for a function $\nu(t)$

$$d(\nu^n(t), \nu(t)) \to 0 \text{ uniformly on } R \text{ as } n \to \infty, \tag{4}$$

because $\nu^n \in \Omega(\mu)$ and $\mu(t)$ is asymptotically almost periodic.

First, we shall show that $\tau_n \to \infty$ as $n \to \infty$. Suppose not. Then we may assume that for a constant $\tau \geq 0, \tau_n \to \tau$ as $n \to \infty$. Since

$$\rho(\sigma(\tau)w, v^n) \leq \rho(\sigma(\tau)w, \sigma(\tau_n)w) + \rho(\sigma(\tau_n)w, v^n),$$

we have by (3)

$$\rho(\sigma(\tau)w, v^n) \to 0 \text{ as } n \to \infty.$$

Here we note that $w(t, s+\tau, \nu(s)) = \nu(s+t)$, because $w(t, s+\tau, \nu(s)) = (\sigma(\tau)w)(t, s, \nu(s))$ $= \lim_{n\to\infty} v^n(t, s, \nu^n(s)) = \lim_{n\to\infty} \nu^n(t+s) = \nu(t+s)$. Since

$$d(\mu(\tau), \nu(0)) \leq d(\mu(\tau), \mu(\tau_n)) + d(\mu(\tau_n), \nu^n(0)) + d(\nu^n(0), \nu(0))$$

and

$$d(\mu(\tau), \mu(\tau_n)) \to 0 \text{ as } n \to \infty,$$

it follows from (2) and (4) that $d(\mu(\tau), \nu(0)) = 0$. Then $\mu(t+\tau) = w(t, \tau, \mu(\tau)) = w(t, \tau, \nu(0)) = \nu(t)$ for all $t \in R^+$. In particular,

$$d(\mu(t_n + \tau), \nu(t_n)) = 0 \text{ for all } n. \tag{5}$$

On the other hand, for sufficiently large n we get

$$\begin{aligned} d(\mu(t_n+\tau), \nu(t_n)) &\geq d(\mu(t_n+\tau_n), \nu^n(t_n)) - d(\mu(t_n+\tau_n), \mu(t_n+\tau)) \\ &\quad - d(\nu^n(t_n), \nu(t_n)) \\ &\geq \varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} \end{aligned}$$

by (1), (4) and the uniform continuity of $\mu(t)$ on R^+. This contradicts (5).

By virtue of the sequential compactness of $H_\sigma(w)$ and the asymptotic almost periodicity of $\mu(t)$, we can assume that

$$(\mu^{\tau_n}, \sigma(\tau_n)w) \to (\eta, v) \text{ compactly on } R \tag{6}$$

for some $(\eta, v) \in \Omega(\mu, w)$. Since

$$d(\eta(0), \nu(0)) \leq d(\eta(0), \mu(\tau_n)) + d(\mu(\tau_n), \nu^n(0)) + d(\nu^n(0), \nu(0)),$$

(2), (4) and (6) imply $\eta(0) = \nu(0)$, and therefore $\eta(t) = v(t, 0, \eta(0)) = v(t, 0, \nu(0)) = \lim_{n\to\infty}(\sigma(\tau_n)w)(t, 0, \nu^n(0)) = \lim_{n\to\infty} v^n(t, 0, \nu^n(0)) = \lim_{n\to\infty} \nu^n(t) = \nu(t)$ by (3), (4) and (6). Hence we have

$$\begin{aligned} d(\mu(\tau_n + t), \nu^n(t)) &\leq d(\mu(\tau_n + t), \eta(t)) + d(\nu(t), \nu^n(t)) \\ &< \varepsilon \end{aligned}$$

for all sufficiently large n, which contradicts (1). This completes the proof of Theorem 1.

Theorem 2 *If the integral $\mu(t)$ on R^+ of the almost periodic process w is asymptotically almost periodic, then there exists an almost periodic integral of the process w.*

Proof Since $w(t, s, x)$ is an almost periodic process, there exists a sequence $\{t_n\}, t_n \to \infty$ as $n \to \infty$, such that $(\sigma(t_n)w)(t, s, x) \to w(t, s, x)$ as $n \to \infty$ uniformly with respect to t, x in bounded sets and $s \in R$. The integral $\mu(t)$ has the decomposition $\mu(t) = \phi(t) + \psi(t)$, where $\phi(t)$ is almost periodic and $\psi(t) \to 0$ as $t \to \infty$. Hence we may assume $\phi(t + t_n) \to \phi^*(t)$ as $n \to \infty$ uniformly on $t \in R$, where we note $\phi^*(t)$ is almost periodic. Since $(\phi^*, w) \in \Omega(\mu, w)$, $\phi^*(t)$ is an almost periodic integral of w.

Lemma 1 *Let $T > 0$. Then for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ with the property that $d(\mu(s), \phi(0)) < \delta(\varepsilon)$ and $\rho(\sigma(s)w, v) < \delta(\varepsilon)$ imply $\phi(t) \in V_\varepsilon(\mu(s + t))$ for $t \in [0, T]$, whenever $s \in R^+$ and $(\phi, v) \in \Omega(\mu, w)$.*

Proof Suppose the contrary. Then, for some $\varepsilon > 0$ there exists sequences $\{s_n\}$, $s_n \in R^+$, $\{\tau_n\}$, $0 < \tau_n < T$, and $(\phi^n, v^n) \in \Omega(\mu, w)$ such that

$$\rho(\sigma(s_n)w, v^n) < \frac{1}{n},$$

$$\phi^n(0) \in V_{1/n}(\mu(s_n)),$$

$$\phi^n(t) \in V_\varepsilon(\mu(s_n + t)) \text{ for } t \in [0, \tau_n)$$

and

$$\phi^n(\tau_n) \in \partial V_\varepsilon(\mu(s_n + \tau_n)).$$

Since $\tau_n \in [0, T]$, we can assume that τ_n converges to a $\tau \in [0, T]$ as $n \to \infty$. Since $\Omega(\mu, w)$ is compact, we may assume $(\phi^n, v^n) \to (\phi, v) \in \Omega(\mu, w)$ and $(\mu^{s_n}, \sigma(s_n)w) \to (\eta, v) \in \Omega(\mu, w)$ as $n \to \infty$, respectively. Then $\phi(\tau) \in \partial V_\varepsilon(\eta(\tau))$. On the other hand, since $\phi(0) = \eta(0)$, we get $\phi(t) = \eta(t)$ on R^+ by (p2). This is a contradiction.

Lemma 2 *Suppose that w is an almost periodic process on $\mathcal{X}$. If the integral $\mu(t)$ on R^+ is UAS, then it is UAS in $\Omega_\sigma(w)$.*

Proof Let $\tau_k \to \infty$ as $k \to \infty$ and $(\mu^{\tau_k}, \sigma(\tau_k)w) \to (\nu, v) \in \Omega_\sigma(\mu, w)$ compactly on R. Let any $\sigma \in R^+$ be fixed. If k is sufficiently large, we get

$$\nu(\sigma) \in V_{\delta(\varepsilon/2)/2}(\mu(\tau_k + \sigma)).$$

Let $y \in V_{\delta(\varepsilon/2)/2}(\nu(\sigma))$. Then $(\sigma(\tau_k)w)(t, \sigma, y) = w(t, \tau_k + \sigma, y) \in V_{\varepsilon/2}(\mu(t + \tau_k + \sigma))$ for $t \geq \sigma$, because $y \in V_{\delta(\varepsilon/2)}(\mu(\tau_k + \sigma))$. Since $(\sigma(\tau_k)w)(t, \sigma, y) \to v(t, \sigma, y)$ and $\mu(t + \tau_k + \sigma) \to \nu(t + \sigma)$, we get $v(t, \sigma, y) \in \overline{V_{\varepsilon/2}(\nu(t + \sigma))}$ for all $t \geq \sigma$, which implies that $\nu(t)$ is US.

Now we shall show that $\nu(t)$ is UAS. Let $y \in V_{\delta_0/2}(\nu(\sigma))$ and $\nu(\sigma) \in V_{\delta_0/2}(\mu(\tau_k + \sigma))$. Since $\mu(t)$ is UAS, we have $w(t, \tau_k + \sigma, y) \in V_{\varepsilon/2}(\mu(t + \tau_k + \sigma))$ for $t \geq t_0(\varepsilon/2)$ because of $y \in V_{\delta_0}(\mu(\tau_k + \sigma))$. Hence $v(t, \sigma, y) \in \overline{V_{\varepsilon/2}(\nu(t + \sigma))}$ for $t \geq t_0(\varepsilon/2)$.

Theorem 3 *Suppose that w is an almost periodic process on $\mathcal{X}$, and let $\mu(t)$ be an integral on R^+ of w such that the set $O^+(\mu)$ is relatively compact in $\mathcal{X}$. If the integral $\mu(t)$ is UAS, then it has Property* (A). *Consequently, it is asymptotically almost periodic.*

Proof Suppose that $\mu(t)$ has not Property (A). Then there are sequences $\{t_n\}, t_n \geq 0, \{r_n\}, r_n > 0, (\phi^n, v^n) \in \Omega(\mu, w)$ and a constant $\delta_1, 0 < \delta_1 < \delta_0/2$, such that

$$\phi^n(0) \in V_{1/n}(\mu(t_n)) \text{ and } \rho(v^n, \sigma(t_n)w) < \frac{1}{n} \tag{7}$$

and

$$\phi^n(r_n) \in \partial V_{\delta_1}(\mu(t_n + r_n)) \text{ and } \phi^n(t) \in V_{\delta_1}(\mu(t + t_n)) \text{ on } [0, r_n), \tag{8}$$

where δ_0 is the one given for the UAS of $\mu(t)$. By Lemma 2, $\mu(t)$ is UAS in $\Omega_\sigma(w)$. Let $\delta(\cdot)$ be the one given for US of $\mu(t)$ in $\Omega_\sigma(w)$. There exists a sequence $\{q_n\}, 0 < q_n < r_n$, such that

$$\phi^n(q_n) \in \partial V_{\delta(\delta_1/2)/2}(\mu(t_n + q_n)) \tag{9}$$

and

$$\phi^n(t) \in \overline{V_{\delta_1}(\mu(t + t_n))} \setminus V_{\delta(\delta_1/2)/2}(\mu(t + t_n)) \text{ on } [q_n, r_n], \tag{10}$$

for a large n by (7) and (8). Suppose that there exists a subsequence of $\{q_n\}$, which we shall denote by $\{q_n\}$ again, such that q_n converges to some $q \in R^+$. It follows from (7) that there exists an $n_0 > 0$ such that for any $n \geq n_0$, $q + 1 \geq q_n \geq 0$ and $\phi^n(t) \in V_{\delta(\delta_1/2)/4}(\mu(t_n+t))$ for $t \in [0, q+1]$ by Lemma 1, which contradicts (9). Therefore, we can see that $q_n \to \infty$ as $n \to \infty$.

Put $p_n = r_n - q_n$ and suppose that $p_n \to \infty$ as $n \to \infty$. Set $s_n = q_n + (p_n/2)$. By (7) and the compactness of $\Omega(\mu, w)$, we may assume that $((\phi^n)^{s_n}, \sigma(s_n)v^n)$ and $(\mu^{t_n+s_n}, \sigma(t_n+s_n)w)$ tend to some (ϕ, v), $(\eta, v) \in \Omega(\mu, w)$ compactly on R as $n \to \infty$, respectively. For any fixed $t > 0$, one can take an $n_1 > 0$ such that for every $n \geq n_1$, $r_n - s_n = p_n/2 > t$, because $p_n \to \infty$ as $n \to \infty$. Therefore for $n \geq n_1$, we have $q_n < t + s_n < r_n$, and

$$\phi^n(t + s_n) \notin V_{\delta(\delta_1/2)/2}(\mu(t + t_n + s_n)) \tag{11}$$

by (10). There exists an $n_2 \geq n_1$ such that for every $n \geq n_2$

$$\phi^n(t + s_n) \in \overline{V_{\delta(\delta_1/2)/8}(\phi(t))} \text{ and } \eta(t) \in \overline{V_{\delta(\delta_1/2)/8}(\mu(t + t_n + s_n))}. \tag{12}$$

It follows from (11) and (12) that for every $n \geq n_2$,

$$\begin{aligned} d(\phi(t), \eta(t)) &\geq d(\phi^n(t + s_n), \mu(t + t_n + s_n)) - d(\mu(t + t_n + s_n), \eta(t)) \\ &\quad - d(\phi^n(t + s_n), \phi(t)) \\ &\geq \delta(\delta_1/2)/4. \end{aligned} \tag{13}$$

However, since $\eta(0) \in V_{\delta_0/2}(\phi(0))$, the UAS of $\mu(t)$ in $\Omega_\sigma(w)$ implies $d(\phi(t), \eta(t)) \to 0$ as $t \to \infty$, which contradicts (13).

Now we may assume that p_n converges to some $p \in R^+$ as $n \to \infty$, and that $0 \le p_n < p+1$ for all n. Moreover, we may assume that $((\phi^n)^{q_n}, \sigma(q_n)v^n)$ and $(\mu^{t_n+q_n}, \sigma(t_n+q_n)w)$ tend to some $(\psi, u), (\nu, u) \in \Omega(\mu, w)$ as $n \to \infty$, respectively. Since $d(\psi(0), \nu(0)) = \delta(\delta_1/2)/2$ by (9), we have $\psi(p) \in V_{\delta_1/2}(\nu(p))$. However, we have a contradiction by (8), because $d(\psi(p), \nu(p)) \ge d(\mu(t_n+r_n), \phi^n(r_n)) - d(\psi(p), \phi^n(q_n+p)) - d(\phi^n(q_n+p_n), \phi^n(q_n+p)) - d(\phi^n(q_n+p_n), \phi^n(r_n)) - d(\mu(t_n+r_n), \nu(p_n)) - d(\nu(p_n), \nu(p)) \ge \delta_1/2$ for all large n. Thus the integral $\mu(t)$ must have Property (A).

4. SEPARATION CONDITIONS

In this section, we shall establish an existence theorem of almost periodic integrals under a separation condition.

Definition 3 $\Omega(\mu, w)$ *is said to satisfy a separation condition if for any* $v \in \Omega_\sigma(w)$, $\Omega_v(\mu)$ *is a finite set and if* ϕ *and* ψ, $\phi, \psi \in \Omega_v(\mu)$, *are distinct integrals of* v, *then there exists a constant* $\lambda(v, \phi, \psi) > 0$ *such that*

$$d(\phi(t), \psi(t)) \ge \lambda(v, \phi, \psi) \text{ for all } t \in R.$$

To make expressions simple, we shall use the following notations. For a sequence $\{\alpha_k\}$, we shall denote it by α and $\beta \subset \alpha$ means that β is a subsequence of α. For $\alpha = \{\alpha_k\}$ and $\beta = \{\beta_k\}, \alpha + \beta$ will denote the sequence $\{\alpha_k + \beta_k\}$. Moreover, $L_\alpha x$ will denote $\lim_{k\to\infty} x(t + \alpha_k)$, whenever $\alpha = \{\alpha_k\}$ and limit exists for each t.

Lemma 3 *Suppose that* $\Omega(\mu, w)$ *satisfies the separation condition. Then one can choose a number* λ_0 *independent of* $v \in \Omega_\sigma(w), \phi$ *and* ψ *for which* $d(\phi(t), \psi(t)) \ge \lambda_0$ *for all* $t \in R$. *The number* λ_0 *is called the separation constant for* $\Omega(\mu, w)$.

Proof Obviously, we can assume that the number $\lambda(v, \phi, \psi)$ is independent of ϕ and ψ. Let v_1 and v_2 are in $\Omega_\sigma(w)$. Then there exists a sequence $r' = \{r'_k\}$ such that

$$v_2(t, s, x) = \lim_{k\to\infty} (\sigma(r'_k)v_1)(t, s, x)$$

uniformly on $R^+ \times S$ for any bounded set S in $R^+ \times \mathcal{X}$, that is, $L_{r'} v_1 = v_2$ uniformly on $R^+ \times S$ for any bounded set S in $R \times \mathcal{X}$. Let $\phi^1(t)$ and $\phi^2(t)$ be integrals in $\Omega_{v_1}(\mu)$. There exist a subsequence $r \subset r'$, $(\psi^1, v_2) \in H(\phi^1, v_1)$ and $(\psi^2, v_2) \in H(\phi^2, v_1)$ such that $L_r \phi^1 = \psi^1$ and $L_r \phi^2 = \psi^2$ in $\mathcal{X}$ compactly on R. Since $H(\phi^i, v_1) \subset \Omega(\mu, w), i = 1, 2, \psi^1$ and ψ^2 also are in $\Omega_{v_2}(\mu)$. Let ϕ^1 and ϕ^2 be distinct integrals. Then

$$\inf_{t \in R} d(\phi^1(t + r_k), \phi^2(t + r_k)) = \inf_{t \in R} d(\phi^1(t), \phi^2(t)) = \alpha_{12} > 0,$$

and hence

$$\inf_{t \in R} d(\psi^1(t), \psi^2(t)) = \beta_{12} \geq \alpha_{12} > 0, \tag{14}$$

which means that ψ^1 and ψ^2 are distinct integrals of the process $v_2(t, s, x)$. Let $p_1 \geq 1$ and $p_2 \geq 1$ be the numbers of distinct integrals of processes $v_1(t, s, x)$ and $v_2(t, s, x)$, respectively. Clearly, $p_1 \leq p_2$. In the same way, we have $p_2 \leq p_1 =: p$.

Now, let $\alpha = \min\{\alpha_{ik} : i, k = 1, 2, \cdots, p, i \neq k\}$ and $\beta = \min\{\beta_{jm} : j, m = 1, 2, \cdots, p, j \neq m\}$. By (14), we have $\alpha \leq \beta$. In the same way, we have $\alpha \geq \beta$. Therfore $\alpha = \beta$, and we may set $\lambda_0 = \alpha = \beta$.

Theorem 4 *Assume that $\mu(t)$ is an integral on R^+ of the almost periodic process w such that the set $O^+(\mu)$ is relatively compact in $\mathcal{X}$, and suppose that $\Omega(\mu, w)$ satisfies the separation condition. Then $\mu(t)$ has Property* (A). *Consequently, $\mu(t)$ is asymptotically almost periodic.*

If for any $v \in \Omega_\sigma(w)$, $\Omega_v(\mu)$ consists of only one element, then $\Omega(\mu, w)$ clearly satisfies the separation condition. Thus, the following result (cf. [2]) is an immediate consequence of Theorems 2 and 4.

Corollary 1 *If for any $v \in \Omega_\sigma(w)$, $\Omega_v(\mu)$ consists of only one element, then there exists an almost periodic integral of the almost periodic process w.*

Proof of Theorem 4 Suppose that $\mu(t)$ has not Property (A). Then there exists an $\varepsilon > 0$ and sequences $(\phi^k, v^k) \in \Omega(\mu, w)$, $\tau_k \geq 0$ and $t_k \geq 0$ such that

$$d(\mu(t_k + \tau_k), \phi^k(t_k)) = \varepsilon (< \lambda_0/2), \tag{15}$$

$$d(\mu(\tau_k), \phi^k(0)) = 1/k \tag{16}$$

and

$$\rho(\sigma(\tau_k)w, v^k) = 1/k, \tag{17}$$

where λ_0 is the separation constant for $\Omega(\mu, w)$.

First, we shall show that $t_k + \tau_k \to \infty$ as $k \to \infty$. Suppose not. Then there exists a subsequence of $\{\tau_k\}$, which we shall denote by $\{\tau_k\}$ again, and a constant $\tau \geq 0$ and that $\tau_k \to \tau$ as $k \to \infty$. Since

$$\rho(\sigma(\tau)w, v^k) \leq \rho(\sigma(\tau)w, \sigma(\tau_k)w) + \rho(\sigma(\tau_k)w, v^k),$$

(17) implies that

$$\rho(\sigma(\tau)w, v^k) \to 0 \text{ as } k \to \infty. \tag{18}$$

Moreover, we can assume that

$$(\phi^k, v^k) \to (\phi, v) \text{ compactly on } R$$

for some $(\phi, v) \in \Omega(\mu, w)$. It follows from (18) that $v = \sigma(\tau)w$. Since

$$d(\mu(\tau), \phi(0)) \leq d(\mu(\tau), \mu(\tau_k)) + d(\mu(\tau_k), \phi^k(0)) + d(\phi^k(0), \phi(0)) \to 0$$

as $k \to \infty$ by (16), we get $\mu(\tau) = \phi(0)$, and hence $\mu(t + \tau) = (\sigma(\tau)w)(t, 0, \mu(\tau)) = (\sigma(\tau)w)(t, 0, \phi(0)) = v(t, 0, \phi(0)) = \phi(t)$. However, we have $d(\mu(t_k+\tau_k), \phi(t_k)) \geq \varepsilon/2$ for a sufficiently large k by (15), because

$$d(\mu(t_k + \tau_k), \phi(t_k)) \geq d(\mu(t_k + \tau_k), \phi^k(t_k)) - d(\phi^k(t_k), \phi(t_k)).$$

This is a contradiction. Thus we must have $t_k + \tau_k \to \infty$ as $k \to \infty$.

Now, set $q_k = t_k + \tau_k$ and $\nu^k(t) = \phi^k(t_k + t)$. Then

$$(\mu^{q_k}, \sigma(q_k)w) \in H(\mu, w) \text{ and } (\nu^k, \sigma(t_k)v^k) \in \Omega(\mu, w),$$

respectively. We may assume that $(\mu^{q_k}, \sigma(q_k)w) \to (\bar{\mu}, \bar{w})$ compactly on R for some $(\bar{\mu}, \bar{w}) \in \Omega(\mu, w)$, because $q_k \to \infty$ as $k \to \infty$. Since

$$\begin{aligned} \rho(\bar{w}, \sigma(t_k)v^k) &\leq \rho(\bar{w}, \sigma(q_k)w) + \rho(\sigma(q_k)w, \sigma(t_k)v^k) \\ &= \rho(\bar{w}, \sigma(q_k)w) + \rho(\sigma(\tau_k)w, v^k), \end{aligned}$$

we see that $\rho(\bar{w}, \sigma(t_k)v^k) \to 0$ as $k \to \infty$ by (17). Hence, we can choose a subsequence $\{\nu^{k_j}\}$ of $\{\nu^k\}$ and a $\bar{\nu} \in \Omega_{\bar{w}}(\mu)$ such that

$$(\nu^{k_j}, \sigma(t_{k_j})v^{k_j}) \to (\bar{\nu}, \bar{w}) \text{ compactly on } R.$$

Since

$$\begin{aligned} & \lim_{j\to\infty}\{d(\mu(t_{k_j}+\tau_{k_j}), \phi^{k_j}(t_{k_j})) - d(\nu^{k_j}(0), \bar{\nu}(0)) - d(\bar{\mu}(0), \mu(q_{k_j}))\} \\ \le\ & d(\bar{\mu}(0), \bar{\nu}(0)) \\ \le\ & \lim_{j\to\infty}\{d(\mu(t_{k_j}+\tau_{k_j}), \phi^{k_j}(t_{k_j})) + d(\nu^{k_j}(0), \bar{\nu}(0)) + d(\bar{\mu}(0), \mu(q_{k_j}))\}, \end{aligned}$$

it follows from (15) that $d(\bar{\mu}(0), \bar{\nu}(0)) = \varepsilon$, which contradicts the separation condition of $\Omega(\mu, w)$.

Theorem 5 *Assume that $\mu(t)$ is an integral on R^+ of the almost periodic process w such that the set $O^+(\mu)$ is relatively compact in $\mathcal{X}$. Then the following statements are equivalent:*

(i) *$\Omega(\mu, w)$ satisfies the separation condition;*

(ii) *there exists a number $\delta_0 > 0$ with the property that for any $\varepsilon > 0$ there exists a $t_0(\varepsilon) > 0$ such that $d(\phi(s), \psi(s)) < \delta_0$ implies $d(\phi(t), \psi(t)) < \varepsilon$ for $t \ge s + t_0(\varepsilon)$, whenever $s \in R, v \in \Omega_\sigma(w)$ and $\phi, \psi \in \Omega_v(\mu)$.*

Consequently, the UAS of the integral $\mu(t)$ on R^+ implies the separation condition on $\Omega(\mu, w)$.

Proof If we set $\delta_0 = \lambda_0$, then (i) clearly implies (ii).

We shall show that (ii) implies (i). First of all, we shall verify that any distinct integrals $\phi(t), \psi(t)$ in $\Omega_v(\mu)$, $v \in \Omega_\sigma(w)$, satisfy

$$\liminf_{t\to-\infty} d(\phi(t), \psi(t)) \ge \delta_0. \tag{19}$$

Suppose not. Then for some $v \in \Omega_\sigma(w)$, there exists two distinct integrals $\phi(t)$ and $\psi(t)$ in $\Omega_v(\mu)$ which satisfy

$$\liminf_{t\to-\infty} d(\phi(t), \psi(t)) < \delta_0. \tag{20}$$

Since $\phi(t)$ and $\psi(t)$ are distinct integrals, we have $d(\phi(s), \psi(s)) = \varepsilon$ at some s and for some $\varepsilon > 0$. Then there is a t_1 such that $t_1 < s - t_0(\varepsilon/2)$ and $d(\phi(t_1), \psi(t_1)) < \delta_0$ by

(20). Then $d(\phi(s),\psi(s)) < \varepsilon/2$, which contradicts $d(\phi(s),\psi(s)) = \varepsilon$. Thus we have (19). Since $\overline{O^+(\mu)}$ is compact, there are a finite number of coverings which consists of m_0 balls with diameter $\delta_0/4$. We shall show that the number of integrals in $\Omega_v(\mu)$ is at most m_0. Suppose not. Then there are m_0+1 integrals in $\Omega_v(\mu), \phi^j(t), j = 1,2,\cdots,m_0+1$, and a t_2 such that

$$d(\phi^j(t_2),\phi^i(t_2)) \geq \delta_0/2 \text{ for } i \neq j, \tag{21}$$

by (19). Since $\phi^j(t_2), j = 1,2,\cdots,m_0+1$, are in $\overline{O^+(\mu)}$, some of these integrals, say $\phi^i(t),\phi^j(t)(i \neq j)$, are in one ball at time t_2, and hence $d(\phi^j(t_2),\phi^i(t_2)) < \delta_0/4$, which contradicts (21). Therefore the number of integrals in $\Omega_v(\mu)$ is $m \leq m_0$. Thus

$$\Omega_v(\mu) = \{\phi^1(t),\phi^2(t),\cdots,\phi^m(t)\} \tag{22}$$

and

$$\liminf_{t\to-\infty} d(\phi^j(t),\phi^i(t)) \geq \delta_0, i \neq j. \tag{23}$$

Consider a sequence $\{\tau_k\}$ such that $\tau_k \to -\infty$ as $k \to \infty$ and $\rho(\sigma(\tau_k)v,v) \to 0$ as $k \to \infty$. For each $j = 1,2,\cdots,m$, set $\phi^{j,k}(t) = \phi^j(t+\tau_k)$. Since $(\phi^{j,k},\sigma(\tau_k)v) \in H(\phi^j,v)$, we can assume that

$$(\phi^{j,k},\sigma(\tau_k)v) \to (\psi^j,v) \text{ compactly on } R$$

for some $(\psi^j,v) \in H(\phi^j,v) \subset \Omega(\mu,w)$. Then it follows from (23) that

$$d(\psi^j(t),\psi^i(t)) \geq \delta_0, \text{ for all } t \in R \text{ and } i \neq j. \tag{24}$$

Since the number of integrals in $\Omega_v(\mu)$ is m, $\Omega_v(\mu)$ consists of $\psi^1(t),\psi^2(t),\cdots,\psi^m(t)$ and we have (24). This shows that $\Omega(\mu,w)$ satisfies the separation condition.

5. APPLICATION

Consider the equation

$$\begin{cases} u_{tt} = & u_{xx} - u_t + f(t,x,u), \quad t > 0,\ 0 < x < 1, \\ u(0,t) = & u(1,t) = 0, \quad t > 0, \end{cases} \tag{25}$$

where $f(t,x,u)$ is continuous in $(t,x,u) \in R \times (0,1) \times R$ and it is an almost periodic function in t uniformly with respect to x and u which satisfies

$$|f(t,x,u)| \le \frac{1}{6}|u| + J$$

and

$$|f(t,x,u_1) - f(t,x,u_2)| \le \frac{1}{6}|u_1 - u_2|$$

for all $(t,x,u), (t,x,u_1), (t,x,u_2) \in R \times (0,1) \times R$ and some constant $J > 0$. We consider a Banach space X given by $X = H_0^1(0,1) \times L^2(0,1)$ equipped with the norm $\|(u,v)\| = \{\|u_x\|_{L^2}^2 + \|v\|_{L^2}^2\}^{1/2} = \{\int_0^1 (u_x^2 + v^2)dx\}^{1/2}$. Then (25) can be considered as an abstract equation

$$\frac{d}{dt}\binom{u}{v} = A\binom{u}{v} + \binom{0}{f(t,x,u)} \tag{26}$$

in X, where A is a (unbounded) linear operator in X defined by

$$A\binom{u}{v} = \binom{v}{u_{xx} - v}$$

for $(u,v) \in H^2(0,1) \times H_0^1(0,1)$. It is well known that A generates a C_0 -semigroup of bounded linear operator on X. In the following, we show that each (mild) solution of (26) is bounded in the future, and that it is UAS. Moreover, we show that each solution of (26) has a compact orbit in X. Consequently, one can apply Theorem 3 or Theorem 5 to the process generated by solutions of (26) and its integrals to conclude that (25) has an almost periodic solution which is UAS.

Now, for any solution $\binom{u(t)}{v(t)}$ of (26), we consider a function $V(t)$ defined by

$$V(t) = \int_0^1 (u_x^2 + 2\lambda uv + v^2)dx$$

with $\lambda = \frac{1}{12}$. Since $\|u\|_{L^2} \le \|u_x\|_{L^2}$ for $u \in H_0^1(0,1)$, we get the inequality

$$\frac{11}{12}(\|u_x\|_{L^2}^2 + \|v\|_{L^2}^2) \le V(t) \le \frac{13}{12}(\|u_x\|_{L^2}^2 + \|v\|_{L^2}^2). \tag{27}$$

Moreover, we get

$$\begin{aligned} \frac{d}{dt}V(t) &\le 2\int_0^1 \{(\lambda - 1)v^2 - \lambda u_x^2 - \lambda uv + (\lambda|u| + |v|)(\frac{|u|}{6} + J)\}dx \\ &\le 2(\lambda - 1)\|v\|_{L^2}^2 - 2\lambda\|u_x\|_{L^2}^2 + \lambda(\varepsilon_1\|u\|_{L^2}^2 + \varepsilon_1^{-1}\|v\|_{L^2}^2) \\ &\quad + \frac{\lambda}{3}\|u\|_{L^2}^2 + \lambda(\varepsilon_2\|u\|_{L^2}^2 + \varepsilon_2^{-1}J^2) + \frac{1}{6}(\varepsilon_3\|u\|_{L^2}^2 + \varepsilon_3^{-1}\|v\|_{L^2}^2) \\ &\quad + (\varepsilon_4\|v\|_{L^2}^2 + \varepsilon_4^{-1}J^2) \end{aligned}$$

for any positive constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4. We set $\varepsilon_1 = \frac{1}{10}, \varepsilon_2 = \frac{1}{10}, \varepsilon_3 = \frac{1}{2}$ and $\varepsilon_4 = \frac{1}{3}$ to get

$$\frac{d}{dt}V(t) \leq -\frac{1}{3}\|v\|_{L^2}^2 - \frac{7}{180}\|u_x\|_{L^2}^2 + 4J^2. \tag{28}$$

By (27) and (28), we get

$$\frac{d}{dt}V(t) \leq -C_1 V(t) + C_2$$

for some positive constants C_1 and C_2. Then $V(t) \leq e^{-C_1(t-t_0)}V(t_0) + \frac{C_2}{C_1}$ for all $t \geq t_0$, and hence $\sup_{t\geq t_0} \|(u(t), v(t))\|_X < \infty$. Thus each solution of (26) is bounded in the future.

Next, for any solutions $\binom{u(t)}{v(t)}$ and $\binom{\bar{u}(t)}{\bar{v}(t)}$ of (26), we consider a function $\bar{V}(t)$ defined by

$$\bar{V}(t) = \int_0^1 \{(u_x - \bar{u}_x)^2 + 2\lambda(u - \bar{u})(v - \bar{v}) + (v - \bar{v})^2\}dx$$

with $\lambda = \frac{1}{12}$. By almost the same arguments as for the function $V(t)$, we get

$$\frac{11}{12}\{\|u_x - \bar{u}_x\|_{L^2}^2 + \|v - \bar{v}\|_{L^2}^2\} \leq \bar{V}(t) \leq \frac{13}{12}\{\|u_x - \bar{u}_x\|_{L^2}^2 + \|v - \bar{v}\|_{L^2}^2\} \tag{29}$$

and

$$\frac{d}{dt}\bar{V}(t) \leq -C_3\bar{V}(t) \tag{30}$$

for some positive constant C_3. Then, by (29) and (30) we get

$$\begin{aligned} \|u_x(t) - \bar{u}_x(t)\|_{L^2}^2 + \|v(t) - \bar{v}(t)\|_{L^2}^2 &\leq \frac{11}{12}V(t) \\ &\leq \frac{11}{12}e^{-C_3(t-t_0)}V(t_0) \\ &\leq e^{-C_3(t-t_0)}\{\|u_x(t_0) - \bar{u}_x(t_0)\|_{L^2}^2 \\ &\quad +\|v(t_0) - \bar{v}(t_0)\|_{L^2}^2\} \end{aligned}$$

for any $t \geq t_0$, which shows that the solution $\binom{u(t)}{v(t)}$ of (26) is UAS.

Finally, we shall show that each bounded solution $(u(t), v(t))$ of (26) has a compact orbit. To do this, it sufficies to show that each increasing sequence $\{\tau_n\} \subset R^+$ such that $\tau_n \to \infty$ as $n \to \infty$ has a subsequence $\{\tau_n'\}$ such that $\{u(\tau_n'), v(\tau_n')\}$ is a Cauchy sequence in X. Taking a subsequence if necessary, we may assume that $\{f(t + \tau_n, x, u)\}$

converges uniformly in (x,u) in bounded sets and $t \in R$. Therefore, for any $\gamma > 0$ there exists an $n_0 > 0$ such that if $n > m \geq n_0$, then

$$\begin{aligned} & \sup_{(t,x,\xi)\in R\times(0,1)\times O^+(u)} |f(t+\tau_n,x,\xi) - f(t+\tau_m,x,\xi)| \\ = & \sup_{(t,x,\xi)\in R\times(0,1)\times O^+(u)} |f(t+\tau_n-\tau_m,x,\xi) - f(t,x,\xi)| < \gamma. \end{aligned}$$

Fix n and m such that $n > m \geq n_0$, and set $u^\tau(t) = u(t+\tau), v^\tau(t) = v(t+\tau)$ and $f^\tau(t,x,\xi) = f(t+\tau,x,\xi)$ with $\tau = \tau_n - \tau_m$. Clearly, (u^τ, v^τ) is a solution of (26) with f^τ instead of f.

Consider a function U defined by

$$U(t) = \int_0^1 \{(u_x - u_x^\tau)^2 + 2\lambda(u-u^\tau)(v-v^\tau) + (v-v^\tau)^2\}dx$$

with $\lambda = \frac{1}{12}$. Then

$$\frac{11}{12}\{\|u_x - u_x^\tau\|_{L^2}^2 + \|v-v^\tau\|_{L^2}^2\} \leq U(t) \leq \frac{13}{12}\{\|u_x - u_x^\tau\|_{L^2}^2 + \|v-v^\tau\|_{L^2}^2\}$$

and

$$\begin{aligned} \frac{d}{dt}U(t) \leq\ & (2\lambda-2)\|v-v^\tau\|_{L^2}^2 - 2\lambda\|u_x - u_x^\tau\|_{L^2}^2 \\ & +2\lambda\int_0^1 |u-u^\tau|\{|v-v^\tau| + |f(t,x,u) - f^\tau(t,x,u^\tau)|\}dx \\ & +2\int_0^1 |v-v^\tau||f(t,x,u) - f^\tau(t,x,u^\tau)|dx. \end{aligned}$$

Since

$$\begin{aligned} |f(t,x,u) - f^\tau(t,x,u^\tau)| \leq\ & |f(t,x,u) - f(t,x,u^\tau)| + |f(t,x,u^\tau) - f^\tau(t,x,u^\tau)| \\ \leq\ & \frac{1}{6}|u-u^\tau| + \gamma, \end{aligned}$$

we get

$$\begin{aligned} \frac{d}{dt}U(t) \leq\ & -\frac{11}{6}\|v-v^\tau\|_{L^2}^2 - \frac{1}{6}\|u_x - u_x^\tau\|_{L^2}^2 \\ & +\frac{1}{12}(\frac{1}{10}\|u-u^\tau\|_{L^2}^2 + 10\|v-v^\tau\|_{L^2}^2) + \frac{1}{36}\|u-u^\tau\|_{L^2}^2 \\ & +\frac{1}{12}(\frac{1}{2}\|u-u^\tau\|_{L^2}^2 + 2\gamma^2) \end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{6}(\varepsilon_5\|v-v^\tau\|_{L^2}^2+\frac{1}{\varepsilon_5}\|u-u^\tau\|_{L^2}^2)+(\varepsilon_6\|v-v^\tau\|_{L^2}^2+\frac{1}{\varepsilon_6}\gamma^2) \\
\leq\ & \|v-v^\tau\|_{L^2}^2(-1+\frac{\varepsilon_5}{6}+\varepsilon_6) \\
& +\|u_x-u_x^\tau\|_{L^2}^2(-\frac{1}{6}+\frac{1}{120}+\frac{1}{36}+\frac{1}{24}+\frac{1}{36\varepsilon_5})+(\frac{1}{6}+\frac{1}{\varepsilon_6})\gamma^2.
\end{aligned}
$$

Putting $\varepsilon_5 = 1$ and $\varepsilon_6 = \frac{1}{3}$, we have

$$
\begin{aligned}
\frac{d}{dt}U(t) &\leq -C(\|u_x-u_x^\tau\|_{L^2}^2+\|v-v_\tau\|_{L^2}^2)+4\gamma^2 \\
&\leq -C_1U(t)+4\gamma^2,
\end{aligned}
$$

where C and C_1 are some constants independent of γ. Then

$$U(\tau_m) \leq e^{-C_1\tau_m}U(0)+\frac{4\gamma^2}{C_1}$$

or

$$\|u_x(\tau_m)-u_x(\tau_n)\|_{L^2}^2+\|v(\tau_m)-v(\tau_n)\|_{L^2}^2 \leq Ke^{-C_1\tau_m}+\frac{8\gamma^2}{C_1},$$

where $K = 2U(0)$. Take $n_1 \geq n_0$ so that $Ke^{-C_1\tau_{n_1}} < \frac{\gamma^2}{C_1}$. Then

$$\|u_x(\tau_m)-u_x(\tau_n)\|_{L^2}^2+\|v(\tau_n)-v(\tau_m)\|_{L^2}^2 < \frac{9}{C_1}\gamma^2$$

if $n \geq m \geq n_1$, which shows that $\{(u(\tau_n), v(\tau_n))\}$ is a Cauchy sequence in X, as required.

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