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ON THE FUNDAMENTAL SOLUTION OF LINEAR DELAY DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAYS

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Abstract. For a class of linear autonomous delay differential equations with parameter α we give upper bounds for the integral $\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t$ of the fundamental solution $X(\cdot,\alpha)$. The asymptotic estimations are sharp at a critical value α_0 where $x=0$ loses stability. We use these results to study the stability properties of perturbed equations.

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1. Introduction

Let $n \geq 1$ be an integer, $r_1, \ldots, r_n$ be real numbers with $r_1 > r_2 > \ldots > r_n \geq 0$, $a_1, \ldots, a_n$ be positive numbers, $\alpha_0 > 0$. In case $n = 1$, we assume $r_1 > 0$. Consider the linear delay differential equation

$$\dot{x}(t) = -\alpha \sum_{j=1}^{n} a_j x(t - r_j) \tag{1.1}$$

with a real parameter $\alpha \in (0, \alpha_0]$.

The natural phase space for Eq. (1.1) is $C = C([-r_1, 0], \mathbb{R})$, the space of all real valued continuous functions defined on $[-r_1, 0]$ equipped with the supremum norm $\|\cdot\|$. For each $\varphi \in C$, there exists a unique solution $x^\varphi : [-r_1, \infty) \to \mathbb{R}$ with $x^\varphi(t) = \varphi(t)$, $-r_1 \leq t \leq 0$.

The characteristic function for (1.1) is

$$\Delta(z,\alpha) = z + \alpha \sum_{j=1}^{n} a_j e^{-r_j z}, \quad z \in \mathbb{C},\ 0 < \alpha \le \alpha_0.$$

The zeros of the characteristic function are the eigenvalues of the generator of the strongly continuous semigroup defined by the solution operators

$$T(t) : C \ni \varphi \mapsto x_t^{\varphi} \in C, \quad t \ge 0,$$

where the segment $x_t^{\varphi} \in C$, $t \ge 0$, is defined by $x_t^{\varphi}(s) = x^{\varphi}(t+s)$, $-r_1 \le s \le 0$.

We assume that the following hypothesis holds throughout the paper.

(H1): For $\alpha = \alpha_0$, there exists a unique pair of purely imaginary and simple eigenvalues $i\nu_0$, $-i\nu_0$ with $\nu_0 > 0$. There exist $\alpha_1 \in (0, \alpha_0)$ and $\gamma < 0$ such that for all $\alpha \in [\alpha_1, \alpha_0)$, there is a unique complex conjugate pair of simple eigenvalues $\lambda = \lambda(\alpha)$, $\overline{\lambda} = \overline{\lambda(\alpha)}$ in $\{z \in \mathbb{C} :\ \gamma < \mathrm{Re} z < 0\}$ with

$$\lim_{\alpha \to \alpha_0} \mathrm{Re}\lambda(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to \alpha_0} \mathrm{Im}\lambda(\alpha) = \nu_0.$$

For $\alpha \in [\alpha_1, \alpha_0]$, all the other eigenvalues are found in $\{z \in \mathbb{C} :\ \mathrm{Re} z < \gamma\}$.

For each α, the fundamental solution of Eq. (1.1) is the function $X(\cdot, \alpha) : [-r_1, \infty) \to \mathbb{R}$ with initial condition

$$X(t,\alpha) = \begin{cases} 0 & \text{if } -r_1 \le t < 0 \\ 1 & \text{if } t = 0, \end{cases} \tag{1.2}$$

that satisfies

$$X(t,\alpha) = 1 - \alpha \sum_{j=1}^{n} a_j \int_0^t X(s - r_j, \alpha)\, \mathrm{d}s \tag{1.3}$$

for all $t \ge 0$. It is clear that X exists uniquely, $X|_{[0,\infty)}$ is continuous and $X|_{(r_1,\infty)}$ is continuously differentiable. It is well known [9] that $\int_0^{\infty} |X(t,\alpha)|\, \mathrm{d}t < \infty$ provided $\mathrm{Re} z < 0$ for all zeros of $\Delta(z,\alpha)$. Our aim is to give explicit estimations for $\int_0^{\infty} |X(t,\alpha)|\, \mathrm{d}t$.

The integral $\int_0^{\infty} |X(t,\alpha)|\, \mathrm{d}t$ has a role via the variation-of-constants formula in perturbation results. For example Eq. (1.1) can appear as the linear variational equation at a stationary point of a nonlinear delay differential equation. If the solution $x = 0$ of Eq. (1.1) is asymptotically stable, then the stationary point of the original nonlinear equation is locally attracting. Integral $\int_0^{\infty} |X(t,\alpha)|\, \mathrm{d}t$ plays an important role in the estimation of the attractivity region of the stationary point [7, 9].

The technique applied to estimate $\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t$ contains a splitting of the spectrum by the vertical line $\mathrm{Re}z = \gamma < 0$ so that there is no eigenvalue on $\mathrm{Re}z = \gamma$. Then the phase space can be decomposed as $C = P \oplus Q$, where P is the realified generalized eigenspace of the generator corresponding to the spectrum in $\mathrm{Re}z > \gamma$, and Q is the realified generalized eigenspace corresponding to the spectrum in $\mathrm{Re}z < \gamma$. The solution operator $T(t)$ is easily estimated on P as it is finite dimensional. On Q it is well known that $\|T(t)\varphi\| \le M(\gamma) e^{\gamma t} \|\varphi\|$ holds for all $\varphi \in Q$ and $t \ge 0$ with some constant $M(\gamma) \ge 1$. An explicit upper bound for $M(\gamma)$ is crucial in our estimation for $\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t$. Giving an optimal upper bound for $M(\gamma)$ is also interesting in the construction of invariant manifolds, in particular when the size of the manifolds is of key importance. E. g., in order to prove that the local attractivity of 0 implies global attractivity for Wright's equation, the estimates for $M(\gamma)$ of this paper are used to find bounds for the size of a center manifold [11].

Although the estimates for $\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t$ seem to be a fundamental technical issue, as far as we know, not much is known except for the results of Győri and Hartung in [5, 6, 7]. In the single delay case $n = 1$ their estimates are sharp for small values of α, but not for α close to the critical value α_0.

This paper is organized as follows. We present the results in Section 2. Section 3 estimates the location of the leading pair of eigenvalues. Sections 4 contains the proofs of the first three theorems. For the single delay case (Theorem 2.4) a different proof is given in Section 5 yielding a sharper result. An example is shown in Section 6 with two delays. Section 7 presents two applications of the results for perturbed equations.

2. Main results

Note that

$$\Delta_z'(\lambda,\alpha) = \frac{\partial}{\partial z}\Delta(\lambda,\alpha) = 1 - \alpha \sum_{j=1}^n a_j r_j e^{-r_j\lambda},$$

and for $\lambda = \mu + i\nu$,

$$|\Delta_z'(\lambda,\alpha)|^2 = \left(1 - \alpha \sum_{j=1}^n a_j r_j e^{-r_j\mu} \cos(r_j\nu)\right)^2 + \left(\alpha \sum_{j=1}^n a_j r_j e^{-r_j\mu} \sin(r_j\nu)\right)^2.$$

According to the results [5, 6] of Győri and Hartung,

$$\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t = O\left(\frac{1}{(\alpha_0-\alpha)^2}\right).$$

Our first theorem gives a sharp asymptotic estimation for $\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t$ as $\alpha \to \alpha_0-$ implying that $\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t = O\left((\alpha_0-\alpha)^{-1}\right)$.

Theorem 2.1. *Under hypothesis (H1)*

$$\lim_{\alpha\to\alpha_0-} (\alpha_0-\alpha)\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t = \frac{4\left|\Delta_z'(i\nu_0,\alpha_0)\right|}{\pi\alpha_0\left(\sum_{j=1}^n a_j r_j \sin(r_j\nu_0)\right)\left(\sum_{j=1}^n a_j \sin(r_j\nu_0)\right)}.$$

From the application point of view it is more important to give an explicit upper bound for $\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t$ on interval $[\alpha_1,\alpha_0)$. This is contained in the next result in terms of α, $\lambda=\lambda(\alpha)$, $\mu=\mu(\alpha)=\mathrm{Re}\lambda(\alpha)$ and $\nu=\nu(\alpha)=\mathrm{Im}\lambda(\alpha)$ guaranteed by (H1).

Theorem 2.2. *Under hypothesis (H1)*

$$\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t \le \frac{2\nu e^{-\frac{\mu}{\nu}\pi}\left(1+e^{-\frac{\mu}{\nu}\pi}\right)}{\left|\Delta_z'(\lambda,\alpha)\right|(\mu^2+\nu^2)\left(1-e^{\frac{\mu}{\nu}\pi}\right)} + \frac{\max\{L_1(\alpha),L_2(\alpha)\}}{-\gamma}$$

for all $\alpha_1 \le \alpha < \alpha_0$ *with*

$$L_1(\alpha) = \frac{1}{2\pi r_1}\left(1+\frac{\omega_0}{2}\right)\left(\frac{8}{\omega_0}+2\int_0^{\omega_0}\frac{d\omega}{\left|\Delta(\gamma+i\omega,\alpha)\right|^2}\right) < \infty, \tag{2.1}$$

$$\omega_0 = 2\alpha_0\sum_{j=1}^n a_j e^{-r_j\gamma},$$

and

$$L_2(\alpha) = e^{-\gamma r_1}\left(e^{\alpha_0 r_1\sum_{j=1}^n a_j} + \frac{2}{\left|\Delta_z'(\lambda,\alpha)\right|}\right) < \infty. \tag{2.2}$$

We are going to check that the upper bound given by Theorem 2.2 is sharp in the sense that this upper estimate multiplied by $(\alpha_0-\alpha)$ has the same limit at α_0- as function $(\alpha_0-\alpha)\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t$.

There is a need for easily computable upper bounds. Our next aim is to give an estimate that is independent of $\lambda=\lambda(\alpha)$.

For simplicity, set $c_1 = 1+\alpha_0\sum_{j=1}^n a_j r_j$. For each $\delta \in (0,\pi/(2r_1))$, set $c_2 = c_2(\delta) = a_1 r_1 \sin(r_1\delta) > 0$ and fix $K=K(\delta)>0$ so large that

$$K > \frac{2c_1^2 e^{-r_1\gamma}}{c_2\delta}$$

holds. We will need the following additional hypothesis besides (H1).

(H2): There exists $\delta \in (0,\pi/(2r_1))$ such that for all $\alpha\in[\alpha_1,\alpha_0]$, we have $\delta \le \mathrm{Im}\lambda \le \pi/r_1-\delta$ and

$$0 < \frac{2c_1^2 e^{-r_1\gamma}}{c_2\delta + c_1\alpha_1^{-1}\mathrm{Re}\lambda} \le K(\delta).$$

The upper bound given by the next result is not sharp, but it is independent of $\lambda = \lambda(\alpha)$.

Theorem 2.3. *If (H1) and (H2) hold, then*

$$(\alpha_0 - \alpha) \int_0^\infty |X(t, \alpha)|\, dt \le K(\delta) \frac{-2\gamma e^{\frac{-\gamma}{\delta}\pi} \left(1 + e^{\frac{-\gamma}{\delta}\pi}\right)}{\alpha_1 c_2 \delta \left(1 - e^{\frac{\gamma}{\delta}\pi}\right)} + \frac{\max\left\{\tilde{L}_1, \tilde{L}_2\right\}(\alpha_0 - \alpha_1)}{-\gamma}$$

for all $\alpha_1 \le \alpha < \alpha_0$*, where* $\tilde{L}_1 = \sup_{\alpha_1 \le \alpha \le \alpha_0} L_1(\alpha)$*,* $L_1(\alpha)$ *is defined by* (2.1) *and*

$$\tilde{L}_2 = e^{-\gamma r_1} \left(e^{\alpha_0 r_1 \sum_{j=1}^n a_j} + \frac{2}{\alpha_1 c_2}\right).$$

The particular case $n = 1$, $r_1 = 1$, $a_1 = 1$ is of special interest as equation

$$\dot{x}(t) = -\alpha x(t-1) \tag{2.3}$$

is the simplest delay differential equation, and it appears as a linearization of famous equations of the form $\dot{x}(t) = f(x(t-1))$. However, surprisingly, little is known about $\int_0^\infty |X(t, \alpha)|\, dt$ even for this simple case when $\alpha \approx \alpha_0$.

Theorem 2.1 is a generalization of a result of Krisztin in [10] saying that for this equation

$$\lim_{\alpha \to \frac{\pi}{2}-} \left(\frac{\pi}{2} - \alpha\right) \int_0^\infty |X(t, \alpha)|\, \mathrm{d}t = \frac{4\sqrt{4 + \pi^2}}{\pi^2} \approx 1.5.$$

The result of Theorem 2.3 can be substantially improved for (2.3). This is essential in the estimation of the attractivity region of $x = 0$ for the Wright's equation for α near the critical value $\pi/2$, see [2].

Theorem 2.4. *If* $X(\cdot, \alpha) : [-r_1, \infty) \to \mathbb{R}$ *is the fundamental solution of Eq.* (2.3)*, then*

$$\left(\frac{\pi}{2} - \alpha\right) \int_0^\infty |X(t, \alpha)|\, dt \le 1.93 + 5.99 \left(\frac{\pi}{2} - \alpha\right) \quad \textit{for } \alpha \in \left[\frac{3}{2}, \frac{\pi}{2}\right).$$

We remark that the upper bound given in [5, 7] for the integral of the fundamental solution of Eq. (2.3) is sharp only for small $\alpha > 0$, in particular for $\alpha \in \left(0, e^{-1}\right]$.

3. The real part of the leading eigenvalues

It is of key importance to understand the behavior of $\mathrm{Re}\lambda(\alpha)$ near the critical value α_0.

For all $z = u + iv \in \mathbb{C}$ and $\alpha \le \alpha_0$, set

$$\begin{aligned} g(u,v,\alpha) &= \mathrm{Re}\Delta(u+iv,\alpha) = u + \alpha \sum_{j=1}^{n} a_j e^{-r_j u} \cos(r_j v), \\ h(u,v,\alpha) &= \mathrm{Im}\Delta(u+iv,\alpha) = v - \alpha \sum_{j=1}^{n} a_j e^{-r_j u} \sin(r_j v). \end{aligned}$$

Then g and h are smooth functions with the following partial derivatives:

$$\frac{\partial g}{\partial u}(u,v,\alpha) = \tfrac{\partial h}{\partial v}(u,v,\alpha) = 1 - \alpha \sum_{j=1}^{n} a_j r_j e^{-r_j u} \cos(r_j v), \tag{3.1}$$

$$\frac{\partial g}{\partial v}(u,v,\alpha) = -\tfrac{\partial h}{\partial u}(u,v,\alpha) = -\alpha \sum_{j=1}^{n} a_j r_j e^{-r_j u} \sin(r_j v), \tag{3.2}$$

$$\frac{\partial g}{\partial \alpha}(u,v,\alpha) = \sum_{j=1}^{n} a_j e^{-r_j u} \cos(r_j v), \tag{3.3}$$

$$\frac{\partial h}{\partial \alpha}(u,v,\alpha) = -\sum_{j=1}^{n} a_j e^{-r_j u} \sin(r_j v). \tag{3.4}$$

Note that

$$\Delta'_z(z,\alpha) = 1 - \alpha \sum_{j=1}^{n} a_j r_j e^{-r_j z} = \frac{\partial g}{\partial u}(u,v,\alpha) - i\frac{\partial g}{\partial v}(u,v,\alpha). \tag{3.5}$$

If $\mu = \mu(\alpha) = \mathrm{Re}\lambda(\alpha)$ and $\nu = \nu(\alpha) = \mathrm{Im}\lambda(\alpha)$, where λ is the leading eigenvalue in (H1), then $g(\mu,\nu,\alpha) = 0$ and $h(\mu,\nu,\alpha) = 0$ for all $\alpha_1 \le \alpha \le \alpha_0$. In particular, $g(0,\nu_0,\alpha_0) = 0$ and $h(0,\nu_0,\alpha_0) = 0$.

By condition (H1), $\lambda(\alpha)$ is a simple zero of $\Delta(\lambda,\alpha)$, that is

$$|\Delta'_z(\lambda,\alpha)|^2 = \left(\frac{\partial g}{\partial u}(\mu,\nu,\alpha)\right)^2 + \left(\frac{\partial g}{\partial v}(\mu,\nu,\alpha)\right)^2 > 0 \tag{3.6}$$

for all $\alpha_1 \le \alpha \le \alpha_0$ with a lower bound independent of α.

The smooth dependence of μ and ν on α is easily guaranteed.

Proposition 3.1. *Assume that condition (H1) holds. Then μ and ν are C^1-smooth functions of α on $[\alpha_1,\alpha_0]$ with*

$$\mu'(\alpha) = \frac{\frac{\partial g}{\partial v}(\mu,\nu,\alpha)\frac{\partial h}{\partial \alpha}(\mu,\nu,\alpha) - \frac{\partial g}{\partial u}(\mu,\nu,\alpha)\frac{\partial g}{\partial \alpha}(\mu,\nu,\alpha)}{\left(\frac{\partial g}{\partial u}(\mu,\nu,\alpha)\right)^2 + \left(\frac{\partial g}{\partial v}(\mu,\nu,\alpha)\right)^2}, \quad \alpha \in [\alpha_1,\alpha_0]. \tag{3.7}$$

Proof. Choose $\alpha \in [\alpha_1,\alpha_0]$ arbitrarily. As $g(\mu,\nu,\alpha) = 0$, $h(\mu,\nu,\alpha) = 0$, and

$$\det\begin{pmatrix} \frac{\partial g}{\partial u}(\mu,\nu,\alpha) & \frac{\partial g}{\partial v}(\mu,\nu,\alpha) \\ \frac{\partial h}{\partial u}(\mu,\nu,\alpha) & \frac{\partial h}{\partial v}(\mu,\nu,\alpha) \end{pmatrix} = \left(\frac{\partial g}{\partial u}(\mu,\nu,\alpha)\right)^2 + \left(\frac{\partial g}{\partial v}(\mu,\nu,\alpha)\right)^2 \neq 0$$

by our initial assumption, the Implicit Function Theorem yields the first assertion.

Differentiating the equations with respect to α, we get

$$\frac{\partial g}{\partial u}(\mu,\nu,\alpha)\,\mu'(\alpha) + \frac{\partial g}{\partial v}(\mu,\nu,\alpha)\,\nu'(\alpha) + \frac{\partial g}{\partial \alpha}(\mu,\nu,\alpha) = 0,$$
$$\frac{\partial h}{\partial u}(\mu,\nu,\alpha)\,\mu'(\alpha) + \frac{\partial h}{\partial v}(\mu,\nu,\alpha)\,\nu'(\alpha) + \frac{\partial h}{\partial \alpha}(\mu,\nu,\alpha) = 0,$$

from which the formula for $\mu'(\alpha)$ easily follows. $\square$

Corollary 3.2. *If (H1) holds, then*

$$\lim_{\alpha\to\alpha_0-}\mu'(\alpha) = \frac{\alpha_0\left(\sum_{j=1}^n a_j r_j \sin(r_j v_0)\right)\left(\sum_{j=1}^n a_j \sin(r_j v_0)\right)}{\left|\Delta_z'(i\nu_0,\alpha_0)\right|^2}.$$

Proof. By hypothesis (H1), $\lim_{\alpha\to\alpha_0}\lambda(\alpha) = i\nu_0$. Proposition 3.1 with (3.1)-(3.5) and relation

$$\frac{\partial g}{\partial \alpha}(0,\nu_0,\alpha_0) = \sum_{j=1}^n a_j\cos(r_j\nu_0) = \frac{g(0,\nu_0,\alpha_0)}{\alpha_0} = 0$$

gives the statement of the corollary. $\square$

If ν is bounded away from 0, and $|\mu|$ is sufficiently small for all α, then we give an upper bound for $(\alpha_0-\alpha)/|\mu(\alpha)|$. The following corollary is needed in the proof of Theorem 2.3.

Corollary 3.3. *Suppose that (H1) and (H2) hold. Then*

$$\frac{\alpha-\alpha_0}{\mu(\alpha)} \le K(\delta) \quad \textit{for each } \alpha\in[\alpha_1,\alpha_0).$$

Proof. Proposition 3.1 gives

$$\mu'(\alpha) = \frac{\left(-\frac{\partial g}{\partial v}(\mu,\nu,\alpha)\right)\left(-\frac{\partial h}{\partial \alpha}(\mu,\nu,\alpha)\right) - \frac{\partial g}{\partial u}(\mu,\nu,\alpha)\frac{\partial g}{\partial \alpha}(\mu,\nu,\alpha)}{\left(\frac{\partial g}{\partial u}(\mu,\nu,\alpha)\right)^2 + \left(-\frac{\partial g}{\partial v}(\mu,\nu,\alpha)\right)^2}, \quad \alpha\in[\alpha_1,\alpha_0].$$

Using this result, we give a positive lower bound for $\mu'(\alpha)$, $\alpha\in[\alpha_1,\alpha_0]$. It clearly follows from (3.1)-(3.2) that for all $\alpha\in[\alpha_1,\alpha_0]$,

$$\frac{\partial g}{\partial u}(\mu,\nu,\alpha) \le \left(1+\alpha\sum_{j=1}^n a_j r_j\right)e^{-r_1\mu} \le c_1 e^{-r_1\mu},$$

$$\frac{\partial g}{\partial u}(\mu,\nu,\alpha) \ge -\alpha\sum_{j=1}^n a_j r_j e^{-r_1\mu} \ge -c_1 e^{-r_1\mu}$$

and
$$-\frac{\partial g}{\partial v}(\mu,\nu,\alpha)\leq\left(\alpha\sum_{j=1}^{n}a_jr_j\right)e^{-r_1\mu}\leq c_1e^{-r_1\mu},$$
with constant c_1 introduced before Theorem 2.3. Note that if (H2) holds, then $\sin(r_j\nu)>0$ for all $j\in\{1,...,n\}$ and $\sin(r_1\nu)>\sin(r_1\delta)$. Hence
$$-\frac{\partial g}{\partial v}(\mu,\nu,\alpha)\geq\alpha a_1r_1e^{-r_1\mu}\sin(r_1\nu)\geq\alpha c_2e^{-r_1\mu}>0,$$
where $c_2=a_1r_1\sin(r_1\delta)$. Equations (3.3), (3.4) with $g(\mu,\nu,\alpha)=0$ and $h(\mu,\nu,\alpha)=0$ give that
$$\frac{\partial g}{\partial\alpha}(\mu,\nu,\alpha)=\frac{-\mu}{\alpha}\quad\text{and}\quad-\frac{\partial h}{\partial\alpha}(\mu,\nu,\alpha)=\frac{\nu}{\alpha}.$$
Therefore
$$\begin{aligned}\mu'(\alpha)&\geq\frac{\alpha c_2e^{-r_1\mu}\frac{\nu}{\alpha}-c_1e^{-r_1\mu}\frac{-\mu}{\alpha}}{2c_1^2e^{-2r_1\mu}}\\&=\frac{c_2\nu+c_1\alpha^{-1}\mu}{2c_1^2e^{-r_1\mu}}\quad\text{for }\alpha\in[\alpha_1,\alpha_0].\end{aligned}$$
Conditions given in (H2) now yield
$$\mu'(\alpha)\geq\frac{c_2\delta+c_1\alpha_1^{-1}\mu}{2c_1^2e^{-r_1\gamma}}>\frac{1}{K(\delta)}.$$
The Lagrange Mean Value Theorem implies that for each $\alpha\in[\alpha_1,\alpha_0)$,
$$-\mu(\alpha)=\mu(\alpha_0)-\mu(\alpha)=\mu'(\xi)(\alpha_0-\alpha)$$
with some $\alpha<\xi<\alpha_0$. Thus the previous result implies$-\mu(\alpha)\geq(\alpha_0-\alpha)/K(\delta)$ for all $\alpha\in[\alpha_1,\alpha_0)$, and the proof is complete. □

4. The Proofs of Theorems 2.1-2.3

Under hypothesis (H1), the phase space $C=C([-r_1,0],\mathbb{R})$ can be decomposed as $C=P\oplus Q$ into the closed subspaces P and Q, where P is the realified generalized eigenspace of the generator associated with the leading eigenvalues $\lambda=\mu+i\nu$, $\bar{\lambda}=\mu-i\nu$, and Q is the realified generalized eigenspace associated with the rest of the spectrum of the generator. Subspace P is spanned by $e^{\mu t}\cos(\nu t)|_{[-r_1,0]}$ and $e^{\mu t}\sin(\nu t)|_{[-r_1,0]}$, therefore $\dim P=2$. Both P and Q are invariant subspaces for the solution segments of Eq. (1.1) in the sense that if $x:[-r_1,\infty)\to\mathbb{R}$ is a solution of Eq. (1.1) and $x_T\in P\ (Q)$ for some $T\geq0$, then $x_t\in P\ (Q)$ for all $t\geq T$.

The decomposition $C=P\oplus Q$ defines a projection Pr_P onto P along Q and a projection Pr_Q onto Q along P.

For all $\alpha \in [\alpha_1, \alpha_0]$, set functions $p(\cdot,\alpha) : [-r_1, \infty) \to \mathbb{R}$ and $q(\cdot,\alpha) : [-r_1, \infty) \to \mathbb{R}$ by

$$p(t,\alpha) = \sum_{z=\lambda,\overline{\lambda}} \operatorname{Res} \frac{e^{zt}}{\Delta(z,\alpha)}, \quad t \geq -r_1,$$

and $q(t,\alpha) = X(t,\alpha) - p(t,\alpha)$, $t \geq -r_1$. For simplicity we also use notations $p = p(\cdot,\alpha)$ and $q = q(\cdot,\alpha)$. As

$$\sum_{z=\lambda,\overline{\lambda}} \operatorname{Res} \frac{e^{zt}}{\Delta(z,\alpha)} = \frac{e^{\lambda t}}{\Delta'_z(\lambda,\alpha)} + \frac{e^{\overline{\lambda} t}}{\Delta'_z(\overline{\lambda},\alpha)} = 2\operatorname{Re} \frac{e^{\lambda t}}{\Delta'_z(\lambda,\alpha)},$$

$p(\cdot,\alpha) : [-r_1, \infty) \to \mathbb{R}$ is a solution of (1.1). Thus it follows from the definition of $X(\cdot,\alpha)$ that $t = 0$ is the only discontinuity of $q(\cdot,\alpha)$, it is differentiable for $t > r_1$ and satisfies

$$q(t,\alpha) = q(0,\alpha) - \alpha \sum_{j=1}^{n} a_j \int_0^t q(s - r_j, \alpha)\, \mathrm{d}s \quad \text{for } t \geq 0.$$

It is a well known result (see [4] of Diekmann et al.), that $p_t = Pr_P X_t \in P$ for all $t \geq r_1$, hence $q_t \in Q$ for all $t \geq r_1$. Moreover, formula

$$q(t,\alpha) = \frac{1}{2\pi} e^{\gamma t} \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{i\omega t}}{\Delta(\gamma + i\omega, \alpha)} \mathrm{d}\omega$$

holds for all $t > 0$ by the Laplace transform technique [4, 9]. In order to estimate $\int_0^\infty |X(t,\alpha)|\, \mathrm{d}t$, we estimate $\int_0^\infty |p(t,\alpha)|\, \mathrm{d}t$ and $\int_0^\infty |q(t,\alpha)|\, \mathrm{d}t$.

Proposition 4.1. *Under hypothesis (H1)*

$$\int_0^\infty |p(t,\alpha)|\, dt \leq \frac{2\nu e^{-\frac{\mu}{\nu}\pi}\left(1 + e^{-\frac{\mu}{\nu}\pi}\right)}{|\Delta'_z(\lambda,\alpha)| (\mu^2 + \nu^2)\left(1 - e^{\frac{\mu}{\nu}\pi}\right)}$$

for all $\alpha_1 \leq \alpha < \alpha_0$. Furthermore,

$$\lim_{\alpha\to\alpha_0-} (\alpha_0 - \alpha) \int_0^\infty |p(t,\alpha)|\, dt = \frac{4\,|\Delta'_z(i\nu_0,\alpha_0)|}{\pi\alpha_0 \left(\sum_{j=1}^n a_j r_j \sin(r_j\nu_0)\right)\left(\sum_{j=1}^n a_j \sin(r_j\nu_0)\right)}.$$

Proof. For all $\alpha \in [\alpha_1, \alpha_0]$ and $t \geq -r_1$,

$$\begin{aligned} p(t,\alpha) &= 2\operatorname{Re} \frac{e^{\lambda t}}{\Delta'_z(\lambda,\alpha)} \\ &= \frac{2e^{\mu t}}{|\Delta'_z(\lambda,\alpha)|^2} \Bigg\{ \cos(\nu t) \left(1 - \alpha \sum_{j=1}^n a_j r_j e^{-r_j\mu} \cos(r_j\nu)\right) \\ &\quad + \sin(\nu t)\left(\alpha \sum_{j=1}^n a_j r_j e^{-r_j\mu} \sin(r_j\nu)\right)\Bigg\}. \end{aligned} \tag{4.1}$$

Since for all $A, B \in \mathbb{R}$ with $A^2 + B^2 > 0$ we have $A\cos(\nu t) + B\sin(\nu t) = \sqrt{A^2+B^2}\sin(\nu t + \eta)$ with $\eta \in [-\pi, \pi)$ satisfying

$$\sin\eta = \frac{A}{\sqrt{A^2+B^2}} \text{ and } \cos\eta = \frac{B}{\sqrt{A^2+B^2}},$$

we obtain that there exists $\eta = \eta(\alpha) \in [-\pi, \pi)$ such that

$$p(t,\alpha) = 2e^{\mu t}\frac{\sin(\nu t+\eta)}{|\Delta_z'(\lambda,\alpha)|}, \quad \alpha \in [\alpha_1, \alpha_0], \, t \geq -r_1. \tag{4.2}$$

Choose $0 < t_1 < t_2 < \ldots < t_n < \ldots$ such that $\nu t_j + \eta = j\pi$ for all $j \geq 1$. Then $\sin(\nu t + \eta) > 0$ for all $t \in (t_{2j}, t_{2j+1})$, $j \geq 1$, and $\sin(\nu t + \eta) < 0$ for all $t \in (t_{2j-1}, t_{2j})$, $j \geq 1$. With this notation,

$$\begin{aligned}
\int_{t_1}^{\infty} |p(t,\alpha)|\,\mathrm{d}t &= \sum_{j=1}^{\infty}(-1)^j\int_{t_j}^{t_{j+1}} p(t,\alpha)\,\mathrm{d}t \\
&= \frac{2}{|\Delta_z'(\lambda,\alpha)|}\sum_{j=1}^{\infty}(-1)^j\int_{t_j}^{t_{j+1}} e^{\mu t}\sin(\nu t+\eta)\,\mathrm{d}t \\
&= \frac{2e^{-\frac{\mu}{\nu}\eta}}{|\Delta_z'(\lambda,\alpha)|\,\nu}\sum_{j=1}^{\infty}(-1)^j\int_{j\pi}^{(j+1)\pi} e^{\frac{\mu}{\nu}s}\sin s\,\mathrm{d}s.
\end{aligned}$$

Since

$$\int e^{\frac{\mu}{\nu}s}\sin s\,\mathrm{d}s = \frac{\nu^2 e^{\frac{\mu}{\nu}s}}{\mu^2+\nu^2}\left(\frac{\mu}{\nu}\sin s - \cos s\right),$$

it follows that

$$\begin{aligned}
\int_{t_1}^{\infty} |p(t,\alpha)|\,\mathrm{d}t &= \frac{2\nu e^{-\frac{\mu}{\nu}\eta}\left(e^{\frac{\mu}{\nu}\pi}+1\right)}{|\Delta_z'(\lambda,\alpha)|\,(\mu^2+\nu^2)}\sum_{j=1}^{\infty} e^{\frac{\mu}{\nu}j\pi} \\
&= \frac{2\nu e^{-\frac{\mu}{\nu}\eta}\left(e^{\frac{\mu}{\nu}\pi}+1\right)}{|\Delta_z'(\lambda,\alpha)|\,(\mu^2+\nu^2)}\frac{e^{\frac{\mu}{\nu}\pi}}{1-e^{\frac{\mu}{\nu}\pi}}
\end{aligned}$$

for all $\alpha_1 \leq \alpha < \alpha_0$.

In addition,

$$\begin{aligned}
\int_0^{t_1} |p(t,\alpha)|\,\mathrm{d}t &= \frac{2}{|\Delta_z'(\lambda,\alpha)|}\int_0^{t_1} e^{\mu t}|\sin(\nu t+\eta)|\,\mathrm{d}t \\
&\leq \frac{2e^{-\frac{\mu}{\nu}\eta}}{|\Delta_z'(\lambda,\alpha)|\,\nu}\left\{-\int_{-\pi}^{0} e^{\frac{\mu}{\nu}s}\sin s\,\mathrm{d}s + \int_0^{\pi} e^{\frac{\mu}{\nu}s}\sin s\,\mathrm{d}s\right\} \\
&= \frac{2\nu e^{-\frac{\mu}{\nu}\eta}\left(e^{\frac{\mu}{\nu}\pi}+2+e^{-\frac{\mu}{\nu}\pi}\right)}{|\Delta_z'(\lambda,\alpha)|\,(\mu^2+\nu^2)}.
\end{aligned}$$

As

$$e^{\frac{\mu}{\nu}\pi}+2+e^{-\frac{\mu}{\nu}\pi}+\frac{e^{\frac{\mu}{\nu}\pi}\left(e^{\frac{\mu}{\nu}\pi}+1\right)}{1-e^{\frac{\mu}{\nu}\pi}}=\frac{1+e^{-\frac{\mu}{\nu}\pi}}{1-e^{\frac{\mu}{\nu}\pi}},$$

the last two results with $\eta\in[-\pi,\pi)$ give the first statement of the proposition.

It is clear that $\int_0^{t_1}|p(t,\alpha)|\,\mathrm{d}t$ is bounded with an upper bound independent of α. Hence

$$\begin{aligned}\lim_{\alpha\to\alpha_0-}(\alpha_0-\alpha)\int_0^\infty|p(t,\alpha)|\,\mathrm{d}t &= \lim_{\alpha\to\alpha_0-}(\alpha_0-\alpha)\int_{t_1}^\infty|p(t,\alpha)|\,\mathrm{d}t\\ &= \frac{4}{|\Delta'_z(i\nu_0,\alpha)|\,\pi}\lim_{\alpha\to\alpha_0-}\left(\frac{\alpha_0-\alpha}{-\mu}\right)\left(\frac{-\frac{\mu}{\nu}\pi}{1-e^{\frac{\mu}{\nu}\pi}}\right).\end{aligned}$$

As $\lim_{x\to0-}(-x)/(1-e^x)=1$ and $-\mu=\mu'(\xi)(\alpha_0-\alpha)$ with some $\xi\in(\alpha,\alpha_0)$, we obtain that

$$\lim_{\alpha\to\alpha_0-}(\alpha_0-\alpha)\int_0^\infty|p(t,\alpha)|\,\mathrm{d}t=\frac{4}{|\Delta'_z(i\nu_0,\alpha)|\,\pi\lim_{\alpha\to\alpha_0-}\mu'(\alpha)}.$$

Now Corollary 3.2 yields the second statement of the proposition. □

It is a well known result that for each α there exists a constant $L(\alpha)>0$ such that $|q(t,\alpha)|<L(\alpha)e^{\gamma t}$ for all $t>0$, see [4, 9]. Next we construct an upper bound for $|q(t,\alpha)|$ that is independent of α.

Proposition 4.2. *If hypothesis (H1) holds, then* $|q(t,\alpha)|<\max\{L_1(\alpha),L_2(\alpha)\}\,e^{\gamma t}$ *for all* $t>0$ *and* $\alpha\in[\alpha_1,\alpha_0]$, *where* $L_1=L_1(\alpha)$ *and* $L_2=L_2(\alpha)$ *are given by* (2.1) *and* (2.2), *respectively. Consequently,*

$$\int_0^\infty|q(t,\alpha)|\,\mathrm{d}t<-\frac{\max\{L_1(\alpha),L_2(\alpha)\}}{\gamma}\quad \textit{for all } \alpha\in[\alpha_1,\alpha_0].$$

Proof. Recall that for $\alpha\in[\alpha_1,\alpha_0]$ and $t>0$,

$$q(t,\alpha)=\frac{e^{\gamma t}}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}\frac{e^{i\omega t}}{\Delta(\gamma+i\omega,\alpha)}\mathrm{d}\omega,$$

where $\Delta(\gamma+i\omega,\alpha)=\gamma+i\omega+\alpha\sum_{j=1}^n a_je^{-r_j(\gamma+i\omega)}$. Partial integration gives that

$$\int_{-T}^{T}\frac{e^{i\omega t}}{\Delta(\gamma+i\omega,\alpha)}\mathrm{d}\omega=\left[\frac{e^{i\omega t}}{it}\frac{1}{\Delta(\gamma+i\omega,\alpha)}\right]_{-T}^{T}+\int_{-T}^{T}\frac{e^{i\omega t}\Delta'_z(\gamma+i\omega,\alpha)}{t\Delta^2(\gamma+i\omega,\alpha)}\mathrm{d}\omega.$$

Since $|\Delta(\gamma+i\omega,\alpha)|\to\infty$ as $|\omega|\to\infty$ and $\left|e^{i\omega t}/t\right|=1/t$ for all $t>0$, we conclude that

$$q(t,\alpha)=\frac{e^{\gamma t}}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}\frac{e^{i\omega t}\Delta'_z(\gamma+i\omega,\alpha)}{t\Delta^2(\gamma+i\omega,\alpha)}\mathrm{d}\omega\quad\text{for } \alpha\in[\alpha_1,\alpha_0] \text{ and } t>0.$$

Hence

$$|q(t,\alpha)| \le \frac{e^{\gamma t}}{2\pi t}\left(1+\alpha\sum_{j=1}^{n} a_j r_j e^{-r_j\gamma}\right)\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{|\Delta(\gamma+i\omega,\alpha)|^2}.$$

It is clear that

$$|\Delta(\gamma+i\omega,\alpha)| \ge |\gamma+i\omega| - \alpha_0\sum_{j=1}^{n} a_j e^{-r_j\gamma} = \sqrt{\gamma^2+\omega^2} - \alpha_0\sum_{j=1}^{n} a_j e^{-r_j\gamma} \ge \frac{\omega}{2}$$

if $\omega \ge \omega_0 = 2\alpha_0\sum_{j=1}^{n} a_j e^{-r_j\gamma}$. Therefore

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{|\Delta(\gamma+i\omega,\alpha)|^2} &= 2\int_{0}^{\infty}\frac{\mathrm{d}\omega}{|\Delta(\gamma+i\omega,\alpha)|^2}\\ &\le 2\left\{\int_{0}^{\omega_0}\frac{\mathrm{d}\omega}{|\Delta(\gamma+i\omega,\alpha)|^2}+\int_{\omega_0}^{\infty}\frac{4}{\omega^2}\mathrm{d}\omega\right\}\\ &= \frac{8}{\omega_0}+2\int_{0}^{\omega_0}\frac{\mathrm{d}\omega}{|\Delta(\gamma+i\omega,\alpha)|^2}.\end{aligned}$$

Thus for $t \ge r_1$ and $\alpha \in [\alpha_1,\alpha_0]$, $|q(t,\alpha)| < L_1(\alpha)\, e^{\gamma t}$, where $L_1(\alpha)$ is defined by (2.1).

We also need an estimate for $t \in (0, r_1)$. It is clear from (1.2)-(1.3) that

$$|X(t,\alpha)| \le 1 + \left(\alpha_0\sum_{j=1}^{n} a_j\right)\int_0^t |X(s,\alpha)|\,\mathrm{d}s, \quad t \in [0,r_1],\ \alpha \in [\alpha_1,\alpha_0],$$

thus $|X(t)| \le \exp\left(\alpha_0 r_1 \sum_{j=1}^{n} a_j\right)$ for $t \in [0, r_1]$ and $\alpha \in [\alpha_1,\alpha_0]$ by Gronwall's Lemma. From (4.2) we see that

$$|p(t,\alpha)| \le \frac{2}{|\Delta_z'(\lambda,\alpha)|} \text{ for } t \in (0,r_1) \text{ and } \alpha \in [\alpha_1,\alpha_0],$$

and this upper bound is finite by (H1). Hence $|q(t,\alpha)| \le |X(t)| + |p(t,\alpha)| < L_2(\alpha)\, e^{\gamma t}$ for $t \in (0,r_1)$ and $\alpha \in [\alpha_1,\alpha_0]$ with $L_2(\alpha)$ defined by (2.2). □

Proof of Theorem 2.1. Proposition 4.2 implies that $\int_0^\infty |q(t,\alpha)|\,\mathrm{d}t$ is bounded on $[\alpha_1,\alpha_0]$. Therefore Theorem 2.1 follows directly from the facts that $X(t,\alpha) = p(t,\alpha) + q(t,\alpha)$, $t \ge -r_1$, $\alpha \in [\alpha_1,\alpha_0]$, and that the same limit holds for

$$(\alpha_0-\alpha)\int_0^\infty |p(t,\alpha)|\,\mathrm{d}t,$$

see Proposition 4.1. □

Note that if $n=2$, $r_1=1$, $r_2=0$, $a_1>a_2\geq 0$ and hypothesis (H1) is satisfied, then

$$a_1\cos\nu_0+a_2=0 \quad\text{and}\quad \sin^2\nu_0=1-\cos^2\nu_0=1-\frac{a_2^2}{a_1^2}.$$

Hence Theorem 2.1 implies

$$\begin{aligned}\lim_{\alpha\to\alpha_0-}(\alpha_0-\alpha)\int_0^\infty|X(t,\alpha)|\,\mathrm{d}t &= \frac{4\sqrt{1-2\alpha_0a_1\cos\nu_0+\alpha_0^2a_1^2}}{\pi\alpha_0\left(a_1\sin\nu_0\right)^2}\\ &= \frac{4\sqrt{1+2\alpha_0a_2+\alpha_0^2a_1^2}}{\pi\alpha_0\left(a_1^2-a_2^2\right)}.\end{aligned}$$

Proof of Theorem 2.2. The upper bound for $\int_0^\infty|X(t,\alpha)|\,\mathrm{d}t$ in the theorem is simply the sum of the upper bounds for $\int_0^\infty|p(t,\alpha)|\,\mathrm{d}t$ and $\int_0^\infty|q(t,\alpha)|\,\mathrm{d}t$ given by Proposition 4.1 and Proposition 4.2, respectively. $\square$

As we have already mentioned, the upper bound given by the Theorem 2.2 is sharp for parameters close to the critical value α_0: the upper estimate multiplied by $(\alpha_0-\alpha)$ has the same limit at α_0- as function $(\alpha_0-\alpha)\int_0^\infty|X(t,\alpha)|\,\mathrm{d}t$. Indeed, as

$$-\mu=\mu'(\xi)(\alpha_0-\alpha) \text{ with some } \xi\in(\alpha,\alpha_0) \text{ and } \lim_{x\to0-}\frac{-x}{1-e^x}=1,$$

we see that

$$\begin{aligned}&\lim_{\alpha\to\alpha_0-}(\alpha_0-\alpha)\left\{\frac{2\nu e^{-\frac{\mu}{\nu}\pi}\left(1+e^{-\frac{\mu}{\nu}\pi}\right)}{|\Delta_z'(\lambda,\alpha)|\left(\mu^2+\nu^2\right)\left(1-e^{\frac{\mu}{\nu}\pi}\right)}+\frac{\max\{L_1,L_2\}}{-\gamma}\right\}\\ &\quad=\lim_{\alpha\to\alpha_0-}\frac{2\nu^2e^{-\frac{\mu}{\nu}\pi}\left(1+e^{-\frac{\mu}{\nu}\pi}\right)}{|\Delta_z'(\lambda,\alpha)|\left(\mu^2+\nu^2\right)\pi}\left(\frac{\alpha_0-\alpha}{-\mu}\right)\left(\frac{-\frac{\mu}{\nu}\pi}{1-e^{\frac{\mu}{\nu}\pi}}\right)\\ &\quad=\frac{4}{|\Delta_z'(i\nu_0,\alpha)|\,\pi}\lim_{\alpha\to\alpha_0-}\frac{1}{\mu'(\alpha)},\end{aligned}$$

which limit is the same as given by Theorem 2.1, see Corollary 3.2.

Proof of Theorem 2.3. Assume that not only hypothesis (H1) but also (H2) is satisfied. In this case $\delta\leq\nu\leq\pi/r_1-\delta$ for all $\alpha\in[\alpha_1,\alpha_0]$, hence $\sin(r_1\nu)>\sin(r_1\delta)$ and $\sin(r_j\nu)>0$ for all $\alpha\in[\alpha_1,\alpha_0]$ and $j\in\{1,\dots,n\}$. In consequence,

$$|\Delta_z'(\lambda,\alpha)|\geq\left|\alpha\sum_{j=1}^n a_jr_je^{-r_j\mu}\sin(r_j\nu)\right|\geq\alpha_1c_2\quad\text{for all }\alpha\in[\alpha_1,\alpha_0] \tag{4.3}$$

with $c_2 = a_1 r_1 \sin(r_1\delta)$. Also, $\mu > \gamma$ and

$$\frac{\alpha - \alpha_0}{\mu} \leq K(\delta) \quad \text{for each } \alpha \in [\alpha_1, \alpha_0]$$

by Corollary 3.3.

With Proposition 4.1, these estimates imply that

$$\begin{aligned}(\alpha_0 - \alpha) \int_0^\infty |p(t,\alpha)| \, \mathrm{d}t &\leq \left(\frac{\alpha_0 - \alpha}{-\mu}\right) \left(\frac{-\frac{\mu}{\nu}\pi}{1 - e^{\frac{\mu}{\nu}\pi}}\right) \frac{2\nu^2 e^{-\frac{\mu}{\nu}\pi}\left(1 + e^{-\frac{\mu}{\nu}\pi}\right)}{\pi \left|\Delta_z'(\lambda,\alpha)\right| (\mu^2 + \nu^2)} \\ &\leq K(\delta) \left(\frac{-\frac{\mu}{\nu}\pi}{1 - e^{\frac{\mu}{\nu}\pi}}\right) \frac{2e^{-\frac{\gamma}{\delta}\pi}\left(1 + e^{-\frac{\gamma}{\delta}\pi}\right)}{\pi \alpha_1 c_2}\end{aligned}$$

for all $\alpha_1 \leq \alpha < \alpha_0$. The function $x \mapsto -x/(1 - e^x)$ is strictly decreasing on $(-\infty, 0)$, hence

$$\frac{-\frac{\mu}{\nu}\pi}{1 - e^{\frac{\mu}{\nu}\pi}} < \frac{-\frac{\gamma}{\delta}\pi}{1 - e^{\frac{\gamma}{\delta}\pi}}$$

and

$$(\alpha_0 - \alpha) \int_0^\infty |p(t,\alpha)| \, \mathrm{d}t \leq K(\delta) \frac{-2\gamma e^{-\frac{\gamma}{\delta}\pi}\left(1 + e^{-\frac{\gamma}{\delta}\pi}\right)}{\alpha_1 c_2 \delta \left(1 - e^{\frac{\gamma}{\delta}\pi}\right)}$$

for all $\alpha_1 \leq \alpha < \alpha_0$.

In addition,

$$L_2(\alpha) = e^{-\gamma r_1} \left(e^{\alpha_0 r_1 \sum_{j=1}^n a_j} + \frac{2}{\left|\Delta_z'(\lambda,\alpha)\right|}\right) \leq e^{-\gamma r_1} \left(e^{\alpha_0 r_1 \sum_{j=1}^n a_j} + \frac{2}{\alpha_1 c_2}\right) = \tilde{L}_2$$

for all $\alpha_1 \leq \alpha < \alpha_0$.

With Proposition 4.2 now we deduce that for all $\alpha_1 \leq \alpha < \alpha_0$,

$$\begin{aligned}(\alpha_0 - \alpha) \int_0^\infty |X(t,\alpha)| \, \mathrm{d}t &\leq (\alpha_0 - \alpha) \int_0^\infty |p(t,\alpha)| \, \mathrm{d}t + (\alpha_0 - \alpha) \int_0^\infty |q(t,\alpha)| \, \mathrm{d}t \\ &\leq K(\delta) \frac{-2\gamma e^{\frac{-\gamma}{\delta}\pi}\left(1 + e^{\frac{-\gamma}{\delta}\pi}\right)}{\alpha_1 c_2 \delta \left(1 - e^{\frac{\gamma}{\delta}\pi}\right)} + \frac{\max\left\{\tilde{L}_1, \tilde{L}_2\right\}(\alpha_0 - \alpha_1)}{-\gamma},\end{aligned}$$

where $\tilde{L}_1 = \sup_{\alpha_1 \leq \alpha \leq \alpha_0} L_1(\alpha)$ and $L_1(\alpha)$ is defined by (2.1). $\square$

5. The proof of Theorem 2.4

For Eq. (2.3) and $\alpha > 1/e$ the eigenvalues are simple and appear in complex conjugate pairs $\left(\lambda_j, \overline{\lambda_j}\right)_{j=0}^\infty$ with

$$\mathrm{Re}\lambda_0 > \mathrm{Re}\lambda_1 > \ldots > \mathrm{Re}\lambda_j \to -\infty \quad (j \to \infty)$$

and

$$\mathrm{Im}\lambda_j \in (2j\pi, (2j+1)\pi) \quad \text{for all } j \geq 0.$$

In coherence with the previous sections, $\lambda = \mu + i\nu$ and $\overline{\lambda} = \mu - i\nu$ denote the leading eigenvalues λ_0 and $\overline{\lambda_0}$. As they are roots of the characteristic function,

$$\mu + \alpha e^{-\mu} \cos \nu = 0 \quad \text{and} \quad \nu - \alpha e^{-\mu} \sin \nu = 0. \tag{5.1}$$

In [10] Krisztin has verified that for $\alpha \in [3/2, \pi/2)$,

$$-\frac{1}{2} < \mu < 0,\ 1.54 < \nu < \frac{\pi}{2},\ |\lambda| > 1,\ \lim_{\alpha \to \frac{\pi}{2}-} \mu = 0 \text{ and } \lim_{\alpha \to \frac{\pi}{2}-} \nu = \frac{\pi}{2},$$

moreover the real parts of the remaining eigenvalues are smaller than -1.

With equations (5.1) Proposition 3.1 implies

$$\begin{aligned} \mu'(\alpha) &= \frac{\alpha \left(e^{-\mu} \sin \nu\right)^2 - \left(1 - \alpha e^{-\mu} \cos \nu\right) e^{-\mu} \cos \nu}{\left(1 - \alpha e^{-\mu} \cos \nu\right)^2 + \left(\alpha e^{-\mu} \sin \nu\right)^2} \\ &= \frac{1}{\alpha} \frac{\nu^2 + (1+\mu)\mu}{(1+\mu)^2 + \nu^2}, \quad \alpha \in [3/2, \pi/2]. \end{aligned} \tag{5.2}$$

For all $\alpha \in [3/2, \pi/2]$, $\nu^2 > 1$ and $-0.25 < (1+\mu)\mu \leq 0$, hence $0 < \mu'(\alpha) < 1/\alpha$. As $-\mu(\alpha) = \mu'(\xi)(\pi/2 - \alpha)$ for all $\alpha \in [3/2, \pi/2)$ with some $\xi \in (\alpha, \pi/2)$, we obtain that

$$-\mu(\alpha) < \frac{1}{\xi}\left(\frac{\pi}{2} - \frac{3}{2}\right) < \frac{2}{3}\left(\frac{\pi}{2} - \frac{3}{2}\right) < 0.05, \quad \alpha \in [3/2, \pi/2).$$

We mention that numerical approximation yields $\mu > -0.033$ for all $\alpha \in [3/2, \pi/2)$, thus conditions (H1) and (H2) hold with $\gamma = -0.1$, $\delta = 1.54$ and $K = 15$. According to Theorem 2.3, $\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t \leq 29$.

In this section we give a better estimate without using any numerical approximation.

Relation (4.1) and equations (5.1) imply that

$$|\Delta_z'(\lambda, \alpha)| = \sqrt{(1+\mu)^2 + \nu^2} \tag{5.3}$$

and

$$p(t, \alpha) = \frac{2e^{\mu t}}{(1+\mu)^2 + \nu^2}\left[(1+\mu)\cos(\nu t) + \nu \sin(\nu t)\right] \tag{5.4}$$

for $t \geq -1$ and $\alpha \in [3/2, \pi/2]$, see also [10].

Proposition 5.1. *For each $\alpha \in [3/2, \pi/2)$,*

$$\left(\frac{\pi}{2} - \alpha\right) \int_0^\infty |p(t, \alpha)|\,\mathrm{d}t \leq 1.93.$$

Proof. Proposition 4.1 and relation (5.3) implies that for all $\alpha \in [3/2, \pi/2)$ we have

$$\left(\frac{\pi}{2}-\alpha\right)\int_0^\infty |p(t,\alpha)|\,\mathrm{d}t \le \frac{2e^{-\frac{\mu}{\nu}\pi}\left(1+e^{-\frac{\mu}{\nu}\pi}\right)}{\pi\left|\Delta_z'(\lambda,\alpha)\right|}\left(\frac{\nu^2}{\mu^2+\nu^2}\right)\left(\frac{-\frac{\mu}{\nu}\pi}{1-e^{\frac{\mu}{\nu}\pi}}\right)\left(\frac{\frac{\pi}{2}-\alpha}{-\mu}\right)$$
$$\le \frac{2e^{-\frac{\mu}{\nu}\pi}\left(1+e^{-\frac{\mu}{\nu}\pi}\right)}{\pi\sqrt{(1+\mu)^2+\nu^2}}\left(\frac{-\frac{\mu}{\nu}\pi}{1-e^{\frac{\mu}{\nu}\pi}}\right)\frac{1}{\mu'(\xi)}$$

with some $\xi \in (\alpha, \pi/2)$.

As $\mu \in [-0.05, 0]$ and $\nu \in [1, 54, \pi/2]$ for each $\alpha \in [3/2, \pi/2)$, it follows from result (5.2) that

$$0 < \frac{1}{\mu'(\alpha)} \le \alpha \max_{(\mu,\nu)\in[-0.05,0]\times[1,54,\pi/2]}\left(1+\frac{1+\mu}{\nu^2+(1+\mu)\mu}\right), \quad \alpha \in \left[\frac{3}{2},\frac{\pi}{2}\right).$$

The expression on the right hand side is strictly decreasing in $\nu \in [1, 54, \pi/2]$. In addition, as

$$\frac{\partial}{\partial\mu}\left(\frac{1+\mu}{\nu^2+(1+\mu)\mu}\right) = \frac{\nu^2-(\mu+1)^2}{\left(\nu^2+(1+\mu)\mu\right)^2} > 0$$

for $(\mu,\nu) \in [-0.05, 0] \times [1, 54, \pi/2]$, it is strictly increasing in $\mu \in [-0.05, 0]$. We deduce that

$$\frac{1}{\mu'(\alpha)} \le \frac{\pi}{2}\left(1+\frac{1}{1.54^2}\right) \quad \text{for all } \alpha \in \left[\frac{3}{2},\frac{\pi}{2}\right).$$

Recall that the function $x \mapsto -x/(1-e^x)$ is strictly decreasing on $(-\infty, 0)$. Hence

$$\left(\frac{\pi}{2}-\alpha\right)\int_0^\infty |p(t,\alpha)|\,\mathrm{d}t \le \frac{2e^{\frac{0.05}{1.54}\pi}\left(1+e^{\frac{0.05}{1.54}\pi}\right)}{\pi\sqrt{0.95^2+1.54^2}}\left(\frac{\frac{0.05}{1.54}\pi}{1-e^{-\frac{0.05}{1.54}\pi}}\right)\frac{\pi}{2}\left(1+\frac{1}{1.54^2}\right)$$

for all $\alpha \in [3/2, \pi/2)$, and this upper bound is smaller than 1.93. □

In order to get a better estimate for $\int_0^\infty |q(t,\alpha)|\,\mathrm{d}t$, we apply an approach different from that of Proposition 4.2. The fact that there is only one delay is crucial here.

We use the discrete Lyapunov functional V of Mallet-Paret and Sell introduced in [12]. $V(\varphi)$ counts the sign changes of $\varphi \in C \setminus \{0\}$ if it is an odd number or infinity, otherwise $V(\varphi)$ is the number of sign changes plus one. Then $V(\varphi) \in \{1, 3, \ldots\} \cup \{\infty\}$. The map $t \mapsto V(x_t)$ is monotone nonincreasing along solutions of Eq. (2.3). In addition, V is upper semi-continuous: for each $\varphi \in C \setminus \{0\}$ and $(\varphi_n)_0^\infty \subset C \setminus \{0\}$ with $\varphi_n \to \varphi$ as $n \to \infty$, $V(\varphi) \le \liminf_{n\to\infty} V(\varphi_n)$.

Proposition 5.2. *For each $\alpha \in [3/2, \pi/2]$ and $t \ge 1$, $V(q_t) \ge 3$, that is q has at least two sign changes on each subinterval $[t-1, t]$ of $[0, \infty)$.*

Proof. Suppose for contradiction that $V(q_s) = 1$ for some $s \ge 1$. Then the monotone property of V gives that $V(q_t) = 1$ for all $t \ge s$. By a result of Cao [3] (see also Arino [1]) and by the Gronwall-Bellmann inequality there exist $C_1 > 0$, $C_2 > 0$

and t_0 such that

$$C_1 \|q_t\| \le \|q_{t+1}\| \le C_2 \|q_t\| \quad \text{for all } t > t_0. \tag{5.5}$$

For $n \ge 1$ define $y^n : [-n, \infty) \to \mathbb{R}$ by $y^n(t) = q(n+t) / \|q_n\|$. Then

$$\dot{y}^n(t) = -\alpha y^n(t-1) \quad \text{for } t > -n+1,$$

and

$$C_1 \|y_t^n\| \le \left\|y_{t+1}^n\right\| \le C_2 \|y_t^n\| \quad \text{if } n+t > t_0.$$

The Arzelà−Ascoli Theorem can be applied to find a subsequence $(y^{n_k})_{k=1}^{\infty}$ and a C^1-function $y : \mathbb{R} \to \mathbb{R}$ so that $y^{n_k}(t) \to y(t)$, $\dot{y}^{n_k}(t) \to y(t)$ as $k \to \infty$ uniformly on compact subintervals of $\mathbb{R}$, moreover

$$\dot{y}(t) = -\alpha y(t-1) \quad \text{for } t \in \mathbb{R}.$$

As Q is closed, $y_0 = \lim_{k\to\infty} y_0^{n_k} \in Q$ with $\|y_0\| = 1$. In addition, $C_1 \|y_t\| \le \|y_{t+1}\| \le C_2 \|y_t\|$ for all $t \in \mathbb{R}$. Hence for all $n \in \{0, 1, 2, \ldots\}$,

$$\|y_{-n}\| \le C_1^{-1} \|y_{-n+1}\| \le C_1^{-2} \|y_{-n+2}\| \le \ldots \le C_1^{-n} \|y_0\| \le \left(1 + C_1^{-1}\right)^n \|y_0\|.$$

Let $t \le 0$ be arbitrary and choose integer n so that $-(n+1) < t \le -n$ holds. Then

$$\|y_t\| \le \left\|y_{-(n+1)}\right\| + \|y_{-n}\| \le \left(1 + C_1^{-1}\right) \|y_{-n}\| \le \left(1 + C_1^{-1}\right)^{n+1} \|y_0\|.$$

As

$$\left(1 + C_1^{-1}\right)^n = e^{n \ln\left(1 + C_1^{-1}\right)} \le e^{-t \ln\left(1 + C_1^{-1}\right)},$$

we conclude that $\|y_t\| \le Ae^{-Bt}$ for all $t \le 0$ with $A = \left(1 + C_1^{-1}\right) \|y_0\| > 0$ and $B = \ln\left(1 + C_1^{-1}\right) > 0$.

Choose $c > B$ so that $\mathrm{Re} z \ne -c$ for all roots of the characteristic function. The space C has the decomposition $C = \hat{P} \oplus \hat{Q}$, where $\hat{P}$ is the realified generalized eigenspace of the generator of the semigroup $(T(t))_{t \ge 0}$ associated with the eigenvalues having real parts greater than $-c$, and $\hat{Q}$ is the realified generalized eigenspace associated with the rest of the spectrum. By [9], there is $M > 0$ so that

$$\|T(t)\varphi\| \le Me^{-ct} \|\varphi\| \quad \text{for all } t \ge 0 \text{ and } \varphi \in \hat{Q}.$$

Let $t \leq 0$ and $\sigma \leq t$. Then

$$\begin{aligned}\left\| Pr_{\hat{Q}} y_t \right\| &= \left\| T(t-\sigma) Pr_{\hat{Q}} y_\sigma \right\| \\ &\leq M e^{-c(t-\sigma)} \left\| Pr_{\hat{Q}} y_\sigma \right\| \\ &\leq M \left\| Pr_{\hat{Q}} \right\| e^{-c(t-\sigma)} \| y_\sigma \| \\ &\leq M \left\| Pr_{\hat{Q}} \right\| e^{-c(t-\sigma)} A e^{-B\sigma} \\ &= M \left\| Pr_{\hat{Q}} \right\| e^{-ct} A e^{-(B-c)\sigma} \to 0\end{aligned}$$

as $\sigma \to -\infty$. It follows that $Pr_{\hat{Q}} y_t = 0$ and $y_t \in \hat{P}$ for all $t \leq 0$. If subspace $\hat{P}$ is trivial, i.e. there are no eigenvalues with real parts greater than $-c$, then the previous result implies $y_0 = 0$, a contradiction to $\|y_0\| = 1$. Otherwise

$$y(t) = \sum_{j=0}^{n} A_j e^{\mathrm{Re}\lambda_j t} \cos(\mathrm{Im}\lambda_j t + B_j), \quad t \leq 0,$$

for some real numbers $A_0, B_0, \ldots, A_n, B_n$ and integer $N > 0$ so that

$$\lambda_0 = \lambda, \overline{\lambda_0} = \overline{\lambda}, \lambda_1, \overline{\lambda_1}, \ldots, \lambda_N, \overline{\lambda_N}$$

are the eigenvalues with real parts greater than $-c$. The upper semi-continuity of V and $V(q_t) = 1$, $t \geq s$, combined yield $V(y_t) = 1$ for all $t \in \mathbb{R}$. As $\mathrm{Im}\lambda_j \in (2j\pi, (2j+1)\pi)$, $j \geq 0$, it follows that $A_1 = A_2 = \ldots = A_N = 0$. This means that $y_t \in P$ for all $t \leq 0$. In particular $y_0 \in P$, a contradiction to $y_0 \in Q \setminus \{0\}$. This completes the proof. □

We can use this result to give an explicit estimate for the growth of q on $[-1, \infty)$. In the next proposition $\lfloor r \rfloor$ denotes the integer part of the positive real number r.

Proposition 5.3. *For each $\alpha \in [3/2, \pi/2]$ and $t \geq -1$,*

$$|q(t)| \leq \left(\frac{\alpha}{2}\right)^k \bar{q}, \quad \textit{where} \quad \bar{q} = \sup_{-1 \leq s \leq \frac{1}{2}} |q(s)| \ \textit{and}\ k = \left\lfloor \frac{2}{3}(t+1) \right\rfloor.$$

Proof. The statement is clearly true for $-1 \leq t < 1/2$.

It is enough to show that if

$$t_0 \geq \frac{1}{2} \quad \text{and} \quad \sup_{s \in \left[t_0 - \frac{3}{2}, t_0\right]} |q(s)| \leq m \text{ for some } m > 0,$$

then

$$|q(t)| \leq \frac{\alpha}{2} m \quad \text{for all } t \in \left[t_0, t_0 + \frac{3}{2}\right],$$

so we confirm this latter assertion.

For all $t_0 \geq 1/2$ and $t \in [t_0, t_0 + 3/2]$, there exists $z \in [t - 1/2, t + 1/2]$ with $q(z) = 0$, see Proposition 5.2. Hence for $t \in [t_0, t_0 + 3/2]$,

$$\begin{aligned} |q(t)| &= |q(t) - q(z)| = \alpha \left| \int_{z-1}^{t-1} q(s)\, \mathrm{d}s \right| \\ &\leq \alpha |t - z| \sup_{s \in \left[t - \frac{3}{2}, t - \frac{1}{2}\right]} |q(s)| \\ &\leq \frac{\alpha}{2} \sup_{s \in \left[t - \frac{3}{2}, t - \frac{1}{2}\right]} |q(s)|. \end{aligned}$$

It follows that

$$\sup_{t \in \left[t_0, t_0 + \frac{1}{2}\right]} |q(t)| \leq \frac{\alpha}{2} m,$$

$$\sup_{t \in \left[t_0 + \frac{1}{2}, t_0 + 1\right]} |q(t)| \leq \frac{\alpha}{2} \sup_{t \in \left[t_0 - 1, t_0 + \frac{1}{2}\right]} |q(t)| \leq \frac{\alpha}{2} \max\left\{m, \frac{\alpha}{2} m\right\} = \frac{\alpha}{2} m$$

and

$$\sup_{t \in \left[t_0 + 1, t_0 + \frac{3}{2}\right]} |q(t)| \leq \frac{\alpha}{2} \sup_{t \in \left[t_0 - \frac{1}{2}, t_0 + 1\right]} |q(t)| \leq \frac{\alpha}{2} \max\left\{m, \frac{\alpha}{2} m\right\} = \frac{\alpha}{2} m.$$

□

The previous statement shows we need an upper bound for $\bar{q}$.

Proposition 5.4. *For all* $\alpha \in [3/2, \pi/2]$, $\bar{q} = \sup_{-1 \leq s \leq 1/2} |q(s)| \leq 1$.

Proof. Set $\alpha \in [3/2, \pi/2]$ arbitrarily. Differentiating (5.4) we get

$$\begin{aligned} p'(t) &= \frac{2e^{\mu t}}{(1+\mu)^2 + \nu^2} \left[\left(\mu + \mu^2 + \nu^2\right) \cos(\nu t) - \nu \sin(\nu t) \right] \\ &= \frac{2e^{\mu t}}{(1+\mu)^2 + \nu^2} \nu \cos(\nu t) \left[\frac{\mu + \mu^2 + \nu^2}{\nu} - \tan(\nu t) \right]. \end{aligned}$$

Note that as $1.54 < \nu \leq \pi/2$ and $-0.05 < \mu \leq 0$, we have $\cos(\nu t) > 0$ for all $t \in [-1, \pi/(2\nu))$, and in addition $\mu + \mu^2 + \nu^2 > 0$. It follows that there exists $t_0 \in (0, \pi/(2\nu))$ such that

$$\tan(\nu t_0) = \frac{\mu + \mu^2 + \nu^2}{\nu},$$

p increases on $[-1, t_0]$ and decreases on $[t_0, \pi/(2\nu))$.

Clearly,

$$p(-1) = \frac{2e^{-\mu}}{(1+\mu)^2 + \nu^2} \left[(1+\mu) \cos\nu - \nu \sin\nu \right] < 0$$

as $(1+\mu)\cos\nu < 1$ and $\nu\sin\nu > 1.54 \cdot \sin(1.54) > 1$ for $\mu \in (-0.05, 0]$ and $\nu \in (1.54, \pi/2]$. Therefore

$$|p(-1)| \leq \frac{2e^{-\mu}\nu}{(1+\mu)^2 + \nu^2}.$$

As the right hand side is decreasing in ν for $\nu \in (1.54, \pi/2]$ and it is decreasing in μ for $\mu \in (-0.05, 0]$, we deduce that

$$|p(-1)| \leq \frac{e^{0.05}1.54}{0.95^2 + 1.54^2} < 1.$$

Also we find that

$$0 < p(0) = \frac{2(1+\mu)}{(1+\mu)^2 + \nu^2} \leq \frac{2}{1+\nu^2} < 1,$$

$$\begin{aligned} 0 < p(1) &= \frac{2e^{\mu}}{(1+\mu)^2 + \nu^2}\left[(1+\mu)\cos\nu + \nu\sin\nu\right], \\ &\leq \frac{2\nu}{(1+\mu)^2 + \nu^2} \leq \frac{\pi}{0.95^2 + 1.54^2} < 1 \end{aligned}$$

and

$$p(t_0) = \frac{2e^{\mu t_0}}{(1+\mu)^2 + \nu^2}\left[(1+\mu) + \frac{\mu + \mu^2 + \nu^2}{\nu}\nu\right]\cos(\nu t_0) = 2e^{\mu t_0}\cos(\nu t_0) \leq 2.$$

Clearly, $p(t) \in (-1, 1)$ for all $t \in [-1, 0]$. In case $t_0 > 1$ one has $p(t) \in (0, 1)$ for all $t \in [0, 1]$. Otherwise $p(t) \in (0, 2)$ for all $t \in [0, 1]$. Using that $q(t) = -p(t)$ for $-1 \leq t < 0$ and $q(t) = 1 - p(t)$ for $0 \leq t \leq 1$, we obtain that $\bar{q} \leq 1$. $\square$

Now we are able to estimate $\int_0^\infty |q(t,\alpha)|\,\mathrm{d}t$. The bound given by the subsequent proposition is substantially better than bound $3e^2$ presented in paper [10] and the bound given by Proposition 4.2.

Proposition 5.5. *For each* $\alpha \in [3/2, \pi/2]$,

$$\int_0^\infty |q(t,\alpha)|\,\mathrm{d}t \leq \frac{2+\pi}{4-\pi}.$$

Proof. By Propositions 5.3 and 5.4, $\sup_{-1\leq s\leq 1/2} |q(s,\alpha)| \leq 1$ for all $\alpha \in [3/2, \pi/2]$, and $|q(t,\alpha)| \leq (\pi/4)^k$ for all $\alpha \in [3/2, \pi/2]$ and $t \geq 0$, where k is the greatest integer with $k \leq 2(t+1)/3$. Thus

$$\int_0^\infty |q(t,\alpha)|\,\mathrm{d}t \leq \frac{1}{2} + \frac{3}{2}\left\{\frac{\pi}{4} + \left(\frac{\pi}{4}\right)^2 + \left(\frac{\pi}{4}\right)^3 + \ldots\right\} = \frac{2+\pi}{4-\pi}.$$

$\square$

Proof of Theorem 2.4. The statement of the theorem follows directly from Proposition 5.1 and Proposition 5.5. □

6. An example with two delays

Consider the linear equation

$$\dot{x}(t) = -\alpha\left(x(t-1) + x(t-2)\right) \text{ with } 0 < \alpha \leq \frac{\pi}{3\sqrt{3}}.$$

In this case the characteristic function is

$$\Delta(z,\alpha) = z + \alpha\left(e^{-z} + e^{-2z}\right). \tag{6.1}$$

For the eigenvalues $z = u + iv$,

$$u + \alpha\left(e^{-u}\cos v + e^{-2u}\cos(2v)\right) = 0 \tag{6.2}$$

and

$$v - \alpha\left(e^{-u}\sin v + e^{-2u}\sin(2v)\right) = 0. \tag{6.3}$$

There are purely imaginary eigenvalues $\pm i\pi/3$ for $\alpha = \pi/\left(3\sqrt{3}\right)$.

We examine the location of the eigenvalues on the complex plane for $0 < \alpha \leq \pi/\left(3\sqrt{3}\right)$.

Suppose $0 < \alpha \leq \pi/\left(3\sqrt{3}\right)$ and $z = u + iv$ is an eigenvalue with $u \leq 0$ and $|v| \geq \pi$. Then it follows from (6.3) that

$$\pi \leq |v| \leq \frac{\pi}{3\sqrt{3}}\left(e^{-u}|\sin v| + e^{-2u}|\sin(2v)|\right) \leq \frac{2\pi}{3\sqrt{3}}e^{-2u},$$

that is

$$u \leq -\frac{1}{2}\ln\frac{3\sqrt{3}}{2} < -0.4.$$

It is also known that there is exactly one pair of eigenvalues $\left(\lambda, \overline{\lambda}\right)$ in the subset $\{z \in \mathbb{C} : |\mathrm{Im} z| < \pi\}$ of the complex plane, see [13]. Numerical approximation gives that for each $\alpha \in \left[0.58, \pi/\left(3\sqrt{3}\right)\right)$, there is an eigenvalue $\lambda = \lambda(\alpha)$ in the rectangle

$$\{z \in \mathbb{C} : \ -0.02 < \mathrm{Re} z < 0 \text{ and } 1.03 \leq \mathrm{Im} z \leq 1.05\},$$

see Fig. 1.

Hence hypotheses (H1) and (H2) are satisfied with $\alpha_0 = \pi/\left(3\sqrt{3}\right)$, $\alpha_1 = 0.58$, $\gamma = -0.04$, $\delta = .52$ and $K = 22$. Theorem 2.1 now gives that

$$\lim_{\alpha \to \frac{\pi}{3\sqrt{3}}-}\left(\frac{\pi}{3\sqrt{3}} - \alpha\right)\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t = \frac{8\sqrt{\left(1 + \frac{\pi}{6\sqrt{3}}\right)^2 + \frac{\pi^2}{4}}}{\sqrt{3}\pi^2} \approx 0.95.$$

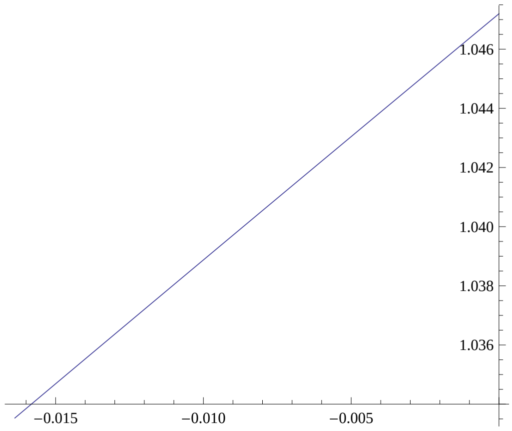

FIGURE 6.1. Curve $\left[0.58, \pi/\left(3\sqrt{3}\right)\right] \ni \alpha \mapsto \lambda(\alpha) \in \mathbb{C}$

In addition, Theorem 2.3 can be applied, and we get

$$\left(\frac{\pi}{3\sqrt{3}} - \alpha\right) \int_0^\infty |X(t, \alpha)| \, dt \le 55$$

for all $0.58 \le \alpha < \pi/\left(3\sqrt{3}\right)$ by Theorem 2.3.

7. Applications

In this section, let $n \ge 1$ be an integer, $r_1, \dots, r_n$ be real numbers with $r_1 > r_2 > \dots > r_n \ge 0$, $a_1, \dots, a_n$ be positive numbers, $\alpha_0 > \alpha_1 > 0$. In case $n = 1$, we assume $r_1 > 0$.

In papers [7, 8] Győri, Hartung and Turi has studied linear equations with perturbed time lags in the form

$$\dot{x}(t) = -\sum_{j=1}^{n} a_j x\left(t - r_j - \eta_j(t)\right), \tag{7.1}$$

where $\eta_j : [0, \infty) \to [0, \infty)$ are piecewise continuous functions for all $j \in \{1, \dots, n\}$. Suppose that the 0 solution of Eq. (7.1) with $\eta_j \equiv 0$, $j \in \{1, \dots, n\}$, is asymptotically stable. Then using estimates on the integral of the fundamental solution, they gave sufficient conditions on η_j to guarantee the asymptotic stability of the 0 solution of Eq. (7.1). We cite the following result as an example: if

$$\sum_{j=1}^{n} a_j \limsup_{t\to\infty} |\eta_j(t)| < \frac{1}{\left(\sum_{j=1}^{n} a_j\right) \int_0^\infty |X(t)| \, dt},$$

then the trivial solution of Eq. (7.1) is asymptotically stable.

We present two more applications in connection with the nonlinear equation

$$\dot{y}(t) = -\alpha \sum_{j=1}^{n} a_j y(t - r_j) + f(y_t, \alpha) \tag{7.2}$$

with parameter $\alpha \in [\alpha_1, \alpha_0)$ and a continuously differentiable nonlinear function $f : C \times [\alpha_1, \alpha_0) \to \mathbb{R}$. We assume the following hypothesis:

(H3): To each $\alpha \in [\alpha_1, \alpha_0)$ there correspond positive constants $k_1 = k_1(\alpha)$ and $\delta_1 = \delta_1(\alpha)$ such that

$$|f(\varphi, \alpha)| \leq k_1(\alpha) \|\varphi\|^2 \quad \text{for all } \alpha \in [\alpha_1, \alpha_0) \text{ and } \varphi \in C \text{ with } \|\varphi\| < \delta_1(\alpha).$$

By the variation-of-constants formula, the solution $y^\varphi : [-1, \infty) \to \mathbb{R}$ of Eq. (7.2) with $y_0^\varphi = \varphi$ satisfies the equation

$$y^\varphi(t) = x^\varphi(t) + \int_0^t X(t-s, \alpha) f(y_s, \alpha) \, \mathrm{d}s \quad \text{for } t \geq 0, \tag{7.3}$$

where $x^\varphi : [-1, \infty) \to \mathbb{R}$ is the solution of the linear variational equation (1.1) with initial segment φ [4, 9]. Using this formula and estimates for the integral of the fundamental solution, the existence of periodic solutions of small amplitude can be excluded for Eq. (7.2).

Theorem 7.1. *Assume that (H1) holds for the linear variational equation* (1.1)*, and $f : \mathbb{R} \times [\alpha_1, \alpha_0) \to \mathbb{R}$ satisfies (H3). If $\alpha \in [\alpha_1, \alpha_0)$ and $p : \mathbb{R} \to \mathbb{R}$ is a nonconstant periodic solution of Eq.* (7.2)*, then*

$$\max_{t \in \mathbb{R}} |p(t)| \geq \min \left\{ \frac{1}{k_1(\alpha) \int_0^\infty |X(t, \alpha)| \, dt}, \delta_1(\alpha) \right\}.$$

Proof. Fix $\alpha \in [\alpha_1, \alpha_0)$ and let $p : \mathbb{R} \to \mathbb{R}$ be a periodic solution of Eq. (7.2) with

$$A = \max_{t \in \mathbb{R}} |p(t)| < \min \left\{ \frac{1}{k_1 \int_0^\infty |X(t, \alpha)| \, \mathrm{d}t}, \delta_1 \right\}.$$

Then clearly $|f(p_t, \alpha)| \leq k_1 \|p_t\|^2 \leq k_1 A^2$ for all $t \in \mathbb{R}$.

By the periodicity of p, there is a sequence $(t_n)_0^\infty$ in $[0, \infty)$ so that $t_n \to \infty$ as $n \to \infty$ and $|p_{t_n}| = A$ for all $n \geq 0$. Then the variation-of-constants formula (7.3) yields

$$A = |p(t_n)| \leq |x^{p_0}(t_n)| + k_1 A^2 \int_0^{t_n} |X(t_n - s, \alpha)| \, \mathrm{d}s \quad \text{for all } n \geq 0.$$

Hypothesis (H1) implies $\lim_{t \to \infty} x^{p_0}(t) = 0$ [9]. Letting $n \to \infty$, we obtain that

$$A \leq k_1 A^2 \int_0^\infty |X(s, \alpha)| \, \mathrm{d}s.$$

Hence

$$A \geq \frac{1}{k_1 \int_0^\infty |X(t,\alpha)| \,\mathrm{d}t},$$

a contradiction to our initial assumption. $\square$

For each $\varphi \in C$, solution $x^\varphi : [-1,\infty) \to \mathbb{R}$ of Eq. (1.1) can be expressed in form

$$x^\varphi(t) = X(t,\alpha)\varphi(0) - \alpha \sum_{j=0}^{n} a_j \int_{-r_j}^{0} X(t - r_j - s, \alpha)\varphi(s)\,\mathrm{d}s \quad \text{for } t \geq 0, \tag{7.4}$$

see [9]. Hence using equality (4.2) and Proposition 4.2, one can determine explicit constants $M = M(\alpha) \geq 1$ and $b = b(\alpha) > 0$ such that $\|x_t^\varphi\| \leq M\|\varphi\| e^{-bt}$ for each $t \geq 0$.

As a second application, we estimate the stability region of the 0 solution of Eq. (7.2) in phase space C.

Theorem 7.2. *Assume that (H1) holds for the linear variational equation* (1.1)*, and $f : \mathbb{R} \times [\alpha_1, \alpha_0) \to \mathbb{R}$ satisfies (H3). In addition, suppose that $\delta_1(\alpha)$ in (H3) is chosen so small that*

$$2k_1(\alpha)\delta_1(\alpha) \int_0^\infty |X(t,\alpha)| \,dt < 1 \quad \text{for all } \alpha \in [\alpha_1, \alpha_0).$$

If $\alpha \in [\alpha_1, \alpha_0)$ and $\varphi \in C$ with $\|\varphi\| \leq \delta_1/(2M(\alpha))$, then for the solution $y^\varphi : [-1,\infty) \to \mathbb{R}$ of Eq. (7.2) *with initial function φ, $\lim_{t\to\infty} y^\varphi(t) = 0$.*

Proof. Set $\alpha \in [\alpha_1, \alpha_0)$ and $\varphi \in C$ with $\|\varphi\| < \delta_1/(2M)$.

First we claim that $|y^\varphi(t)| < \delta_1$ for all $t \geq -1$. Suppose for contradiction that there is a minimal $t_0 > 0$ with $|y^\varphi(t_0)| = \delta_1$. Then the variation-of-constants formula (7.3) gives that

$$\begin{aligned} \delta_1 = |y^\varphi(t_0)| &\leq |x^\varphi(t_0)| + \int_0^{t_0} |X(t_0 - s, \alpha) f(y_s, \alpha)| \,\mathrm{d}s \\ &\leq M\|\varphi\| e^{-bt} + k_1\delta_1^2 \int_0^{t_0} |X(t_0 - s, \alpha)| \,\mathrm{d}s, \end{aligned}$$

where $x^\varphi : [-1,\infty) \to \mathbb{R}$ is the solution of the linear variational equation (1.1) with $x_0^\varphi = \varphi$. It follows that

$$\delta_1 \left(1 - k_1\delta_1 \int_0^\infty |X(t,\alpha)| \,\mathrm{d}t\right) \leq M\|\varphi\| e^{-bt} \leq M\|\varphi\|.$$

As $1 - k_1\delta_1 \int_0^\infty |X(t,\alpha)| \,\mathrm{d}t > 1/2$, we obtain that $\|\varphi\| > \delta_1/(2M)$, which contradicts our initial assumption. So $|y^\varphi(t)| < \delta_1$ for all $t \geq -1$.

Set $A = \limsup_{t\to\infty} |y^\varphi(t)|$. The previous step implies that $A \in [0, \delta_1]$. We have to show that $A = 0$.

Suppose for contradiction that $A = \delta_1$. Choose a sequence $(t_n)_0^\infty$ in $[0, \infty)$ so that $t_n \to \infty$ and $|y^\varphi(t_n)| \to \delta_1$ as $n \to \infty$. Then by the variation-of-constants formula (7.3),

$$|y^\varphi(t_n)| \le |x^\varphi(t_n)| + k_1 \delta_1^2 \int_0^{t_n} |X(t_n - s, \alpha)| \, \mathrm{d}s \quad \text{for all } n \ge 0.$$

Clearly, $\lim_{t\to\infty} x^\varphi(t) = 0$. Letting $n \to \infty$, we obtain that

$$\delta_1 \le k_1 \delta_1^2 \int_0^\infty |X(s, \alpha)| \, \mathrm{d}s,$$

that is $1 \le k_1 \delta_1 \int_0^\infty |X(s, \alpha)| \, \mathrm{d}s$, which contradicts the choice of δ_1. So $A < \delta_1$.

Suppose that $A \in (0, \delta_1)$. Set $\varepsilon \in \left(0, \sqrt{2} - 1\right)$ so small that inequality $(1 + \varepsilon) A < \delta_1$ also holds. As before, there exists a sequence $(t_n)_0^\infty$ in $[0, \infty)$ so that $t_n \to \infty$ and $|y^\varphi(t_n)| \to A$ as $n \to \infty$. In addition, there is a threshold number $T > 0$ so that $y^\varphi(t) < (1 + \varepsilon) A$ for $t \ge T - 1$. Then hypothesis (H3) implies that $|f(y_t^\varphi, \alpha)| \le k_1 \delta_1^2$ for all $0 \le t < T$, and $|f(y_t^\varphi, \alpha)| \le k_1 (1 + \varepsilon)^2 A^2$ for all $t \ge T$. By the variation-of-constants formula,

$$\begin{aligned} |y^\varphi(t_n)| \le\ & |x^\varphi(t_n)| + k_1 \delta_1^2 \int_0^T |X(t_n - s, \alpha)| \, \mathrm{d}s + \\ & k_1 (1 + \varepsilon)^2 A^2 \int_T^{t_n} |X(t_n - s, \alpha)| \, \mathrm{d}s \\ =\ & |x^\varphi(t_n)| + k_1 \delta_1^2 \int_{t_n - T}^{t_n} |X(s, \alpha)| \, \mathrm{d}s + k_1 (1 + \varepsilon)^2 A^2 \int_0^{t_n - T} |X(s, \alpha)| \, \mathrm{d}s \end{aligned}$$

for all sufficiently large n. Since $\int_0^\infty |X(s, \alpha)| \, \mathrm{d}s < \infty$ for all $\alpha \in [\alpha_1, \alpha_0)$, we see that $\int_{t_n - T}^{t_n} |X(s, \alpha)| \, \mathrm{d}s \to 0$ as $n \to \infty$. So letting $n \to \infty$, we conclude that

$$A \le k_1 (1 + \varepsilon)^2 A^2 \int_0^\infty |X(s, \alpha)| \, \mathrm{d}s,$$

that is

$$A \ge \frac{1}{k_1 (1 + \varepsilon)^2 \int_0^\infty |X(t, \alpha)| \, \mathrm{d}t} > \frac{2}{(1 + \varepsilon)^2} \delta_1.$$

This result contradicts the fact that $A < \delta_1$.

It follows that $A = \limsup_{t\to\infty} |y^\varphi(t)| = 0$, and the proof is complete. □

As an example, consider Wright's equation

$$\dot{y}(t) = -\alpha \left(e^{y(t-1)} - 1\right) \tag{7.5}$$

with parameter $\alpha \in [3/2, \pi/2)$. Now the linear variational equation along the 0 solution is Eq. (2.3). For the fundamental solution X of Eq. (2.3), Theorem 2.4

gives that

$$(\pi/2-\alpha)\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t \le 2.36$$

for all $\alpha \in [3/2, \pi/2)$.

Eq. (7.5) can be written in form (7.2) with

$$f : C \times \left[\frac{3}{2}, \frac{\pi}{2}\right) \ni (\varphi,\alpha) \mapsto -\alpha\left(e^{\varphi(-1)} - 1 - \varphi(-1)\right) \in \mathbb{R}.$$

For $\alpha \in [3/2, \pi/2)$ and $\varphi \in C$ with $\|\varphi\| < \delta_1 < 3$,

$$|f(\varphi,\alpha)| \le \alpha \sum_{k=2}^{\infty} \frac{\|\varphi\|^k}{k!} \le \alpha \frac{\|\varphi\|^2}{2} \sum_{k=0}^{\infty} \left(\frac{\delta_1}{3}\right)^k \le \frac{\pi}{4}\frac{3}{3-\delta_1}\|\varphi\|^2.$$

So (H3) holds with

$$\delta_1(\alpha) \in (0,3) \quad \text{and} \quad k_1(\alpha) \ge \frac{\pi}{4}\frac{3}{3-\delta_1}.$$

Set $\delta_1(\alpha) \equiv 0.04$ and $k_1(\alpha) \equiv 0.8$ for example. Then it follows from Theorem 7.1 that whenever $p : \mathbb{R} \to \mathbb{R}$ is a nonconstant periodic solution of Eq. (7.2) for some $\alpha \in [3/2, \pi/2)$, then

$$\max_{t\in\mathbb{R}} |p(t)| \ge 0.52\left(\frac{\pi}{2} - \alpha\right).$$

We note that it is verified in [2] that

$$\max_{t\in\mathbb{R}} p(t) \ge \frac{2}{\pi}\left(\frac{\pi}{2}-\alpha\right) \approx 0.64\left(\frac{\pi}{2}-\alpha\right).$$

To apply Theorem 7.2, we decrease $\delta_1(\alpha)$ so that

$$2k_1(\alpha)\,\delta_1(\alpha)\int_0^\infty |X(t,\alpha)|\,\mathrm{d}t \le 2\cdot 0.8\cdot \delta_1(\alpha)\,\frac{2.36}{\frac{\pi}{2}-\alpha} < 1$$

for all $\alpha \in [3/2, \pi/2)$; we set $\delta_1(\alpha) = 0.26(\pi/2-\alpha)$ for each $\alpha \in [3/2, \pi/2)$.

At last we need constants $M(\alpha) > 1$ and $b(\alpha) > 0$ such that for the solution $x^\varphi : [-1,\infty) \to \mathbb{R}$ of Eq. (2.3), $\|x_t^\varphi\| \le M(\alpha)\|\varphi\| e^{-b(\alpha)t}$ for all $t \ge 0$.

Recall that for $\alpha \in [3/2, \pi/2)$,

$$-0.05 < \mu < 0 \quad \text{and} \quad 1.54 < \nu < \frac{\pi}{2},$$

where μ and ν are the real and imaginary parts of the leading eigenvalue $\lambda = \lambda(\alpha)$ in the spectrum of the generator of the semigroup defined by the solution operators of Eq.(2.3). Hence by (5.4),

$$|p(t,\alpha)| \le \frac{2(1+\mu+\nu)}{(1+\mu)^2+\nu^2}e^{\mu t} \le \frac{2\left(1+\frac{\pi}{2}\right)}{0.95^2+1.54^2}e^{\mu t} < 1.58e^{\mu t}$$

for all $\alpha \in [3/2, \pi/2)$ and $t \geq -1$. In addition, Proposition 5.3 and Proposition 5.4 imply

$$|q(t,\alpha)| \leq \left(\frac{\pi}{4}\right)^{\left\lfloor \frac{2}{3}(t+1) \right\rfloor} \leq \left(\frac{\pi}{4}\right)^{\frac{2}{3}t - \frac{1}{3}} = \sqrt[3]{\frac{4}{\pi}} e^{\frac{2}{3}\ln\left(\frac{\pi}{4}\right)\cdot t} \leq 1.09 e^{-0.16t}$$

for all $\alpha \in [3/2, \pi/2)$ and $t \geq 0$. It follows that

$$|X(t,\alpha)| \leq |p(t,\alpha)| + |q(t,\alpha)| \leq (1.58 + 1.09)e^{\mu t} = 2.67 e^{\mu t}$$

for all $\alpha \in [3/2, \pi/2)$ and $t \geq 0$. Recall that $X(t,\alpha) = 0$ for all $t \in [-1, 0)$, hence the previous estimate can be used for all $t \geq -1$.

According to formula (7.4),

$$\begin{aligned} |x^{\varphi}(t)| &\leq |X(t,\alpha)| \, |\varphi(0)| + \alpha \int_{-1}^{0} |X(t-1-s,\alpha)| \, |\varphi(s)| \, \mathrm{d}s \\ &\leq 2.67 \left(1 + \frac{\pi}{2} e^{0.05}\right) \|\varphi\| \, e^{\mu t} \leq 7.08 \, \|\varphi\| \, e^{\mu t} \end{aligned}$$

for each $t \geq 0$. Therefore we may set $M(\alpha) = 7.08$ and $b(\alpha) = -\mu(\alpha)$ for $\alpha \in [3/2, \pi/2)$.

According to Theorem 7.2, if $\varphi \in C$ with $\|\varphi\| \leq 0.018(\pi/2 - \alpha)$, then for the solution $y^{\varphi} : [-1, \infty) \to \mathbb{R}$ of Eq. (7.2) with initial function φ, $\lim_{t\to\infty} y^{\varphi}(t) = 0$.

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