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Observation problems posed for the Klein-Gordon equation

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Abstract. Transversal vibrations $u = u(x,t)$ of a string of length l with fixed ends are considered, where u is governed by the Klein-Gordon equation

$$u_{tt}(x,t) = a^2 u_{xx}(x,t) + cu(x,t), \qquad (x,t) \in [0,l] \times \mathbb{R}, \quad a > 0, \ c < 0.$$

Sufficient conditions are obtained that guarantee the solvability of each of four observation problems with given state functions f, g at two distinct time instants $-\infty < t_1 < t_2 < \infty$. The essential conditions are the following: smoothness of f, g as elements of a corresponding subspace $D^{s+i}(0,l)$ (introduced in [2]) of a Sobolev space $H^{s+i}(0,l)$, where $i = 1,2$ depending on the type of the observation problem, and the representability of $t_2 - t_1$ as a rational multiple of $\frac{2l}{a}$. The reconstruction of the unknown initial data $(u(x,0), u_t(x,0))$ as the elements of $D^{s+1}(0,l) \times D^s(0,l)$ are given by means of the method of Fourier expansions.

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1. BACKGROUND AND KNOWN RESULTS

In control theory - which is closely related to the subject of this paper - numerous monographies and articles dealt with the accessability of a final state (position and speed) of oscillations (in particular string oscillations) in the time interval $0 \le t \le T < \infty$; see for example, [1] - [10]. Although, only the short communication [11] dealt with observability of the string oscillations on the interval $0 \le x \le l$, and it treated just the case when the observation instants t_1 and t_2 are small, namely $0 \le t_1 \le t_2 \le \frac{2l}{a}$, where a is the speed of the wave propagation. Furthermore, it is assumed in [11] that the initial data are known on some subinterval $[h_1, h_2] \subset [0,l]$. We reconstruate the initial data in each of the four observation problems related to the Klein-Gordon equation for arbitrary large t_1 and t_2. Our preassumptions are only that $(t_2 - t_1)\frac{a}{2l}$ is rational and the given state functions are smooth enough. The cases $f, g \in D^s$ with arbitrary $s \in \mathbb{R}$ are also admitted.

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Let $\Omega = \{(x,t) \ : \ 0 < x < l, \ t \in \mathbb{R}\}$. Consider the problem (at first in the classical sense) of the vibrating $[0,l]$ string with fixed ends when there is an elastic withdrawing force proportional to the transversal deflection $u(x,t)$ of the point x of the string at the instant denoted by t. This phenomenon is described by the Klein-Gordon equation as follows:

$$(1) \qquad u_{tt}(x,t) = a^2 u_{xx}(x,t) + cu(x,t), \qquad (x,t) \in \overline{\Omega}, \ a,c \in \mathbb{R}, \ 0 < a, \ 0 > c,$$

with the initial conditions

$$(2) \qquad u(x,0) = \varphi(x), \ u_t(x,0) = \psi(x), \qquad 0 \le x \le l,$$

and the homogeneous boundary conditions of the first kind

$$(3) \qquad u(0,t) = 0, \ u(l,t) = 0, \qquad t \in \mathbb{R}.$$

We recall, that the function u is said to be a classical solution of this problem, if $u \in C^2(\overline{\Omega})$ and conditions $(1) - (3)$ are satisfied.

It is well known that if

$$(4) \quad \varphi \in C^2[0,l], \psi \in C^1[0,l] \quad \text{and} \quad \varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = \psi(0) = \psi(l) = 0,$$

then the Fourier method gives the classical solution u of the problem $(1) - (3)$ posed for the Klein-Gordon equation, which is of the following form:

$$(5) \qquad u(x,t) = \sum_{n=1}^{\infty} [\alpha_n \cos(t\omega_n) + \beta_n \sin(t\omega_n)] \sin(\frac{n\pi}{l}x), \qquad (x,t) \in \overline{\Omega},$$

where

$$(6) \qquad \omega_n = \sqrt{(\frac{n\pi}{l}a)^2 - c}, \qquad n \in \mathbb{N},$$

$$(7) \quad \varphi(x) = u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin(\frac{n\pi}{l}x) \Rightarrow \alpha_n = \frac{2}{l}\int_0^l \varphi(x)\sin(\frac{n\pi}{l}x)dx, \qquad n \in \mathbb{N},$$

$$(8) \ \psi(x) = u_t(x,0) = \sum_{n=1}^{\infty} \omega_n \, \beta_n \sin(\frac{n\pi}{l}x) \Rightarrow \beta_n = \frac{1}{\omega_n}\frac{2}{l}\int_0^l \psi(x)\sin(\frac{n\pi}{l}x)dx, \quad n \in \mathbb{N}.$$

The uniqueness of the solution is a consequence of the law of conservation of energy.

To have a wider class of functions for φ, ψ and f, g, we shall consider certain generalized solutions of the problem $(1) - (3)$. Namely, by using the suggestions of the referee, we introduce the spaces $D^s(0,l)$, $s \in \mathbb{R}$ mentioned in the abstract (see [2]). Given an arbitrary real number s, on the linear span D of the functions $\sin \frac{n\pi}{l}x$, $n = 1, 2, ...$, consider the following Euclidean norm:

$$\left\| \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{l}x) \right\|_s := \left(\sum_{n=1}^{\infty} n^{2s} |c_n|^2 \right)^{\frac{1}{2}}.$$

Completing D with respect to this norm, we obtain a Hilbert space D^s. One can readily verify that for $s \geq 0$, D^s is a closed subspace of the Sobolev space $H^s(0,l)$, namely

$$D^s = \{u \in H^s(0,l) \ : \ u^{(2i)}(0) = u^{(2i)}(l) = 0, \ i = 0, 1, ..., [(s-1)/2]\}.$$

If we identify $D^0 = L^2(0,l)$ with its dual, then D^{-s} is the dual space of D^s. Some of the results of [2] (see Section 1.1-1.3) and [10] say that for arbitrary $s \in \mathbb{R}$ with $(\varphi, \psi) \in D^{s+1} \times D^s$ the generalized mixed problem $(1)-(3)$ has a unique solution u satisfying

$$u \in C(\mathbb{R}, D^{s+1}) \cap C^1(\mathbb{R}, D^s) \cap C^2(\mathbb{R}, D^{s-1})$$

given by the Fourier series (5) with coefficients α_n, β_n defined by (7) and (8). Here and below all Fourier expansions for φ, ψ, f, g and u are understood in the spaces $D^s(0,l)$.

2. NEW RESULTS

Definition 1. The observation problem posed for the Klein-Gordon equation is the following. The initial functions φ, ψ are unknown, but such functions $f(x)$ and $g(x)$ are given for which one of the following four conditions holds:

$$u(x,t_1) = f(x), \ u(x,t_2) = g(x), \qquad 0 \leq x \leq l; \tag{9}$$

$$u_t(x,t_1) = f(x), \ u(x,t_2) = g(x), \qquad 0 \leq x \leq l; \tag{10}$$

$$u(x,t_1) = f(x), \ u_t(x,t_2) = g(x), \qquad 0 \leq x \leq l; \tag{11}$$

$$u_t(x,t_1) = f(x), \ u_t(x,t_2) = g(x), \qquad 0 \leq x \leq l. \tag{12}$$

Here u is the solution of the generalized problem $(1)-(3)$, and the given functions f, g are said to be the partial state of the string at distinct time instants t_1 and t_2, $-\infty < t_1 < t_2 < \infty$. Now the problem is to find the initial functions φ, ψ in terms of $f(x)$, $g(x)$.

Theorem 1. *Suppose that*

$$f \in D^{s+2}, \ g \in D^{s+2}, \qquad \textit{where } s \in \mathbb{R}, \tag{13}$$

$$t_2 - t_1 = \frac{p}{q}\frac{2l}{a}, \tag{14}$$

where p, q *are positive integers and they are relative primes. In addition, suppose that*

$$\sin\left((t_2 - t_1)\sqrt{(\frac{n\pi}{l}a)^2 - c}\right) \neq 0, \qquad \forall n \in \mathbb{N}. \tag{15}$$

Then the observation problem $(1) - (3)$ *under condition* (9) *has a unique solution for* $(\varphi, \psi) \in D^{s+1} \times D^s$*. They are represented by their Fourier expansions in the proof below.*

Theorem 2. *Suppose that*

$$f \in D^{s+1}, \ g \in D^{s+2}, \qquad \textit{where } s \in \mathbb{R}, \tag{16}$$

condition (14) *holds and*

$$\cos\left((t_2 - t_1)\sqrt{(\frac{n\pi}{l}a)^2 - c}\right) \neq 0, \qquad \forall n \in \mathbb{N}. \tag{17}$$

Then the observation problem $(1) - (3)$ *under condition* (10) *has a unique solution for* $(\varphi, \psi) \in D^{s+1} \times D^s$*. They are represented by their Fourier expansions in the proof below.*

Theorem 3. *Suppose that*

$$f \in D^{s+2}, \ g \in D^{s+1}, \qquad \textit{where } s \in \mathbb{R}, \tag{18}$$

and conditions (14) *and* (17) *hold. Then the observation problem* $(1) - (3)$ *under condition* (11) *has a unique solution for* $(\varphi, \psi) \in D^{s+1} \times D^s$*. They are represented by their Fourier expansions in the proof below.*

Theorem 4. *Suppose that*

$$f \in D^{s+1}, \ g \in D^{s+1}, \qquad \textit{where } s \in \mathbb{R}, \tag{19}$$

and conditions (14) *and* (15) *hold. Then the observation problem* $(1) - (3)$ *under condition* (12) *has a unique solution for* $(\varphi, \psi) \in D^{s+1} \times D^s$*. They are represented by their Fourier expansions in the proof below.*

3. AUXILIARY RESULTS

Lemma 1. *If condition* (14) *holds, then there exist* $N \in \mathbb{N}$ *and* $m \in \mathbb{R}$ *such that*

$$\frac{1}{|\sin(\omega_n(t_2-t_1))|} < \frac{n}{m}, \qquad \forall n > N.$$

Proof. First, we deal with the denominator of the left-hand side of the inequality

$$\sin(\omega_n(t_2-t_1)) = \sin\left((t_2-t_1)\frac{n\pi}{l}a + (t_2-t_1)\left[\omega_n - \frac{n\pi}{l}a\right]\right) = \tag{20}$$

$$= \sin\left((t_2-t_1)\frac{n\pi}{l}a + (t_2-t_1)\frac{\omega_n^2 - (\frac{n\pi}{l}a)^2}{\omega_n + \frac{n\pi}{l}a}\right) = \sin\left((t_2-t_1)\frac{n\pi}{l}a + (t_2-t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right).$$

It follows from the condition (14) that

$$(t_2-t_1)\frac{n\pi}{l}a = \frac{p}{q}2n\pi,$$

and that it takes on at most q different values (mod 2π) as n varies. Let

$$z_n := (t_2-t_1)\frac{n\pi}{l}a \qquad \text{and} \qquad d_1 := \min_{n,\ \sin z_n \neq 0} \{|\sin(z_n)|\}.$$

Due to the absolute value bars, there is a real number d_2 such that

$$\sin(d_2) = d_1, \qquad 0 < d_2 \leq \frac{\pi}{2}.$$

It is easy to see, that

$$(t_2-t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a} = O(\frac{1}{n}) \quad \text{as } n \to \infty.$$

Therefore, there exist constants $N \in \mathbb{N}$, $m \in \mathbb{R}^+$ such that

$$\frac{\pi m}{2n} < \left|(t_2-t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right| < \frac{d_2}{2} \text{ and } \frac{m}{n} < \sin\left(\frac{d_2}{2}\right), \qquad \forall n > N. \tag{21}$$

So, if $\sin\left((t_2-t_1)\frac{n\pi}{l}a\right) \neq 0$, then

$$\left|\sin\left((t_2-t_1)\frac{n\pi}{l}a + (t_2-t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right)\right| > \left|\sin\left(d_2 - \frac{d_2}{2}\right)\right| = \sin\left(\frac{d_2}{2}\right) > \frac{m}{n},$$

whenever $n > N$, by virtue of (21).

On the other hand, if $\sin\left((t_2-t_1)\frac{n\pi}{l}a\right) = 0$, then

$$\left|\sin\left((t_2-t_1)\frac{n\pi}{l}a + (t_2-t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right)\right| = \left|\sin\left((t_2-t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right)\right| >$$
$$> \frac{2}{\pi}\left|(t_2-t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right| > \frac{m}{n}, \qquad \forall n > N,$$

due to (21) and the inequality

$$\text{(22)} \qquad |\sin t| > \frac{2}{\pi}|t|, \qquad \text{if } 0 < |t| < \frac{\pi}{2}.$$

Combining the two cases just above, we get that

$$\left|\sin\left((t_2-t_1)\frac{n\pi}{l}a + (t_2-t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right)\right| > \frac{m}{n}, \qquad \forall n > N.$$

□

Lemma 2. *If condition* (14) *holds, then there exist* $N \in \mathbb{N}$ *and* $m \in \mathbb{R}$ *such that*

$$\frac{1}{|\cos(\omega_n(t_2-t_1))|} < \frac{n}{m}, \qquad \forall n > N.$$

Proof. Similarly to (20) in the proof of Lemma 1, now we obtain that

$$\cos(\omega_n(t_2-t_1)) = \cos\left((t_2-t_1)\frac{n\pi}{l}a + (t_2-t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right).$$

Let

$$z_n := (t_2-t_1)\frac{n\pi}{l}a \qquad \text{and} \qquad d_1 := \min_{n,\ \cos z_n \neq 0}\{|\cos z_n|\}.$$

Due to the absolute value bars, there is a real number d_2 such that

$$\cos(d_2) = d_1, \qquad 0 \le d_2 < \frac{\pi}{2}.$$

Similarly to (21) in the proof of Lemma 1, there exist constants $N \in \mathbb{N}$ and $m \in \mathbb{R}^+$ such that

$$\text{(23)} \quad \frac{\pi m}{2n} < \left|(t_2-t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right| < \frac{\frac{\pi}{2} - d_2}{2} \ \text{ and } \ \frac{m}{n} < \cos\left(\frac{\frac{\pi}{2}+d_2}{2}\right), \qquad \forall n > N.$$

In this manner, if $\cos\left((t_2-t_1)\frac{n\pi}{l}a\right) \neq 0$, we obtain again that

$$\left|\cos\left((t_2-t_1)\frac{n\pi}{l}a + (t_2-t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right)\right| > \left|\cos\left(d_2 + \frac{\frac{\pi}{2}-d_2}{2}\right)\right| = \cos\left(\frac{\frac{\pi}{2}+d_2}{2}\right) > \frac{m}{n},$$

whenever $n > N$, by virtue of (23).

On the other hand, in the case when $\cos\left((t_2-t_1)\frac{n\pi}{l}a\right) = 0$, we get

$$\left|\cos\left((t_2-t_1)\frac{n\pi}{l}a + (t_2-t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right)\right| = \left|\sin\left(t_2-t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right)\right| >$$
$$> \frac{2}{\pi}\left|(t_2-t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right| > \frac{m}{n}, \qquad \forall n > N,$$

due to (22) and (23).

Combining the two cases just above, we get that

$$\left|\cos\left((t_2-t_1)\frac{n\pi}{l}a+(t_2-t_1)\frac{-c}{\omega_n+\frac{n\pi}{l}a}\right)\right|>\frac{m}{n},\qquad \forall n>N.$$

□

4. PROOFS OF THE THEOREMS 1 − 4

Proof of Theorem 1. Since any of the solutions u of problem (1)−(3) has representation (5) with some coefficients $\alpha_n,\ \beta_n;\ n\in\mathbb{N}$, the observation problem can be reduced to the problem of the appropriate choices of α_n and β_n such that (9) is satisfied. For this reason, we substitute t_1 and t_2 into (5), and use the two conditions in (9). As a result, we get the following necessary conditions for $\alpha_n,\ \beta_n$:

(24) $$f(x)=u(x,t_1)=\sum_{n=1}^{\infty}[\alpha_n\cos(\omega_n t_1)+\beta_n\sin(\omega_n t_1)]\sin(\frac{n\pi}{l}x),\qquad x\in[0,l],$$

(25) $$g(x)=u(x,t_2)=\sum_{n=1}^{\infty}[\alpha_n\cos(\omega_n t_2)+\beta_n\sin(\omega_n t_2)]\sin(\frac{n\pi}{l}x),\qquad x\in[0,l],$$

where ω_n is defined in (6).

The assumption (13) guarantees that the coefficients of the sine Fourier expansions of the functions $f(x),\ g(x)$ are unambiguously determined and comparing these Fourier series with (24) and (25), for $\alpha_n,\ \beta_n$ we get the following conditions:

(26) $$\begin{aligned}\alpha_n\cos(\omega_n t_1)+\beta_n\sin(\omega_n t_1)&=\frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l}x)dx,\qquad n\in\mathbb{N},\\ \alpha_n\cos(\omega_n t_2)+\beta_n\sin(\omega_n t_2)&=\frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi}{l}x)dx,\qquad n\in\mathbb{N}.\end{aligned}$$

The linear system (26) can be uniquely solved for the unknown coefficients α_n and β_n due to assumption (15):

(27) $$\begin{aligned}\alpha_n&=\frac{\sin(\omega_n t_2)\frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l}x)dx-\sin(\omega_n t_1)\frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi}{l}x)dx}{\sin(\omega_n(t_2-t_1))},\\ \beta_n&=\frac{-\cos(\omega_n t_2)\frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l}x)dx+\cos(\omega_n t_1)\frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi}{l}x)dx}{\sin(\omega_n(t_2-t_1))}.\end{aligned}$$

So the unknown initial functions φ and ψ are uniquely determined and found in the form of (7) and (8). It remains to show that $\varphi,\ \psi$ are from the classes $D^{s+1},\ D^s$, respectively, i. e. to show that the following inequality holds:

(28) $$\max\{\|\varphi\|_{s+1}^2,\|\psi\|_s^2\}<\infty.$$

We introduce the following notations for the sake of transparency:

$$D_n := \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l}x)dx,$$
$$E_n := \frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi}{l}x)dx.$$

Since $(f, g) \in D^{s+2} \times D^{s+2}$, we have the following inequality:

$$\sum_{n=1}^{\infty} n^{2s+4} \max\{|D_n|^2, |E_n|^2\} < \infty. \tag{29}$$

By using Lemma 1, for every $n > N$ we get

$$|\alpha_n| = \left|\frac{\sin(\omega_n t_2)D_n - \sin(\omega_n t_1)E_n}{\sin(\omega_n(t_2 - t_1))}\right| < \left|\frac{n}{m}D_n\right| + \left|\frac{n}{m}E_n\right|,$$

$$|\beta_n| = \left|\frac{-\cos(\omega_n t_2)D_n + \cos(\omega_n t_1)E_n}{\sin(\omega_n(t_2 - t_1))}\right| < \left|\frac{n}{m}D_n\right| + \left|\frac{n}{m}E_n\right|,$$

which means that

$$\max\{|\alpha_n|, |\beta_n|\} < c_1 n \max\{|D_n|, |E_n|\} \qquad n \in \mathbb{N}, \tag{30}$$

with a suitable constant c_1.

Let $M \geq 1$ be a constant such that $\omega_n < Mn$, $\forall n \in \mathbb{N}$. Combining (29), (30) and the definition of the norm $\|.\|_s$ we get the desired inequality (28):

$$\max\{\|\varphi\|_{s+1}^2, \|\psi\|_s^2\} = \max\{\sum_{n=1}^{\infty} n^{2s+2}|\alpha_n|^2, \sum_{n=1}^{\infty} n^{2s}|\omega_n\beta_n|^2\} \leq$$
$$\leq \sum_{n=1}^{\infty} M^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\} < c_1^2 M^2 \sum_{n=1}^{\infty} n^{2s+4} \max\{|D_n|^2, |E_n|^2\} < \infty.$$

□

Remark 1. In the classical case when the given state functions are continuously differentiable, according to Theorem 1, the initial functions are also continuously differentiable. More precisely, if

$$u(x, t_1) = f(x) \in C^4[0, l], \quad u(x, t_2) = g(x) \in C^4[0, l], \quad f, g|_{0,l} = f'', g''|_{0,l} = 0,$$

then $f, g \in D^4$ and the observation problem has a unique classical solution

$$u(x, 0) = \varphi(x) \in D^3 \subset C^2, \quad u_t(x, 0) = \psi(x) \in D^2 \subset C^1.$$

Remark 2. Taking into account (20), condition (15) can be written into the following form:

$$\sin\left((t_2-t_1)\omega_n\right) = \sin\left((t_2-t_1)\frac{n\pi}{l}a + (t_2-t_1)\frac{-c}{\sqrt{(\frac{n\pi}{l}a)^2-c}+\frac{n\pi}{l}a}\right) \neq 0 \tag{31}$$

for all $n \in \mathbb{N}$. Analysing the proof of Lemma 1, it is easy to see that the above condition is certainly satisfied for all n large enough, say $n > N$.
If we want to get an easily verifiable condition instead of (15), which is not necessary then

$$(t_2-t_1)\frac{-c}{\sqrt{(\frac{\pi}{l}a)^2-c}+\frac{\pi}{l}a} < \frac{\pi}{q} \tag{32}$$

is such a sufficient condition. We justify this claim as follows. The first term in the argument of the sine function in (31) is either 0 (mod 2π), or its distance is at least $\frac{\pi}{q}$ from its zeroes, and the second term in the argument of the sine function in (31) is positive and monotone decreasing function of n. So, if we assume that the second term is already smaller than $\frac{\pi}{q}$ for $n = 1$, which is actually the case in (32), then condition (31) is satisfied for each $n \geq 1$.

Nevertheless, we can see from this simpler condition (32), that if the parameters $|c|$ and a in equation (1) are such that either c is small or a is great enough, then condition (31) is always satisfied. Similar observations can be made in the following Theorems $2-4$.

Proof of Theorem 2. In an analogous way as in the proof of Theorem 1, now we start with the following equalities:

$$f(x) = u_t(x,t_1) = \sum_{n=1}^{\infty}[-\alpha_n\omega_n\sin(\omega_n t_1) + \beta_n\omega_n\cos(\omega_n t_1)]\sin(\frac{n\pi}{l}x), \qquad x \in [0,l],$$

$$g(x) = u(x,t_2) = \sum_{n=1}^{\infty}[\alpha_n\cos(\omega_n t_2) + \beta_n\sin(\omega_n t_2)]\sin(\frac{n\pi}{l}x), \qquad x \in [0,l].$$

Hence we get the following necessary conditions for the coefficients α_n, β_n:

$$-\alpha_n\omega_n\sin(\omega_n t_1) + \beta_n\omega_n\cos(\omega_n t_1) = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l}x)dx, \qquad n \in \mathbb{N},$$

$$\alpha_n\cos(\omega_n t_2) + \beta_n\sin(\omega_n t_2) = \frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi}{l}x)dx, \qquad n \in \mathbb{N}.$$

The linear equations just received can be uniquely solved for the unknown coefficients α_n and β_n, due to assumption (17):

$$\alpha_n = \frac{-\sin(\omega_n t_2)\frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l}x)dx + \cos(\omega_n t_1)\omega_n\frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi}{l}x)dx}{\omega_n\cos(\omega_n(t_2-t_1))},$$

$$\beta_n = \frac{\cos(\omega_n t_2)\frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l}x)dx + \sin(\omega_n t_1)\omega_n\frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi}{l}x)dx}{\omega_n\cos(\omega_n(t_2-t_1))}.$$

So the unknown initial functions φ and ψ are uniquely determined and found in the form of (7) and (8). It remains to show that φ, ψ are from the classes D^{s+1}, D^s, respectively. To this effect, it is enough to show that (28) holds.

Again, let

$$D_n := \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l}x)dx,$$
$$E_n := \frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi}{l}x)dx.$$

Since $(f,g)\in D^{s+1}\times D^{s+2}$, we have that the inequality (29′) holds:

$$(29') \qquad \sum_{n=1}^{\infty} n^{2s+4}\max\{|\frac{1}{n}D_n|^2, |E_n|^2\} < \infty.$$

By using Lemma 2, for every $n > N$ we have

$$|\alpha_n| = \left|\frac{-\sin(\omega_n t_2)D_n + \cos(\omega_n t_1)\omega_n E_n}{\omega_n\cos(\omega_n(t_2-t_1))}\right| < \left|\frac{1}{\omega_n}\frac{n}{m}D_n\right| + \left|\frac{n}{m}E_n\right|,$$

$$|\beta_n| = \left|\frac{\cos(\omega_n t_2)D_n + \sin(\omega_n t_1)\omega_n E_n}{\omega_n\cos(\omega_n(t_2-t_1))}\right| < \left|\frac{1}{\omega_n}\frac{n}{m}D_n\right| + \left|\frac{n}{m}E_n\right|,$$

which means that

$$(30') \qquad \max\{|\alpha_n|, |\beta_n|\} < c_2 n\max\{|\frac{1}{n}D_n|, |E_n|\} \qquad n\in\mathbb{N},$$

with a suitable constant c_2.

Combining (29′), (30′) and the definition of the norm $\|.\|_s$ we get the desired inequality (28):

$$\max\{\|\varphi\|_{s+1}^2, \|\psi\|_s^2\} = \max\{\sum_{n=1}^{\infty} n^{2s+2}|\alpha_n|^2, \sum_{n=1}^{\infty} n^{2s}|\omega_n\beta_n|^2\} \leq$$
$$\leq \sum_{n=1}^{\infty} M^2 n^{2s+2}\max\{|\alpha_n|^2, |\beta_n|^2\} < c_2^2 M^2\sum_{n=1}^{\infty} n^{2s+4}\max\{|\frac{1}{n}D_n|^2, |E_n|^2\} < \infty. \qquad \square$$

Proof of Theorem 3. This proof goes along the same lines as that of Theorem 2, except that here we have to interchange the roles of the coefficients α_n and β_n. □

Proof of Theorem 4. Now, we have

$$f(x)=u_t(x,t_1)=\sum_{n=1}^{\infty}[-\alpha_n\omega_n\sin(\omega_n t_1)+\beta_n\omega_n\cos(\omega_n t_1)]\sin(\frac{n\pi}{l}x),\qquad x\in[0,l],$$

$$g(x)=u_t(x,t_2)=\sum_{n=1}^{\infty}[-\alpha_n\omega_n\sin(\omega_n t_2)+\beta_n\omega_n\cos(\omega_n t_2)]\sin(\frac{n\pi}{l}x),\qquad x\in[0,l],$$

whence the necessary conditions for the coefficients α_n, β_n are the following:

$$-\alpha_n\omega_n\sin(\omega_n t_1)+\beta_n\omega_n\cos(\omega_n t_1)=\frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l}x)dx,\qquad n\in\mathbb{N},$$

$$-\alpha_n\omega_n\sin(\omega_n t_2)+\beta_n\omega_n\cos(\omega_n t_2)=\frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi}{l}x)dx,\qquad n\in\mathbb{N}.$$

The linear equations just received can be uniquely solved for the unknown coefficients α_n and β_n, due to assumption (15):

$$\alpha_n=\frac{\cos(\omega_n t_2)\frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l}x)dx-\cos(\omega_n t_1)\frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi}{l}x)dx}{\omega_n\sin(\omega_n(t_2-t_1))},$$

$$\beta_n=\frac{\sin(\omega_n t_2)\frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l}x)dx-\sin(\omega_n t_1)\frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi}{l}x)dx}{\omega_n\sin(\omega_n(t_2-t_1))}.$$

So the unknown initial functions φ and ψ are uniquely determined and found in the form of (7) and (8). It remains to show that φ, ψ are from the classes D^{s+1}, D^s, respectively. To this effect, it is enough to show that (28) holds.

Again, let

$$D_n:=\frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l}x)dx,$$
$$E_n:=\frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi}{l}x)dx.$$

Since $(f,g)\in D^{s+1}\times D^{s+1}$, we have that the inequality (29″) holds:

$$\sum_{n=1}^{\infty}n^{2s+2}\max\{|D_n|^2,|E_n|^2\}<\infty. \tag{29″}$$

By using Lemma 1, for every $n>N$ we get

$$|\alpha_n|=\left|\frac{\cos(\omega_n t_2)D_n-\cos(\omega_n t_1)E_n}{\omega_n\sin(\omega_n(t_2-t_1))}\right|<\left|\frac{1}{\omega_n}\frac{n}{m}D_n\right|+\left|\frac{1}{\omega_n}\frac{n}{m}E_n\right|,$$

$$|\beta_n|=\left|\frac{\sin(\omega_n t_2)D_n-\sin(\omega_n t_1)E_n}{\omega_n\sin(\omega_n(t_2-t_1))}\right|<\left|\frac{1}{\omega_n}\frac{n}{m}D_n\right|+\left|\frac{1}{\omega_n}\frac{n}{m}E_n\right|,$$

which means that

$$\max\{|\alpha_n|,|\beta_n|\}<c_4\max\{|D_n|,|E_n|\}\qquad n\in\mathbb{N}, \tag{30″}$$

with a suitable constant c_4.

Combining (29″), (30″) and the definition of the norm $\|.\|_s$ we get the desired inequality (28):

$$\max\{\|\varphi\|_{s+1}^2, \|\psi\|_s^2\} = \max\{\sum_{n=1}^{\infty} n^{2s+2}|\alpha_n|^2, \sum_{n=1}^{\infty} n^{2s}|\omega_n\beta_n|^2\} \leq$$
$$\leq \sum_{n=1}^{\infty} M^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\} < c_4^2 M^2 \sum_{n=1}^{\infty} n^{2s+2} \max\{|D_n|^2, |E_n|^2\} < \infty. \qquad \square$$

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