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Eigenvalue problems for a class of singular quasilinear elliptic equations in weighted spaces*

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Abstract: In this paper, by using the Galerkin method and the generalized Brouwer's theorem, some problems of the higher eigenvalues are studied for a class of singular quasiliner elliptic equations in the weighted Sobolev spaces. The existence of weak solutions is obtained for this problem.

Keywords: Weighted Sobolev space; Singular quasilinear elliptic equation; Eigenvalue.

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1 Introduction

In this paper, we consider the existence of weak solutions in weighted Sobolev spaces for the singular quasilinear elliptic equation

$$\begin{cases} \mathcal{L}u = \lambda_J u\rho + f(x,u)\rho - G, & in \ \ \Omega, \\ u = 0, & on \ \ \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\mathcal{L}u = -\sum_{i,j=1}^{N} D_i(p_i^{\frac{1}{2}} p_j^{\frac{1}{2}} a_{ij}(x) D_j u) + a_0(x) q u, \tag{1.2}$$

and $\lambda_J (J > 1)$ is J-th eigenvalue of the operator (1.2) of multiplicity J_1.

Equation (1.1) is singular arises from the fact that Ω may be unbounded or that p_i may equal zero or infinity on part or all of the boundary of Ω.

Working in Sobolev spaces, there are many existence results for linear or quasilinear elliptic equation. For example, we can refer to [1]-[4]. However, there are seem to be relatively few papers

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that consider the quasilinear elliptic equations in weighted Sobolev spaces, because the compact embedding theorem cannot be obtained easily.

In 2001, V.L. Shapiro[5] established a new weighted compact Sobolev embedding theorem, and proved a series of existence problems for weighted quasilinear elliptic equations and parabolic equations.

In 2005, working in Sobolev space $H^1_{p,\rho}(\Omega,\Gamma)$ only for the first eigenvalue, A. Rumbos and V. L. Shapiro[6] on the basis of [7] by using the generalized Landesman-Lazer conditions[8] discussed the existence of the solutions for weighted quasilinear elliptic equations

$$\begin{cases} \mathcal{P}u-\lambda_1 u\rho=-a(x,u)u^-\rho+g(x,u)\rho+h, & x\ in\ \Omega, \\ u=0, & x\ on\ \Omega, \end{cases}$$

where

$$\mathcal{P}u=-\sum_{i,j=1}^{N} D_i(p_i^{\frac{1}{2}}p_j^{\frac{1}{2}}b_{ij}(x)D_ju)+\rho(x)c(x)u.$$

The problems what we discussed have physical background. In fact, equation (1.1) is one of the most useful sets of Navier-Stokes equations, which describe the motion of viscous fluid substances liquids and gases.

The purpose of this paper is to obtain an existence result of the weakly result for problem (1.1). Our results are bases on the Galerkin-type techniques[9] and the generalized Brouwer's theorem[10] and other methods.

2 Preliminaries and Fundamental Lemmas

Let Ω denote a bounded domain in $R^N(N\geq 1)$, $p_i(x)$ and $\rho(x)\in C^0(\Omega)$ be positive functions, $q(x)\in C^0(\Omega)$ be a nonnegative function, with the property that

$$\int_\Omega q(x)dx<\infty, \int_\Omega \rho(x)dx<\infty, \int_\Omega p_i(x)dx<\infty, i=1,2,\cdots,N. \tag{2.1}$$

Let $\Gamma\subset\partial\Omega$ be a fixed closed set (it may be the empty set), $q(x)$ maybe identically zero, $D_iu=\frac{\partial u}{\partial x_i}$, $i=1,2,\cdots,N$. We consider the following pre-Hilbert spaces

$$C^0_\rho(\Omega)=\left\{u\in C^0(\Omega)\left|\int_\Omega |u|^2\rho dx<\infty\right.\right\}, \tag{2.2}$$

with inner-product $\langle u,v\rangle_\rho=\int_\Omega uv\rho dx,\ \forall u,v\in C^0_\rho(\Omega)$, and

$$C^1_{p,q,\rho}(\Omega,\Gamma)=\left\{u\in C^0(\bar\Omega)\cap C^2(\Omega)\left|u(x)=0,\forall x\in\Gamma;\int_\Omega\left[\sum_{i=1}^N|D_iu|^2p_i+u^2(q+\rho)\right]dx<\infty\right.\right\}, \tag{2.3}$$

with inner-product

$$\langle u,v\rangle_{p,q,\rho}=\int_\Omega\left[\sum_{i=1}^N p_iD_iuD_iv+(q+\rho)uv\right]dx. \tag{2.4}$$

Let $L^2_\rho(\Omega)$ denote the Hilbert space obtained through the completion of $C^0_\rho(\Omega)$ by using the method of Cauchy sequences with respect to the norm $||u||_\rho = \langle u, u\rangle_\rho^{\frac{1}{2}}$, $H^1_{p,q,\rho}(\Omega,\Gamma)$ denote the completion of the space $C^1_{p,q,\rho}(\Omega,\Gamma)$ with respect to the norm $||u||_{p,q,\rho} = \langle u, u\rangle^{\frac{1}{2}}_{p,q,\rho}$. Obviously, $H^1_{p,q,\rho}(\Omega,\Gamma)$ is a weighted Sobolev spaces. In a similar manner we have the spaces $L^2_{p_i}(\Omega), (i = 1, 2, \cdots, N)$ and $L^2_q(\Omega)$. Hance we see from (2.4) that

$$\langle u, v\rangle_{p,q,\rho} = \sum_{i=1}^{N} \langle D_i u, D_i v\rangle_{p_i} + \langle u, v\rangle_\rho + \langle u, v\rangle_q. \tag{2.5}$$

Next, we make the following assumptions for the functions $a_{ij}(x)$ and $a_0(x)$

(a-1) $a_{ij}(x), a_0(x) \in L^\infty(\Omega)$, $i, j = 1, 2, \cdots, N$, and $a_0(x) \geq 0,\ a.e. x \in \Omega$;

(a-2) $a_{ij}(x) = a_{ji}(x),\ \forall x \in \Omega,\ i, j = 1, 2, \cdots, N$;

(a-3) $\sum\limits_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2$, $\forall x \in \Omega,\ \xi \in R^N$, where $c_0 > 0$.

$f(x, s)$ will meet the following conditions:

(f-1) $f(x, s)$ satisfies Caratheodory conditions[9];

(f-2) There exists a nonnegative function $f_0(x) \in L^2_\rho$, for $\forall s \in R$, such that $|f(x, s)| \leq 2\gamma^{'}|s| + f_0(x)$, where $0 < \gamma^{'} < \gamma, \gamma = (\lambda_{J+J_1} - \lambda_J)/2$;

(f-3) $f(x, s) \geq -f_0(x)$, $s > 0$ and a.e. $x \in \Omega$; $f(x, s) \leq f_0(x)$, $s \leq 0$ and a.e.$x \in \Omega$, where $f_0(x) \in L^2_\rho$ is similar with (f-2).

Hence (f-2) and (f-3) together imply

$$|f(x, s) - \gamma^{'} s| \leq \gamma^{'}|s| + f_0(x),\ \forall s \in R,\ a.e. x \in \Omega. \tag{2.6}$$

Now we introduce the two-form for the operator $\mathcal{L}$

$$\mathcal{L}(u, v) = \sum_{i,j=1}^{N} \int_\Omega p_i^{\frac{1}{2}} p_j^{\frac{1}{2}} a_{ij}(x) D_j u D_i v + \langle a_0 u, v\rangle_q, \tag{2.7}$$

$\forall u, v \in H^1_{p,q,\rho}(\Omega,\Gamma)$. We say $\mathcal{L}$ satisfies $V_L(\Omega,\Gamma)$ conditions if the following two facts obtain:

$(V_L - 1)$ There exists a complete orthonormal system $\{\varphi_n\}_{n=1}^\infty \subset L^2_\rho$, and for arbitrary n, $\varphi_n \in H^1_{p,q,\rho}(\Omega,\Gamma) \cap C^2(\Omega)$;

$(V_L - 2)$ There exists a sequence of eigenvalues $\{\lambda_n\}_{n=1}^\infty$ with $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \to \infty$, such that

$$\mathcal{L}(\varphi_n, v) = \lambda_n \langle \varphi_n, v\rangle_\rho,\ \forall v \in H^1_{p,q,\rho}(\Omega,\Gamma).$$

Let S_n be the subspace of $H^1_{p,q,\rho}(\Omega,\Gamma)$ spanned by $\varphi_1, \varphi_2, \cdots, \varphi_n$. For $u_n \in S_n$, $\exists(\alpha_1, \alpha_2, \cdots, \alpha_n) \in R^n$, s.t.

$$u_n = \sum_{k=1}^{J-1} \alpha_k\varphi_k + \sum_{k=J}^{J+J_1-1} \alpha_k\varphi_k + \sum_{k=J+J_1}^{n} \alpha_k\varphi_k.$$

From $(V_L - 1)$, we see that $||u_n||^2_\rho = |\alpha|^2$. Setting

$$v_n = \sum_{k=1}^{J-1} \alpha_k\varphi_k + \sum_{k=J+J_1}^{n} \alpha_k\varphi_k, \quad w_n = \sum_{k=J}^{J+J_1-1} \alpha_k\varphi_k, \tag{2.8}$$

then $u_n = w_n + v_n$, and $\langle v_n, \varphi_j \rangle = 0,\ j = J, J+1, \cdots, J+J_1-1$, and w_n is an eigenfunction of the operator $\mathcal{L}$ corresponding to eigenvalue λ_J.

We say functional G satisfies G^*- conditions if the following two facts obtain:

(G-1) $G \in (H^1_{p,q,\rho}(\Omega, \Gamma))^*$, the dual of $H^1_{p,q,\rho}(\Omega, \Gamma)$, that is, $G : H^1_{p,q,\rho}(\Omega, \Gamma) \to R$ linearly and $|G(u)| \le K_0 ||u||_{p,q,\rho}$, $\forall u \in H^1_{p,q,\rho}(\Omega, \Gamma)$, where K_0 is a constant;

(G-2) For $u_n \in S_n$, $u_n = w_n + v_n$ (same as (2.8)), if

$$\lim_{n\to\infty} \frac{||v_n||_{p,q,\rho}}{||u_n||_{p,q,\rho}} \to 0,$$

then

$$\limsup_{n\to\infty} [(1 - n^{-1})\langle f(x, u_n), w_n \rangle_\rho - G(w_n)] > 0. \tag{2.9}$$

We need the following lemmas in section three.

Lemma 2.1 Assume that operator $\mathcal{L}$ is given by (1.2) and that the conditions (a-1)-(a-3) are valid, and that $V_L(\Omega, \Gamma)$ conditions hold, and that $v \in L^2_\rho(\Omega)$. Set $\hat{v}(n) = \langle v, \varphi_n \rangle_\rho, n = 1, 2, \cdots$, then $v \in H^1_{p,q,\rho}(\Omega, \Gamma)$ if and only if

$$\sum_{n=1}^{\infty} \lambda_n |\hat{v}(n)|^2 < \infty.$$

Furthermore if $v \in H^1_{p,q,\rho}(\Omega, \Gamma)$, then

$$\mathcal{L}(v, v) = \sum_{n=1}^{\infty} \lambda_n |\hat{v}(n)|^2.$$

Lemma 2.2 Assume that operator $\mathcal{L}$ is given by (1.2) and that the conditions (a-1)-(a-3) are valid, for $\forall u \in H^1_{p,q,\rho}(\Omega, \Gamma)$. Then, there exist constants $K_1 > 0$ and $K_2 > 0$, such that

$$K_1 ||u||^2_{p,q,\rho} \le \mathcal{L}(u, u) \le K_2 ||u||^2_{p,q,\rho}.$$

Lemma 2.3 Assume that operator $\mathcal{L}$ is given by (1.2) and that the conditions (a-1)-(a-3) are valid, and that $V_L(\Omega, \Gamma)$ conditions hold. Then $H^1_{p,q,\rho}(\Omega, \Gamma)$ is compactly imbedded in $L^2_\rho(\Omega)$.

Proofs of the Lemma 2.1 and Lemma 2.2 can be refer to [6,P.9-10], Lemma 2.3 can be refer to [5, P.38], so the proofs are omitted.

3 Main Results and Their Proofs

In this section, we prove that problem (1.1) has at least one solution, which is the main result of this paper stated as Theorem 3.3.

In order to prove the problem (1.1) has a weak solution, we first discuss it in a finite dimension space S_n, where S_n is the subspace of $H^1_{p,q,\rho}(\Omega, \Gamma)$ spanned by $\varphi_1, \varphi_2, \cdots, \varphi_n$, then we extend the result to the infinite dimension space $H^1_{p,q,\rho}(\Omega, \Gamma)$.

Theorem 3.1 Let Ω denote a bounded domain in $R^N (N \ge 1)$, $p_i(x)$ and $\rho(x) \in C^0(\Omega)$ be positive functions, $q(x) \in C^0(\Omega)$ be a nonnegative function and assume that (2.1) holds. Let

$\Gamma \subset \partial\Omega$ be a fixed closed set (it may be the empty set), the operator $\mathcal{L}$ be given by (1.2) and assume (a-1)-(a-3), that $V_L(\Omega,\Gamma)$ conditions hold. Suppose that $f(x,u)$ satisfies (f-1)-(f-3), that the functional G satisfies (G-1). Then if $n \geq n_0 = J + J_1 + 1$, there exists $u_n^* \in S_n$ with the property that

$$\mathcal{L}(u_n^*, v) = (\lambda_J + \gamma' n^{-1})\langle u_n^*, v\rangle_\rho + (1 - n^{-1})\langle f(x, u_n^*), v\rangle_\rho - G(v), \ \forall v \in S_n. \tag{3.1}$$

Proof We only consider the situation for $J > 1$. The case $J = 1$ has already been treated in [6]. We set

$$u = \sum_{k=1}^{n} \alpha_k \varphi_k, \ \ \tilde{u} = \sum_{k=1}^{n} \delta_k \alpha_k \varphi_k, \tag{3.2}$$

$$\delta_k = \begin{cases} -1, & k = 1, \cdots, J + J_1 - 1, \\ \ \ 1, & k = J + J_1, J + J_1 + 1, \cdots, n, \end{cases} \tag{3.3}$$

where $n \geq n_0$, and $n_0 = J + J_1 + 1$. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in R^n$, from (3.2) and (3.3), we see that

$$||u||_\rho^2 = ||\tilde{u}||_\rho^2 = |\alpha|^2 = \sum_{i=1}^{n} \alpha_i^2.$$

We define

$$F_k(\alpha) = \mathcal{L}(u, \delta_k\varphi_k) - (\lambda_J + \gamma' n^{-1})\langle u, \delta_k\varphi_k\rangle_\rho - (1 - n^{-1})\langle f(x, u), \delta_k\varphi_k\rangle_\rho + G(\delta_k\varphi_k). \tag{3.4}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned} \sum_{k=1}^{n} F_k(\alpha)\alpha_k &= \mathcal{L}(u, \tilde{u}) - (\lambda_J + \gamma' n^{-1})\langle u, \tilde{u}\rangle_\rho - (1 - n^{-1})\langle f(x, u), \tilde{u}\rangle_\rho + G(\tilde{u}) \\ &= \mathcal{L}(u, \tilde{u}) - (\lambda_J + \gamma')\langle u, \tilde{u}\rangle_\rho - (1 - n^{-1})\langle f(x, u) - \gamma' u, \tilde{u}\rangle_\rho + G(\tilde{u}). \end{aligned} \tag{3.5}$$

First of all,

$$\mathcal{L}(u, \tilde{u}) - (\lambda_J + \gamma')\langle u, \tilde{u}\rangle_\rho = \sum_{k=1}^{J+J_1-1} (\lambda_J + \gamma' - \lambda_k)\alpha_k^2 + \sum_{k=J+J_1}^{n} (\lambda_k - \lambda_J - \gamma')\alpha_k^2,$$

since $2\gamma' < \lambda_{J+J_1} - \lambda_J$ and $||u||_\rho^2 = ||\tilde{u}||_\rho^2 = |\alpha|^2$, hence we obtain

$$\mathcal{L}(u, \tilde{u}) - (\lambda_J + \gamma')\langle u, \tilde{u}\rangle_\rho \geq \gamma' |\alpha|^2. \tag{3.6}$$

Secondly, from (2.6), Hölder inequality and Minkowski inequality, we get

$$\begin{aligned} \langle f(x, u) - \gamma' u, \tilde{u}\rangle_\rho &= \int (f(x, u) - \gamma' u)\tilde{u}\rho dx \\ &\leq \int (\gamma' |u| + f_0(x))|\tilde{u}|\rho dx \\ &\leq ||\gamma' |u| + f_0(x)||_\rho ||u||_\rho \\ &\leq (\gamma' |\alpha| + ||f_0||_\rho)|\alpha|. \end{aligned} \tag{3.7}$$

In addition, according to (G-1), Lemma 2.2, $\mathcal{L}(u,u)=\sum_{k=1}^{n}\lambda_k|\hat{u}(k)|^2$, and $\sum_{k=1}^{n}|\hat{u}(k)|^2=||u||_\rho^2$ for fixed n, we have

$$\begin{aligned}
&|G(\tilde{u})|\le K_0||\tilde{u}||_{p,q,\rho}\le K_0^{'}[L(\tilde{u},\tilde{u})]^{1/2}\\
&=K_0^{'}\left[\sum_{k=1}^{n}\lambda_k|\hat{u}(k)|^2\right]^{1/2}\le K_0^{'}\left[\lambda_n\sum_{k=1}^{n}|\hat{u}(k)|^2\right]^{1/2}=K_3|\alpha|,
\end{aligned}\tag{3.8}$$

where $K_0,K_0^{'},K_3$ are positive constants. We observe from (3.5), (3.6), (3.7) and (3.8) that there exists $t>0$, such that

$$\begin{aligned}
\sum_{k=1}^{n}F_k(\alpha)\alpha_k &\ge \gamma^{'}|\alpha|^2-(1-n^{-1})[(\gamma^{'}|\alpha|+||f_0||_\rho)|\alpha|]-K_3|\alpha|\\
&\ge n^{-1}\gamma^{'}|\alpha|^2-[(1-n^{-1})||f_0||_\rho+K_3]|\alpha|\\
&\ge \gamma^{'}|\alpha|^2/2n>0,\quad |\alpha|\ge t.
\end{aligned}$$

By the generalized Brouwer's theorem[10], there exists $\alpha^*=(\alpha_1^*,\ldots,\alpha_n^*)$ satisfying $F_k(\alpha^*)=0, k=1,2,\cdots,n$. Thus, taking $u_n^*=\sum_{k=1}^{n}\alpha_k^*\varphi_k$, we obtain from(3.3) and (3.4) that

$$\mathcal{L}(u_n^*,\varphi_k)=(\lambda_J+\gamma^{'}n^{-1})\langle u_n^*,\varphi_k\rangle_\rho+(1-n^{-1})\langle f(x,u_n^*),\varphi_k\rangle_\rho-G(\varphi_k),\quad k=1,2,\cdots,n,$$

and the proof of Theorem 3.1 is complete by the definition of S_n .

Theorem 3.2 The sequence $\{u_n^*\}$ obtained in Theorem 3.1 is uniformly bounded in $H_{p,q,\rho}^1$ with respect to the norm $||u||_{p,q,\rho}=\langle u,u\rangle_{p,q,\rho}^{\frac{1}{2}}$.

Proof From Theorem 3.1, for $u_n^*\in S_n$, we have

$$\mathcal{L}(u_n^*,v)=(\lambda_J+\gamma^{'}n^{-1})\langle u_n^*,v\rangle_\rho+(1-n^{-1})\langle f(x,u_n^*),v\rangle_\rho-G(v),\forall v\in S_n,\tag{3.9}$$

where $0<\gamma^{'}<\gamma,\ \gamma=\frac{\lambda_{J+J_1}-\lambda_J}{2},\ n\ge n_0,\ \ n_0=J+J_1+1$.

For ease of notation, we represent the sequence $\{u_n^*\}_{n\ge n_0}$ by $\{u_n\}_{n\ge n_0}$. In order to prove Theorem 3.2, we need to prove that there exists constant K_4 for above $u_n\in S_n$, such that

$$||u_n||_{p,q,\rho}\le K_4,\quad \forall n\ge n_0.\tag{3.10}$$

Suppose to the contrary that (3.10) does not hold, then there exists subsequence (for ease of notation, we still denoted by $\{u_n\}$), such that

$$\lim_{n\to\infty}||u_n||_{p,q,\rho}=\infty.\tag{3.11}$$

Taking $v=u_n$ in (3.9), from Lemma 2.2, $G\in(H_{p,q,\rho}^1(\Omega,\Gamma))^*$ and the methods of (3.7), there exists $K_5>0$, such that

$$\begin{aligned}
K_5||u_n||_{p,q,\rho}^2 &\le (\lambda_J+\gamma^{'}n^{-1})\langle u_n,u_n\rangle_\rho+(1-n^{-1})|\langle f(x,u_n),u_n\rangle_\rho|+|G(u_n)|\\
&\le (\lambda_J+\gamma^{'})||u_n||_\rho^2+(1-n^{-1})(\gamma^{'}||u_n||_\rho^2+||f_0||_\rho||u_n||_\rho)+K_0||u_n||_{p,q,\rho}.
\end{aligned}$$

Hence dividing both sides of above inequality by $||u_n||^2_{p,q,\rho}$ and taking the limit as $n \to \infty$, then there exists positive integer $n_1(n_1 \geq n_0)$, when $n \geq n_1$ that

$$\frac{K_5}{2(\lambda_J + 2\gamma')} \leq \frac{||u_n||^2_\rho}{||u_n||^2_{p,q,\rho}} \leq 1.$$

From (3.11), we have

$$\lim_{n\to\infty} ||u_n||_\rho = \infty, \tag{3.12}$$

and when $n \geq n_1$ that

$$K_6||u_n||_{p,q,\rho} \leq ||u_n||_\rho. \tag{3.13}$$

Set

$$u_n = u_{n1} + u_{n2} + u_{n3},\ \bar{u}_n = -u_{n1} - u_{n2} + u_{n3},$$
$$u_{n1} = \sum_{k=1}^{J-1} \hat{u}_n(k)\varphi_k,\quad u_{n2} = \sum_{k=J}^{J+J_1-1} \hat{u}_n(k)\varphi_k,\quad u_{n3} = \sum_{k=J+J_1}^{n} \hat{u}_n(k)\varphi_k. \tag{3.14}$$

In the following, for $\forall n \geq n_1$, we propose to show the fact

$$\lim_{n\to\infty} \frac{[||u_{n1}||_{p,q,\rho} + ||u_{n3}||_{p,q,\rho}]}{||u_n||_\rho} = 0. \tag{3.15}$$

In matter of fact, we obtain from (3.9) and (3.14) that

$$\mathcal{L}(u_n, \bar{u}_n) - (\lambda_J + \gamma^{'})\langle u_n, \bar{u}_n\rangle_\rho = (1 - n^{-1})\langle f(x, u_n) - \gamma^{'} u_n, \bar{u}_n\rangle_\rho - G(\bar{u}_n),$$

and

$$\sum_{k=1}^{J+J_1-1} (\lambda_J + \gamma^{'} - \lambda_k)|\hat{u}_n(k)|^2 + \sum_{k=J+J_1}^{n} (\lambda_k - \lambda_J - \gamma^{'})|\hat{u}_n(k)|^2$$
$$= (1 - n^{-1})\langle f(x, u_n) - \gamma^{'} u_n, \bar{u}_n\rangle_\rho - G(\bar{u}_n). \tag{3.16}$$

From (2.6) and (3.14), we conclude that

$$|\langle f(x, u_n) - \gamma^{'} u_n, \bar{u}_n\rangle_\rho| \leq \gamma^{'}||u_n||^2_\rho + ||f_0||_\rho||u_n||_\rho. \tag{3.17}$$

Taking $\delta = \gamma - \gamma^{'}$, from $\gamma = (\lambda_{J+J_1} - \lambda_J)/2$, we also obtain from (3.16), (3.17), $\sum_{k=1}^{n} |\hat{u}_n(k)|^2 = ||u_n||^2_\rho$ and $G \in (H^1_{p,q,\rho}(\Omega, \Gamma))^*$ that

$$\gamma^{'}||u_n||^2_\rho + \sum_{k=1}^{J-1}(\lambda_J - \lambda_k)|\hat{u}_n(k)|^2 + \sum_{k=J+J_1}^{n} (\lambda_k - \lambda_{J+J_1} + \delta)|\hat{u}_n(k)|^2$$
$$\leq K_0||u_n||_{p,q,\rho} + (1 - n^{-1})(\gamma^{'}||u_n||^2_\rho + ||f_0||_\rho||u_n||_\rho)$$
$$\leq K_0||u_n||_{p,q,\rho} + \gamma^{'}||u_n||^2_\rho + ||f_0||_\rho||u_n||_\rho. \tag{3.18}$$

It is clear that for fixed n, $\exists\gamma^{''} > 0$, such that

$$\gamma^{''}(1 + \lambda_k) \leq \lambda_J - \lambda_k, k = 1, 2, \cdots, J - 1;\ \gamma^{''}(1 + \lambda_k) \leq (\lambda_k - \lambda_{J+1}) + \delta, k \geq J + J_1. \tag{3.19}$$

Since $f_0(x) \in L^2_\rho$, it is follows from Lemma 2.1, Lemma 2.2, (3.18) and (3.19) that

$$\gamma^* \left[||u_{n1}||^2_{p,q,\rho} + ||u_{n3}||^2_{p,q,\rho}\right] \le K_7||u_n||_\rho + K_0||u_n||_{p,q,\rho}, \tag{3.20}$$

where $\forall n \ge n_1$, K_0, K_7 and γ^* are positive constants. Dividing both sides of (3.20) by $||u_n||^2_\rho$ and taking the limit as $n \to \infty$, from (3.12) and (3.13), we obtain (3.15).

Next, setting

$$w_n = u_{n2},\ v_n = u_{n1} + u_{n3}, \tag{3.21}$$

then $u_n = w_n + v_n$. Observe that $\langle v_n, \varphi_j\rangle_\rho = 0$ for $j = J, J+1, \cdots, J+J_1-1$ and that w_n is a λ_J-eigenfunction of $\mathcal{L}$, also from (3.15) that

$$\lim_{n\to\infty} ||v_n||_{p,q,\rho}/||u_n||_\rho = 0. \tag{3.22}$$

Taking $v = w_n$ in (3.9), from (V_L-2), we obtain $\mathcal{L}(u_n, w_n) = \lambda_n\langle u_n, w_n\rangle_\rho$. Hence, for $\forall n \ge n_1$, we have

$$-\gamma' n^{-1}||w_n||^2_\rho = (1-n^{-1})\langle f(x,u_n), w_n\rangle_\rho - G(w_n). \tag{3.23}$$

Therefore, we infer from (3.23) that

$$(1-n^{-1})\langle f(x,u_n), w_n\rangle_\rho - G(w_n) \le 0. \tag{3.24}$$

Consequently,

$$\limsup_{n\to\infty}[(1-n^{-1})\langle f(x,u_n), w_n\rangle_\rho - G(w_n)] \le 0. \tag{3.25}$$

We obtain that (3.25) is contrary to (2.9). Hence (3.10) is true.

Theorem 3.3 Let Ω denote a bounded domain in $R^N (N \ge 1)$, $p_i(x)$, $\rho(x) \in C^0(\Omega)$ be positive functions, $q(x) \in C^0(\Omega)$ be a nonnegative function and satisfy (2.1), $\Gamma \subset \partial\Omega$ be a closed set(it may be the empty set), operator $\mathcal{L}$ be given by (1.2) and assume (a-1)-(a-3), that $\mathcal{L}$ satisfies $V_L(\Omega,\Gamma)$ conditions. Let $f(x,u)$ satisfy (f-1)-(f-3) and functional G satisfy G^*- conditions. Then problem (1.1) at least has a weak solution, i.e. there exits $u^* \in H^1_{p,q,\rho}(\Omega,\Gamma)$ with the property that

$$\mathcal{L}(u^*, v) = \lambda_J\langle u^*, v\rangle_\rho + \langle f(x,u^*), v\rangle_\rho - G(v), \forall v \in H^1_{p,q,\rho}(\Omega,\Gamma). \tag{3.26}$$

Proof Since $H^1_{p,q,\rho}(\Omega,\Gamma)$ is a separable Hilbert space, we obtain from (3.10) and Lemma 2.3 that $\{u^*_n\}$ what we described above in $H^1_{p,q,\rho}(\Omega,\Gamma)$ is uniformly bounded, and that there exists a weak convergence subsequence (still denote by $\{u^*_n\}$) and a function $u^* \in H^1_{p,q,\rho}(\Omega,\Gamma)$, such that

$$\lim_{n\to\infty} ||u^*_n - u^*||_\rho = 0, \tag{3.27}$$

$$\exists k(x) \in L^2_\rho,\quad |u^*_n(x)| \le k(x),\ a.e. x \in \Omega, \tag{3.28}$$

$$\lim_{n\to\infty} u_n^*(x) = u^*(x),\ \ a.e. x \in \Omega, \tag{3.29}$$

$$\lim_{n\to\infty} \langle D_i u_n^*, v\rangle_{p_i} = \langle D_i u^*,\ v\rangle_{p_i}, \forall v \in L^2_{p_i},\ \ i = 1, 2, \cdots, N, \tag{3.30}$$

$$\lim_{n\to\infty} \langle a_0 u_n^*,\ v\rangle_q = \langle a_0 u^*, v\rangle_q,\ \ \forall v \in L^2_q. \tag{3.31}$$

From (2.7), (3.9), (3.27), (3.30) and (3.31), for $v_j \in S_j (j \geq n_0)$, we obtain that

$$\lim_{n\to\infty} \mathcal{L}(u_n^*, v_j) = \mathcal{L}(u^*, v_j). \tag{3.32}$$

We see from (f-1) and (3.29) that

$$\lim_{n\to\infty} f(x, u_n^*) = f(x, u^*),\ \ a.e. x \in \Omega. \tag{3.33}$$

From (f-2) and (3.28), we get

$$|f(x, u_n^*)| \leq 2\gamma' k(x) + f_0(x),\ \ a.e. x \in \Omega. \tag{3.34}$$

Consequently, we conclude from (3.33), (3.34) and the Lebesgue dominated convergence theorem[10] that

$$\lim_{n\to\infty} \langle f(x, u_n), v_j\rangle_\rho = \langle f(x, u^*), v_j\rangle_\rho, \tag{3.35}$$

where $0 < \gamma' < \gamma$. Taking $v = v_j$ in (3.9) and letting $n \to \infty$, we have

$$\mathcal{L}(u^*, v_j) = \lambda_J \langle u^*, v_j\rangle_\rho + \langle f(x, u^*), v_j\rangle_\rho - G(v_j). \tag{3.36}$$

Since $P_j v = \sum_{k=1}^{j} \hat{v}(k)\varphi_k \in S_j$, replacing v_j by $P_j v$ in (3.36), we observe

$$\mathcal{L}(u^*, P_j v) = \lambda_J \langle u^*, P_j v\rangle_\rho + \langle f(x, u^*), P_j v\rangle_\rho - G(P_j v). \tag{3.37}$$

From Lemma 2.1 and Lemma 2.2, we have

$$\lim_{j\to\infty} ||P_j v - v||_{p,q,\rho} = 0,\ \ \forall v \in H^1_{p,q,\rho}(\Omega, \Gamma).$$

Consequently, it follows that

$$\begin{aligned}
&\lim_{j\to\infty} \mathcal{L}(u^*, P_j v) = \mathcal{L}(u^*, v),\\
&\lim_{j\to\infty} \langle u^*, P_j v\rangle_\rho = \langle u^*, v\rangle_\rho,\\
&\lim_{j\to\infty} \langle f(x, u^*), P_j v\rangle_\rho = \langle f(x, u^*), v\rangle_\rho,\\
&\lim_{j\to\infty} G(P_j v) = G(v).
\end{aligned}$$

Passing to the limit as $j \to \infty$ on both side of (3.37) and using the above established facts, for $\forall v \in H^1_{p,q,\rho}(\Omega, \Gamma)$, we obtain

$$\mathcal{L}(u^*, v) = \lambda_J \langle u^*, v\rangle_\rho + \langle f(x, u^*), v\rangle_\rho - G(v).$$

Hence the proof of Theorem 3.3 is complete.

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References

[1] **V.L. SHAPIRO**, Quasilinear ellipticity and the 1th eigenvalue, Comm. Partial Differential Equations, 16(1991), 1819-1855.

[2] **S.B. ROBINSON, A. RUMBOS, V.L. SHAPIRO**, One-Sided resonance problems for quasilinear elliptic operators, Journal of Mathematical Analysis and Applications, 256(2001), 636-649.

[3] **M. CONTRERAS**, Double resonance for quasilinear elliptic higher order partial differential equations between the first and second eigenvalues, Nonlinear Analysis, 12(1995), 1257-1281.

[4] **S. BENMEHIDI, B. KHODJA**, Some results of nontrivial solutions for a nonlinear PDE in Sobolev space, Electronic Journal of Qualitative Theory of Differential Equations, 44(2009), 1-14.

[5] **V.L. SHAPIRO**, Singular Quasilinearity and Higher Eigenvalues, Providence, Rhode Island: Memoirs of the American Mathematical Society, 2001.

[6] **A. RUMBOS, V.L. SHAPIRO**, Jumping nonlinearities and weighted Sobolev space, Journal of Differential Equations, 214(2005), 326-357.

[7] **H. BERESTYCKI, D. FIGUEIREDO**, Double resonance in semilinear elliptic problems, Comm. Partial Differential Equations, 6(1981), 91-120.

[8] **E.M. LANDESMAN, A.C. LAZE**, Nonlinear perturbations of a linear elliptic boundary value problem at resonance, J Math. Mech., 19(1970), 609-623.

[9] **W.D. LU**, Variational Methods in Differential Equations, Science Press, Beijing, 2003.

[10] **S. KESAVAN**, Topics in Functional Analysis and Applications, John Wiley and Sons, New York, 1989.

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