EXISTENCE RESULTS FOR FIRST ORDER IMPULSIVE SEMILINEAR EVOLUTION INCLUSIONS

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Abstract

In this paper the concepts of lower mild and upper mild solutions combined with a fixed point theorem for condensing maps and the semigroup theory are used to investigate the existence of mild solutions for first order impulsive semilinear evolution inclusions.

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1. INTRODUCTION

This paper is concerned with the existence of mild solutions for the impulsive semilinear evolution inclusion of the form:

$$y' - A(t)y \in F(t, y), \quad t \in J = [0, b], \; t \neq t_k, \quad k = 1, \dots, m, \tag{1.1}$$

$$y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.2}$$

$$y(0) = a, \tag{1.3}$$

where $F : J \times E \longrightarrow 2^E$ is a closed, bounded and convex valued multivalued map, $a \in E, A(t), t \in J$ a linear closed operator from a dense subspace $D(A(t))$ of E into

E and E a real ordered Banach space with the norm $|\cdot|$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $I_k \in C(E, E)$ $(k = 1, 2, \ldots, m)$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively.

Impulsive differential equations have become more important in recent years in some mathematical models of real world phenomena, specially in the biological or medical domain (see the monographs of Lakshmikantham et al [11], and Samoilenko and Perestyuk [17], and the papers of Agur et al [1], Erbe et al [5], Goldbeter et al [6], Kirane and Rogovchenko [10], Liu et al [13] and Liu and Zhang [14]).

This paper will be divided into three sections. In Section 2 we will recall briefly some basic definitions and preliminary facts from multivalued analysis which will be used throughout Section 3. In Section 3 we establish two existence theorems for (1.1)-(1.3). The first result of this section is new even for the multivalued problem without impulses (see [3]); i.e., $I_k(y) = y$ for $k = 1, \ldots, m$. Our approach is based on the concepts of upper mild and lower mild solutions combined with a fixed point theorem for condensing maps due to Martelli [15] and the semigroup theory [16].

The notions of lower mild and upper mild solutions for differential equations in ordered Banach spaces can be found in the book of Heikkila and Lakshmikantham [8].

In our results we do not assume any type of monotonicity condition on $I_k, k = 1, \ldots, m$ which is usually the situation in the literature, see for instance, [5], [10] and [13].

2. PRELIMINARIES

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel.

Let the Banach space E be partially ordered by a cone P of E, i.e., $u \leq v$ if and only if $v - u \in P$. Let $K := \{u \in C(J, E); u(t) \geq 0 \text{ for all } t \in J\}$. Then K is a cone in the space $C(J, E)$, and $C(J, E)$ is partially ordered by K, that is $u \leq v$ if and only if $v - u \in K$: i.e., $u(t) \leq v(t)$ for all $t \in J$. The properties of the cone and the partial order may found in the monograph by Guo and Lakshmikantham [7].

$B(E)$ denotes the Banach space of bounded linear operators from E into E.

A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. For properties of the Bochner integral, we refer to Yosida [18].

$L^1(J, E)$ denotes the Banach space of functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt \quad \text{for all} \quad y \in L^1(J, E).$$

Let $(X, |\cdot|)$ be a Banach space. A multivalued map $G : X \longrightarrow 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semicontinuous (u.s.c.) on X if for each $x_* \in X$ the set $G(x_*)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x_*)$, there exists an open neighbourhood V of x_* such that $G(V) \subseteq B$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In the following $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G : J \longrightarrow BCC(E)$ is said to be measurable if for each $x \in E$ the function $Y : J \longrightarrow \mathbb{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

is measurable. An upper semi-continuous map $G : X \longrightarrow 2^X$ is said to be condensing [2] if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompacteness [2]. We remark that a completely continuous multivalued map is the easiest example of a condensing map. For more details on multivalued maps see the books of Deimling [4] and Hu and Papageorgiou [9].

Definition 2.1 *A multivalued map $F : J \times E \longrightarrow 2^E$ is said to be an L^1-Carathéodory if*

(i) $t \longmapsto F(t, y)$ *is measurable for each* $y \in E$;

(ii) $y \longmapsto F(t, y)$ *is upper semicontinuous for almost all* $t \in J$;

(iii) For each $\rho > 0$, *there exists* $h_\rho \in L^1(J, \mathbb{R}_+)$ *such that* $\|F(t, y)\| = \sup\{|v| : v \in F(t, y)\} \le h_\rho(t)$ *for all* $|y| \le \rho$ *and for almost all* $t \in J$.

In order to define the solution of (1.1)-(1.3) we shall consider the following space

$$\Omega = \{y \ : \ J = [0, b] \longrightarrow E : y_k \in C(J_k, E), k = 0, \ldots, m \text{ and there exist}$$

$$y(t_k^-) \text{ and } y(t_k^+), k = 1, \ldots, m \text{ with } y(t_k^-) = y(t_k)\}\}$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_\infty, k = 0, \ldots, m\},$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}],\ k = 0, \ldots, m$.

So let us start by defining what we mean by a solution of problem (1.1)-(1.3).

Definition 2.2 *A function $y \in \Omega$ is said to be a mild solution of (1.1)-(1.3) (see [16]) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y(t))$ a.e. on J_k, and*

$$y(t) = \begin{cases} T(t,0)a + \displaystyle\int_0^t T(t,s)v(s)ds, & t \in J_0 \\ I_k(y(t_k^-)) + \displaystyle\int_{t_k}^t T(t,s)v(s)ds, & t \in J_k. \end{cases}$$

Now we introduce the notions of lower mild and upper mild solutions for the problem (1.1)-(1.3), which are the basic tools in the approach that follows.

Definition 2.3 *A function $\underline{y} \in \Omega$ is said to be a lower mild solution of (1.1)-(1.3) if there exists a function $v_1 \in L^1(J,E)$ such that $v_1(t) \in F(t,\underline{y}(t))$ a.e. on J_k, and*

$$\underline{y}(t) \leq \begin{cases} T(t,0)a + \displaystyle\int_0^t T(t,s)v_1(s)ds, & t \in J_0 \\ I_k(\underline{y}(t_k^-)) + \displaystyle\int_{t_k}^t T(t,s)v_1(s)ds, & t \in J_k. \end{cases}$$

Similarly a function $\overline{y} \in \Omega$ is said to be an upper mild solution of (1.1)-(1.3) if there exists a function $v_2 \in L^1(J,E)$ such that $v_2(t) \in F(t,\overline{y}(t))$ a.e. on J_k, and

$$\overline{y}(t) \geq \begin{cases} T(t,0)a + \displaystyle\int_0^t T(t,s)v_2(s)ds, & t \in J_0 \\ I_k(\overline{y}(t_k^-)) + \displaystyle\int_{t_k}^t T(t,s)v_2(s)ds, & t \in J_k. \end{cases}$$

For the multivalued map F and for each $y \in C(J_k,E)$ we define $S^1_{F,y}$ by

$$S^1_{F,y} = \{v \in L^1(J_k,E) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in J_k\}.$$

Our main result is based on the following:

Lemma 2.1 *[12]. Let I be a compact real interval and X be a Banach space. Let $F : I \times X \longrightarrow BCC(X); (t,y) \to F(t,y)$ measurable with respect to t for any $y \in X$ and u.s.c. with respect to y for almost each $t \in I$ and $S^1_{F,y} \neq \emptyset$ for any $y \in C(I,X)$ and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$, then the operator*

$$\Gamma \circ S^1_F : C(I,X) \longrightarrow BCC(C(I,X)), \ y \longmapsto (\Gamma \circ S^1_F)(y) := \Gamma(S^1_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 2.2 *[15]. Let $G : X \longrightarrow BCC(X)$ be a condensing map. If the set*

$$\mathcal{M} := \{y \in X : \lambda y \in G(y) \ \ \textit{for some} \ \ \lambda > 1\}$$

is bounded, then G has a fixed point.

3. MAIN RESULTS

We are now in a position to state and prove our first existence result for problem (1.1)-(1.3).

Theorem 3.1 *Let $t_0 = 0$, $t_{m+1} = b$ and assume that $F : J \times E \longrightarrow BCC(E)$ is an L^1-Carathéodory multivalued map. In addition suppose:*

(H1) *$A(t)$, $t \in J$ is continuous such that*

$$A(t)y = \lim_{h\to 0^+} \frac{T(t+h,t)y - y}{h}, \quad y \in D(A(t)),$$

where $T(t,s) \in B(E)$ for each $(t,s) \in \gamma := \{(t,s); 0 \le s \le t \le b\}$, satisfying

(i) *$T(t,t) = I$ (I is the identity operator in E),*

(ii) *$T(t,s)T(s,r) = T(t,r)$ for $0 \le r \le s \le t \le b$,*

(iii) *the mapping $(t,s) \longmapsto T(t,s)y$ is strongly continuous in γ for each $y \in E$;*

Suppose that $|T(t,s)| \le M$ for $(t,s) \in \gamma$;

(H2) *$F : J \times E \longrightarrow BCC(E)$ is an L^1-Carathéodory multivalued map and for each fixed $y \in C(J_k, E)$ the set*

$$S^1_{F,y} = \Big\{ v \in L^1(J_k, E) : \ v(t) \in F(t, y(t)) \text{ for a.e. } \ t \in J_k \Big\}$$

is nonempty;

(H3) *There exist $\underline{y}$, $\overline{y}$ repectively lower mild and upper mild solution for (1.1)-(1.3) such that $\underline{y} \le \overline{y}$;*

(H4) $\underline{y}(t_k^+) \le \min\limits_{[\underline{y}(t_k^-), \overline{y}(t_k^-)]} I_k(y) \le \max\limits_{[\underline{y}(t_k^-), \overline{y}(t_k^-)]} I_k(y) \le \overline{y}(t_k^+)$, $k = 1, \dots, m$.

(H5) *$T(t,s)$ is order-preserving for all $(t,s) \in \gamma$;*

(H6) *For each bounded set $B \subseteq C(J_k, E)$ and for each $t \in J_k$ the set*

$$\Big\{ \int_{t_k}^{t} T(t,s)v(s)ds : v \in S^1_{F,B} \Big\}$$

is relatively compact in E, where $S^1_{F,B} = \cup\{S^1_{F,y} : y \in B\}$ and $k = 0, \dots, m$.

Then the problem (1.1)-(1.3) has at least one mild solution $y \in \Omega$ with

$$\underline{y}(t) \leq y(t) \leq \overline{y}(t) \quad \textit{for all} \;\; t \in J.$$

Remark 3.1 *(i) If $dimE < \infty$, then for each $y \in C(J_k, E)$, $S^1_{F,y} \neq \emptyset$ (see Lasota and Opial [12]).*
(ii) If $dimE = \infty$ and $y \in C(J_k, E)$ the set $S^1_{F,y}$ is nonempty if and only if the function $Y : J \longrightarrow \mathbb{R}$ defined by

$$Y(t) := \inf\{|v| : v \in F(t, y)\}$$

belongs to $L^1(J, \mathbb{R})$ (see Hu and Papageorgiou [9]).
(iii) If $T(t,s)$, $(t,s) \in \gamma$ is completely continuous then (H6) is automatically satisfied.

Proof. The proof is given in several steps.

Step 1. Consider the problem (1.1)-(1.3) on $J_0 := [0, t_1]$

$$y' - A(t)y \in F(t, y(t)), \quad \text{a.e.} \;\; t \in J_0, \tag{3.1}$$

$$y(0) = a. \tag{3.2}$$

We transform this problem into a fixed point problem. Let $\tau : C(J_0, E) \longrightarrow C(J_0, E)$ be the truncation operator defined by

$$(\tau y)(t) = \begin{cases} \underline{y}(t), & \text{if} \;\; y < \underline{y}(t); \\ y(t), & \text{if} \;\; \underline{y}(t) \leq y \leq \overline{y}(t); \\ \overline{y}(t)), & \text{if} \;\; \overline{y}(t) < y. \end{cases}$$

Consider the modified problem

$$y' - A(t)y \in F(t, (\tau y)(t)), \quad \text{a.e.} \;\; t \in J_0, \tag{3.3}$$

$$y(0) = a. \tag{3.2}$$

Set

$$C_0(J_0, E) := \{y \in C(J_0, E) : y(0) = a\}.$$

A solution to (3.3)-(3.2) is a fixed point of the operator $G : C_0(J_0, E) \longrightarrow 2^{C_0(J_0,E)}$ defined by

$$G(y) := \left\{ h \in C_0(J_0, E) : h(t) = T(t,0)a + \int_0^t T(t,s)v(s)ds : \; v \in \tilde{S}^1_{F,\tau y} \right\}$$

where

$$\tilde{S}^1_{F,\tau y} = \{v \in S^1_{F,\tau y} : v(t) \geq v_1(t) \text{ a.e. on } \; A_1 \;\; \text{and} \;\; v(t) \leq v_2(t) \text{ a.e. on } \; A_2\}$$

and

$$S^1_{F,\tau y} = \{v \in L^1(J_0, E) : v(t) \in F(t, (\tau y)(t)) \text{ for a.e. } t \in J_0\},$$
$$A_1 = \{t \in J : y(t) < \underline{y}(t) \leq \overline{y}(t)\}, \quad A_2 = \{t \in J : \underline{y}(t) \leq \overline{y}(t) < y(t)\}.$$

Remark 3.2 *For each* $y \in C(J,E)$ *the set* $\tilde{S}^1_{F,\tau y}$ *is nonempty. Indeed, by (H2) there exists* $v \in S^1_{F,y}$*. Set*

$$w = v_1\chi_{A_1} + v_2\chi_{A_2} + v\chi_{A_3},$$

where

$$A_3 = \{t \in J : \underline{y}(t) \le y(t) \le \overline{y}(t)\}.$$

Then by decomposability $w \in \tilde{S}^1_{F,\tau y}$*.*

We shall show that G satisfies the assumptions of Lemma 2.2.

Claim 1: *$G(y)$ is convex for each $y \in C_0(J_0,E)$.*

This is obvious since $\tilde{S}^1_{F,\tau y}$ is convex (because F has convex values).

Claim 2: *G sends bounded sets into relatively compact sets in $C_0(J_0,E)$.*

This is a consequence of the boundedness of $T(t,s),\ (t,s) \in \gamma$ and the L^1-Carathédory character of F. As a consequence of Claim 2 together with the Ascoli-Arzela theorem we can conclude that $G : C_0(J_0,E) \longrightarrow 2^{C_0(J_0,E)}$ is a compact multivalued map, and therefore, a condensing map.

Claim 3: *G has a closed graph.*

Let $y_n \longrightarrow y_*,\ h_n \in G(y_n)$ and $h_n \longrightarrow h_*$. We shall prove that $h_* \in G(y_*)$. $h_n \in G(y_n)$ means that there exists $v_n \in \tilde{S}_{F,\tau y_n}$ such that

$$h_n(t) = T(t,0)a + \int_0^t T(t,s)v_n(s)ds, \quad t \in J_0.$$

We must prove that there exists $v_* \in \tilde{S}^1_{F,\tau y_*}$ such that

$$h_*(t) = T(t,0)a + \int_0^t T(t,s)v_*(s)ds, \quad t \in J_0.$$

Consider the linear continuous operator $\Gamma : L^1(J_0,E) \longrightarrow C(J_0,E)$ defined by

$$(\Gamma v)(t) = \int_0^t T(t,s)v(s)ds.$$

We have

$$\|(h_n - T(t,0)a) - (h_* - T(t,0)a)\|_\infty \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty.$$

From Lemma 2.1, it follows that $\Gamma \circ \tilde{S}^1_F$ is a closed graph operator.
Also from the definition of Γ we have that

$$h_n(t) - T(t,0)a \in \Gamma(\tilde{S}^1_{F,\tau y_n}).$$

Since $y_n \longrightarrow y_*$, it follows from Lemma 2.1 that

$$h_*(t) = T(t,0)a + \int_0^t T(t,s)v_*(s)ds, \quad t \in J_0$$

for some $v_* \in \tilde{S}^1_{F,\tau y_*}$.

Claim 4: *Now, we show that the set*

$$\mathcal{M} := \{y \in C_0(J_0, E) : \lambda y \in G(y) \quad \textit{for some} \quad \lambda > 1\}$$

is bounded.

Let $y \in \mathcal{M}$, then $\lambda y \in G(y)$ for some $\lambda > 1$. Thus there exists $v \in \tilde{S}^1_{F,\tau y}$ such that

$$y(t) = \lambda^{-1}T(t,0)a + \lambda^{-1}\int_0^t T(t,s)v(s)ds, \quad t \in J_0.$$

Thus

$$|y(t)| \leq M|a| + M\int_0^t |v(s)|ds, \quad t \in J_0.$$

From the definition of τ there exists $\phi \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t,(\tau y)(t))\| = \sup\{|v| : v \in F(t,(\tau y)(t))\} \leq \phi(t) \text{ for each } y \in C(J,E).$$

Thus we obtain

$$\|y\|_\infty \leq M|a| + M\|\phi\|_{L^1}.$$

This shows that $\mathcal{M}$ is bounded. Hence, Lemma 2.2 applies and G has a fixed point which is a mild solution to problem (3.3)-(3.2).

Claim 5: *We shall show that the solution y of (3.3)-(3.2) satisfies*

$$\underline{y}(t) \leq y(t) \leq \overline{y}(t) \quad \textit{for all} \quad t \in J_0.$$

Let y be a solution to (3.3)-(3.2). We prove that

$$\underline{y}(t) \leq y(t) \quad \text{for all} \quad t \in J_0.$$

Suppose not. Then there exist $e_1, e_2 \in J_0,\ e_1 < e_2$ such that $\underline{y}(e_1) = y(e_1)$ and

$$\underline{y}(t) > y(t) \quad \text{for all} \quad t \in (e_1, e_2).$$

In view of the definition of τ one has

$$y(t) \in T(t,e_1)y(e_1) + \int_{e_1}^t T(t,s)F(s,\underline{y}(s))ds \quad \text{a.e. on} \quad (e_1,e_2).$$

Thus there exists $v(t) \in F(t, \underline{y}(t))$ a.e. on (e_1, e_2) with $v(t) \geq v_1(t)$ a.e. on (e_1, e_2) such that

$$y(t) = T(t, e_1)y(e_1) + \int_{e_1}^{t} T(t,s)v(s)ds \quad t \in (e_1, e_2).$$

Since $\underline{y}$ is a lower mild solution to (1.1)-(1.2), then

$$\underline{y}(t) - T(t, e_1)\underline{y}(e_1) \leq \int_{e_1}^{t} T(t,s)v_1(s)ds, \quad t \in (e_1, e_2).$$

Since $y(e_1) = \underline{y}(e_1)$ and $v(t) \geq v_1(t)$, it follows that

$$\underline{y}(t) \leq y(t) \quad \text{for all} \quad t \in (e_1, e_2)$$

which is a contradiction since $y(t) < \underline{y}(t)$ for all $t \in (e_1, e_2)$. Consequently

$$\underline{y}(t) \leq y(t) \quad \text{for all} \quad t \in J_0.$$

Analogously, we can prove that

$$y(t) \leq \overline{y}(t) \quad \text{for all} \quad t \in J_0.$$

This shows that the problem (3.1)-(3.2) has a mild solution in the interval $[\underline{y}, \overline{y}]$. Since $\tau(y) = y$ for all $y \in [\underline{y}, \overline{y}]$, then y is a mild solution to (1.1)-(1.3). Denote this solution by y_0.

Step 2. Consider now the following problem on $J_1 := [t_1, t_2]$

$$y' - A(t)y \in F(t, y(t)), \quad \text{a.e.} \quad t \in J_1, \tag{3.4}$$

$$y(t_1^+) = I_1(y_0(t_1^-)), \tag{3.5}$$

and the modified problem

$$y'(t) \in F(t, (\tau y)(t)), \quad \text{a.e.} \quad t \in J_1, \tag{3.6}$$

$$y(t_1^+) = I_1(y_0(t_1^-)). \tag{3.5}$$

Since $y_0(t_1^-) \in [\underline{y}(t_1^-), \overline{y}(t_1^-)]$, then (H4) implies that

$$\underline{y}(t_1^+) \leq I_1(y_0(t_1^-)) \leq \overline{y}(t_1^+),$$

that is

$$\underline{y}(t_1^+) \leq y(t_1^+)) \leq \overline{y}(t_1^+).$$

Using the same reasoning as that used for problem (3.1)-(3.2) we can conclude to the existence of at least one mild solution y to (3.6)-(3.5).

We now show that this solution satisfies

$$\underline{y}(t) \le y(t) \le \overline{y}(t) \quad \text{for all} \quad t \in J_1.$$

We first show that

$$\underline{y}(t) \le y(t) \quad \text{on} \quad J_1.$$

Assume this is false, then since $y(t_1^+) \ge \underline{y}(t_1^+)$, there exist $e_3, e_4 \in J_1$ with $e_3 < e_4$ such that $y(e_3) = \underline{y}(e_3)$ and $y(t) < \underline{y}(t)$ on (e_3, e_4).

Consequently,

$$y(t) - T(e_3, t)y(e_3) = \int_{e_3}^{t} T(t,s)v(s)ds, \quad t \in (e_3, e_4),$$

where $v(t) \in F(t, \underline{y}(t))$ a.e. on J_1 with $v(t) \ge v_1(t)$ a.e. on (e_3, e_4).

Since $\underline{y}$ is a lower mild solution to (1.1)-(1.3), then

$$\underline{y}(t) - T(e_3, t)\underline{y}(e_3) \le \int_{e_3}^{t} v_1(s)ds, \quad t \in (e_3, e_4).$$

It follows that

$$\underline{y}(t) \le y(t) \quad \text{on} \quad (e_3, e_4),$$

which is a contradiction. Similarly we can show that $y(t) \le \overline{y}(t)$ on J_1. Hence y is a solution of (1.1)-(1.3) on J_1. Denote this by y_1.

Step 3. Continue this process and construct solutions $y_k \in C(J_k, E), k = 2, \ldots, m$ to

$$y' - A(t) \in F(t, (\tau y)(t)), \quad \text{a.e.} \quad t \in J_k, \tag{3.7}$$

$$y(t_k^+) = I_k(y(t_k^-)), \tag{3.8}$$

with $\underline{y}(t) \le y_k(t) \le \overline{y}(t), t \in J_k := [t_k, t_{k+1}]$. Then

$$y(t) = \begin{cases} y_0(t), & t \in [0, t_1] \\ y_1(t), & t \in (t_1, t_2] \\ . & \\ . & \\ . & \\ y_{m-1}(t), & t \in (t_{m-1}, t_m] \\ y_m(t), & t \in (t_m, b] \end{cases}$$

is a mild solution of (1.1)-(1.3). □

Using the same reasoning as that used in the proof of Theorem 3.1 we can obtain the following result.

Theorem 3.2 *Let $t_0 = 0,\ t_{m+1} = b$ and assume that $F : J \times E \longrightarrow BCC(E)$ is an L^1-Carathéodory multivalued map. In addition to (H1), (H2) and (H6) suppose that the following hypotheses hold*

(H7) there exist functions $\{r_k\}_{k=0}^{k=m}$ and $\{s_k\}_{k=0}^{k=m}$ with $r_k, s_k \in C(J_k, E)$, $s_0(0) \le a \le r_0(0)$ and $s_k(t) \le r_k(t)$ for $t \in J_k, k = 0, \ldots, m$ and

$$\begin{aligned} s_{k+1}(t_{k+1}^+) &\le \min_{[s_k(t_{k+1}^-), r_k(t_{k+1}^-)]} I_{k+1}(y) \\ &\le \max_{[s_k(t_{k+1}^-), r_k(t_{k+1}^-)]} I_{k+1}(y) \le r_{k+1}(t_{k+1}^+),\ \ k = 0, \ldots, m-1. \end{aligned}$$

(H8) there exist $v_{1,k}, v_{2,k} \in L^1(J_k, E)$, with $v_{1,k}(t) \in F(t, s_k(t)), v_{2,k}(t) \in F(t, r_k(t))$ a.e. on J_k such that for each $k = 0, \ldots, m$

$$\int_{z_k}^{t} T(t,s)v_{1,k}(s)ds \ge s_k(t) - s_k(z_k),$$

$$\int_{z_k}^{t} T(t,s)v_{2,k}(s)ds \ge r_k(t) - r_k(z_k),\ \ \text{with } t, z_k, \in J_k.$$

Then the problem (1.1)-(1.3) has at least one mild solution.

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