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Oscillation theorems for certain third order nonlinear delay dynamic equations on time scales *

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Abstract: In this paper, we establish some new oscillation criteria for the third order nonlinear delay dynamic equations

$$\left(b(t)\left([a(t)(x^{\Delta}(t))^{\alpha_1}]^{\Delta}\right)^{\alpha_2}\right)^{\Delta}+q(t)x^{\alpha_3}(\tau(t))=0$$

on a time scale $\mathbb{T}$ unbounded above, where α_i are ratios of positive odd integers, $i=1,\ 2,\ 3$, b, a and q are positive real-valued rd-continuous functions defined on $\mathbb{T}$, and the so-called delay function $\tau:\mathbb{T}\to\mathbb{T}$ is a strictly increasing function such that $\tau(t)\le t$ for $t\in\mathbb{T}$ and $\tau(t)\to\infty$ as $t\to\infty$. By using the Riccati transformation technique and integral averaging technique, some new sufficient conditions which insure that every solution oscillates or tends to zero are established. Our results are new for third order nonlinear delay dynamic equations and complement the results established by Yu and Wang in J. Comput. Appl. Math., 2009, and Erbe, Peterson and Saker in J. Comput. Appl. Math., 2005. Some examples are given here to illustrate our main results.

Keywords: Oscillation; Third order; Nonlinear delay dynamic equations; Time scales
Mathematics Subject Classification 2010: 34K11, 39A21, 34N05

1 Introduction

In this paper, we are concerned with the oscillation criteria for the following certain third order nonlinear delay dynamic equations

$$\left(b(t)\left([a(t)(x^{\Delta}(t))^{\alpha_1}]^{\Delta}\right)^{\alpha_2}\right)^{\Delta}+q(t)x^{\alpha_3}(\tau(t))=0 \tag{1.1}$$

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on a time scale $\mathbb{T}$. Throughout this paper and without further mention, we assume that the following conditions are satisfied:
(C_1) α_i are ratios of positive odd integers, $i = 1,\ 2,\ 3$;
(C_2) $b,\ a$ and q are positive, real-valued, rd-continuous functions defined on $\mathbb{T}$ and

$$\int_{t_0}^{\infty} \left(\frac{1}{b(t)}\right)^{\frac{1}{\alpha_2}} \Delta t = \infty, \quad \int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha_1}} \Delta t = \infty; \tag{1.2}$$

(C_3) $\tau : \mathbb{T} \to \mathbb{T}$, $\tau(t) \le t$, $\tau^{\Delta}(t) > 0$ for all $t \in \mathbb{T}$ and $\tau(t) \to \infty$ as $t \to \infty$.

The theory of time scales, which has recently received a lot of attention, was originally introduced by Stefan Hilger [1] in his Ph. D. Thesis in 1988, in order to unify, extend and generalize continuous and discrete analysis. The book on the subject of time scales by Bohner and Peterson [2] summarizes and organizes much of time scale calculus and many applications. In recent years, there has been increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to the papers [3–21]. To the best of our knowledge, it seems to have much research activity concerning the oscillation results for third order dynamic equations; see, for example, [3–12].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ and since we are interested in the oscillatory behavior of solutions near infinity, we make the assumption throughout this paper that the time scale $\mathbb{T}$ is unbounded above. We assume $t_0 \in \mathbb{T}$ and it is convenient to assume $t_0 > 0$. We define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \bigcap \mathbb{T}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward and the backward jump operators are defined by:

$$\sigma(t) = \inf\,\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\,\{s \in \mathbb{T} : s < t\},$$

where $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided g is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$ and for any function $f : \mathbb{T} \to \mathbb{R}$, the notation f^{σ} denotes $f \circ \sigma$.

By a solution of Eq. (1.1), we mean a nontrivial real-valued function $x \in C^1_{rd}[t_x, \infty)_{\mathbb{T}}$, $t_x \ge t_0$, which has the property that $a(x^{\Delta})^{\alpha_1} \in C^1_{rd}[t_x, \infty)_{\mathbb{T}}$, $b((a(x^{\Delta})^{\alpha_1})^{\Delta})^{\alpha_2} \in C^1_{rd}[t_x, \infty)_{\mathbb{T}}$ and satisfies Eq. (1.1) on $[t_x, \infty)_{\mathbb{T}}$. A solution of Eq. (1.1) is said to be oscillatory on $[t_x, \infty)_{\mathbb{T}}$ in case it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Eq. (1.1) is said to be oscillatory in case all its solutions are oscillatory. Our attention is restricted to those solutions of Eq. (1.1) which exist on some half line $[t_x, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge t_x$.

Recently, Erbe et al. [7–9] studied the oscillatory behavior of third order dynamic equations

$$\left(c(t)\left[a(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\Delta} + q(t)f(x(t)) = 0, \quad t \in \mathbb{T}, \tag{1.3}$$

$$x^{\Delta\Delta\Delta}(t) + p(t)x(t) = 0, \quad t \in \mathbb{T},$$

and

$$\left(a(t)\left\{[r(t)x^{\Delta}(t)]^{\Delta}\right\}^{\gamma}\right)^{\Delta} + f(t, x(t)) = 0, \quad t \in \mathbb{T}.$$

Hassan [10] and Li et al. [5] considered the oscillation of third order nonlinear delay dynamic equations on time scales

$$\left(a(t)\left\{[r(t)x^{\Delta}(t)]^{\Delta}\right\}^{\gamma}\right)^{\Delta} + f(t, x(\tau(t))) = 0. \tag{1.4}$$

[5] established some new oscillation criteria for (1.4) that can be applied on any time scale $\mathbb{T}$ and the results of [5] are different and complement the results established by [10].

Han et al. [3] considered the oscillation of third order nonlinear delay dynamic equations on time scales

$$\left(\left(x^{\Delta\Delta}(t)\right)^{\gamma}\right)^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0, \quad t \in \mathbb{T}, \tag{1.5}$$

where $\gamma > 0$ is a quotient of odd positive integers, p is positive, real-valued and rd-continuous function defined on $\mathbb{T}$, $\tau : \mathbb{T} \to \mathbb{T}$ is an rd-continuous function such that $\tau(t) \le t$ and $\tau(t) \to \infty$ as $t \to \infty$, and established some new oscillation criteria for (1.5) which guarantee that every solution of (1.5) oscillates or converges as $t \to \infty$.

Yu and Wang [11] studied asymptotic behavior of solutions to more general third-order nonlinear dynamic equations

$$\left(\frac{1}{a_2(t)}\left(\left(\frac{1}{a_1(t)}(x^{\Delta}(t))^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta} + q(t)f(x(t)) = 0, \quad t \in \mathbb{T}, \tag{1.6}$$

and they showed that if

$$\alpha_1\alpha_2 = 1, \quad \int_{t_0}^{\infty} [a_i(s)]^{\frac{1}{\alpha_i}} \Delta s = \infty, \quad i = 1, 2,$$

there exists a positive Δ-differentiable function r on $\mathbb{T}$, for all $M > 0$ and sufficiently large $t_1,\ t_2$ with $t_2 > t_1$,

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[Mr(s)q(s) - \frac{(r^{\Delta}(s))^2}{4Q(s)} \right] \Delta s = \infty,$$

$$Q(t) = r(t)\left[a_1(t)\delta(t,t_1)\right]^{\frac{1}{\alpha_1}}, \quad \delta(t,t_1) = \int_{t_1}^{t} [a_2(s)]^{\frac{1}{\alpha_2}} \Delta s,$$

then every solution x of (1.6) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite). In addition to

$$\int_{t_0}^{\infty} q(s)\Delta s = \infty, \tag{1.7}$$

every solution x of (1.6) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Clearly, (1.1) is a special case of the above equations. In this paper, we will use a different Riccati transformation with the above papers. The purpose of this paper is to establish some new oscillation criteria for (1.1) which guarantee that every solution x of (1.1) oscillates or converges to zero as $t \to \infty$. Our results are new for third order nonlinear delay dynamic equations and complement the results established in literature.

The paper is organized as follows: In Section 2, we present some lemmas which will be used in the proof of our main results. In Section 3, by developing a Riccati transformation technique, integral averaging technique and inequalities, we give some sufficient conditions which guarantee that every solution of Eq. (1.1) oscillates or converges to zero. In Section 4, we give two examples to illustrate Corollary 3.1 and Theorem 3.2, respectively.

2 Some preliminary lemmas

In this section, by employing the Riccati transformation technique, we state the main results which guarantee that every solution of Eq. (1.1) oscillates or converges to zero.

It will be convenient to make the following notations:

$$d_+(t) := \max\{0, d(t)\}, \quad d_-(t) := \max\{0, -d(t)\}.$$

Before stating our main results, we begin with the following lemmas which will play important roles in the proof of the main results.

Lemma 2.1 *[21] Let $a,\ b \in \mathbb{T}$ and $\tau \in C^1_{rd}([a,b]_{\mathbb{T}}, \mathbb{T})$ be a strictly increasing function and $x \in C^1_{rd}([\tau(a), \tau(b)]_{\mathbb{T}}, \mathbb{R})$. Then for $t \in [a,b]_{\mathbb{T}}$,*

$$(x(\tau(t)))^{\Delta} = x^{\Delta}(\tau(t))\tau^{\Delta}(t). \tag{2.1}$$

Lemma 2.2 *Assume that* (1.2) *holds. Furthermore, assume that x is an eventually positive solution of* (1.1). *Then there are only two possible cases for $t \geq t_0$ sufficiently large:*

$$(I)\quad x(t) > 0, \quad x^{\Delta}(t) > 0, \quad (a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} > 0;$$

or

$$(II)\quad x(t) > 0, \quad x^{\Delta}(t) < 0, \quad (a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} > 0.$$

Proof. Let x be an eventually positive solution of (1.1). Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. From (C_2) and (1.1), it is clear that

$$\left(b(t)\left((a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta}\right)^{\alpha_2}\right)^{\Delta} = -q(t)x^{\alpha_3}(\tau(t)) < 0, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Then $b(t)\left((a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta}\right)^{\alpha_2}$ is decreasing on $[t_1, \infty)_{\mathbb{T}}$, thus $(a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta}$ is eventually of one sign. We claim that $(a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} > 0$. Otherwise, there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$(a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} < 0, \quad t \in [t_2, \infty)_{\mathbb{T}}.$$

Then $a(t)(x^{\Delta}(t))^{\alpha_1}$ is decreasing and there exist constants d and $t_3 \in [t_2, \infty)_{\mathbb{T}}$, such that

$$b(t)\left((a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta}\right)^{\alpha_2} \leq d < 0, \quad t \in [t_3, \infty)_{\mathbb{T}}.$$

Dividing by $b(t)$ and integrating from t_3 to t, we get

$$a(t)(x^{\Delta}(t))^{\alpha_1} \leq a(t_3)(x^{\Delta}(t_3))^{\alpha_1} + d^{\frac{1}{\alpha_2}} \int_{t_3}^{t} \left(\frac{1}{b(s)}\right)^{\frac{1}{\alpha_2}} \Delta s. \tag{2.2}$$

Letting $t \to \infty$ in (2.2), we obtain $a(t)(x^{\Delta}(t))^{\alpha_1} \to -\infty$ by (1.2). Thus, there exist constants c and $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that

$$a(t)(x^{\Delta}(t))^{\alpha_1} \leq a(t_4)(x^{\Delta}(t_4))^{\alpha_1} = c < 0, \quad t \in [t_4, \infty)_{\mathbb{T}}.$$

Dividing by $a(t)$ and integrating the previous inequality from t_4 to t, we have

$$x(t) - x(t_4) \leq c^{\frac{1}{\alpha_1}} \int_{t_4}^{t} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha_1}} \Delta s, \tag{2.3}$$

which implies that $x(t) \to -\infty$ as $t \to \infty$ by (1.2), a contradiction with the fact that $x(t) > 0$. We conclude that $(a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} > 0$ for large t and we get (I) or (II). This completes the proof.

Lemma 2.3 *Assume that* (1.2) *holds. If x is an eventually positive solution of* (1.1) *satisfying Case (I) of Lemma 2.2, then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$x^{\Delta}(\tau(t)) \geq \left(\frac{\delta(\tau(t), t_1)}{a(\tau(t))}\right)^{\frac{1}{\alpha_1}} \left(b(\sigma(t))\left((a(\sigma(t))(x^{\Delta}(\sigma(t)))^{\alpha_1})^{\Delta}\right)^{\alpha_2}\right)^{\frac{1}{\alpha_1\alpha_2}}, \quad t \in [t_1, \infty)_{\mathbb{T}}, \tag{2.4}$$

where $\delta(t, t_1) = \int_{t_1}^{t} b^{-1/\alpha_2}(s)\Delta s$.

Proof. Let x is an eventually positive solution of (1.1) satisfying Case (I) of Lemma 2.2. Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and from Eq. (1.1), we have

$$x^{\Delta}(t) > 0, \quad (a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} > 0, \quad \left(b(t)\left((a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta}\right)^{\alpha_2}\right)^{\Delta} < 0 \ \text{ for } \ t \in [t_1, \infty)_{\mathbb{T}}.$$

So $b(t)\left((a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta}\right)^{\alpha_2}$ is decreasing on $[t_1, \infty)_{\mathbb{T}}$. For $t \in [t_1, \infty)_{\mathbb{T}}$, we have

$$a(t)(x^{\Delta}(t))^{\alpha_1} = a(t_1)(x^{\Delta}(t_1))^{\alpha_1} + \int_{t_1}^{t} \frac{b^{\frac{1}{\alpha_2}}(s)(a(s)(x^{\Delta}(s))^{\alpha_1})^{\Delta}}{b^{\frac{1}{\alpha_2}}(s)} \Delta s$$

$$\geq b^{\frac{1}{\alpha_2}}(t)(a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta}\delta(t,t_1),$$

that is,

$$x^{\Delta}(t) \geq \left(\frac{\delta(t,t_1)b^{\frac{1}{\alpha_2}}(t)}{a(t)}\left(a(t)(x^{\Delta}(t))^{\alpha_1}\right)^{\Delta}\right)^{\frac{1}{\alpha_1}}.$$

Since $b(t)\left((a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta}\right)^{\alpha_2}$ is decreasing on $[t_1,\infty)_{\mathbb{T}}$, we obtain

$$x^{\Delta}(\tau(t)) \geq \left(\frac{\delta(\tau(t),t_1)}{a(\tau(t))}b^{\frac{1}{\alpha_2}}(\sigma(t))\left(a(\sigma(t))(x^{\Delta}(\sigma(t)))^{\alpha_1}\right)^{\Delta}\right)^{\frac{1}{\alpha_1}}$$

and this leads to (2.4). The proof is complete.

3 Main results

Now, we are in a position to state and prove the main results which guarantee that every solution of Eq. (1.1) oscillates or converges to zero.

Theorem 3.1 *Assume that* (1.2) *and* $\alpha_1\alpha_2 \geq 1$ *hold. Furthermore, assume that there exists a positive function* $r \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that for sufficiently large* $t_1,\ t_2$ *with* $t_2 \geq t_1 \geq t_0$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[r(s)q(s)\xi(s) - \frac{(r^{\Delta}(s))^2}{4MQ(s)\tau^{\Delta}(s)}\right]\Delta s = \infty \tag{3.1}$$

and

$$\int_{t_0}^{\infty}\left[\frac{1}{a(u)}\int_{u}^{\infty}\left[\frac{1}{b(v)}\int_{v}^{\infty}q(s)\Delta s\right]^{\frac{1}{\alpha_2}}\Delta v\right]^{\frac{1}{\alpha_1}}\Delta u = \infty, \tag{3.2}$$

where M *is a positive constant,* δ *is defined as in Lemma 2.3,* $Q(t) = r(t)\left[\delta(\tau(t),t_1)/a(\tau(t))\right]^{1/\alpha_1}$,

$$\xi(t) = \begin{cases} m_1, \quad m_1 \text{ is any positive constant}, \quad \text{if } \alpha_3 > 1,\\ 1, \quad \text{if } \alpha_3 = 1,\\ m_2\eta^{\alpha_3-1}(t,t_2), \quad m_2 \text{ is any positive constant}, \quad \text{if } \alpha_3 < 1\end{cases}$$

and $\eta(t,t_2) = \int_{t_2}^{t}(\delta(s)/a(s))^{1/\alpha_1}\Delta s$. *Then every solution of* (1.1) *is either oscillatory or converges to zero.*

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.1). We may assume without loss of generality that there exists a number $t_1 \in [t_0,\infty)_{\mathbb{T}}$, such that $x(t) > 0,\ x(\tau(t)) > 0$ and the conclusions of Lemmas 2.1 and 2.2 hold for all $t \in [t_1,\infty)_{\mathbb{T}}$. We only consider the case when x is eventually positive, since the case when x is eventually negative is similar. Since (1.2) holds, in view of Lemma 2.1, there are two possible cases.

Case (I): $x^{\Delta}(t) > 0,\quad (a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} > 0,\quad t \in [t_1,\infty)_{\mathbb{T}}$.
Define the function ω by

$$\omega(t) = r(t)\frac{b(t)\left((a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta}\right)^{\alpha_2}}{x(\tau(t))},\quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{3.3}$$

Then $\omega(t) > 0$. Using the product rule, we have

$$\omega^{\Delta}(t) = b(\sigma(t))\left((a(\sigma(t))(x^{\Delta}(\sigma(t)))^{\alpha_1})^{\Delta}\right)^{\alpha_2}\left(\frac{r(t)}{x(\tau(t))}\right)^{\Delta} + \frac{r(t)}{x(\tau(t))}(b(t)\left((a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta}\right)^{\alpha_2})^{\Delta}.$$

By the quotient rule and applying (1.1) to the above equality, we get

$$\omega^{\Delta}(t) = b(\sigma(t))\left((a(\sigma(t))(x^{\Delta}(\sigma(t)))^{\alpha_1})^{\Delta}\right)^{\alpha_2}\frac{r^{\Delta}(t)x(\tau(t)) - r(t)(x(\tau(t)))^{\Delta}}{x(\tau(t))x(\tau(\sigma(t)))} - r(t)q(t)x^{\alpha_3-1}(\tau(t)).$$

From (2.1) and (3.3), it follows that

$$\omega^{\Delta}(t) = -r(t)q(t)x^{\alpha_3-1}(\tau(t)) + \frac{r^{\Delta}(t)}{r(\sigma(t))}\omega(\sigma(t))$$

$$-b(\sigma(t))\left((a(\sigma(t))(x^{\Delta}(\sigma(t)))^{\alpha_1})^{\Delta}\right)^{\alpha_2}\frac{r(t)x^{\Delta}(\tau(t))\tau^{\Delta}(t)}{x(\tau(t))x(\tau(\sigma(t)))}. \tag{3.4}$$

Since $b(t)\left((a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta}\right)^{\alpha_2}$ is decreasing on $[t_1,\infty)_{\mathbb{T}}$, there exists a constant $b_1 > 0$ such that

$$b(\sigma(t))\left((a(\sigma(t))(x^{\Delta}(\sigma(t)))^{\alpha_1})^{\Delta}\right)^{\alpha_2} \le b(t)\left((a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta}\right)^{\alpha_2} \le b_1, \quad t \ge t_1, \tag{3.5}$$

where $b_1 = b(t_1)\left((a(t_1)(x^{\Delta}(t_1))^{\alpha_1})^{\Delta}\right)^{\alpha_2}$. Applying (3.5) to (2.4) and noting that $\alpha_1\alpha_2 \ge 1$, we obtain

$$x^{\Delta}(\tau(t)) \ge M\left(\frac{\delta(\tau(t),t_1)}{a(\tau(t))}\right)^{\frac{1}{\alpha_1}} b(\sigma(t))\left((a(\sigma(t))(x^{\Delta}(\sigma(t)))^{\alpha_1})^{\Delta}\right)^{\alpha_2}, \tag{3.6}$$

where $M = b_1^{1/(\alpha_1\alpha_2)-1}$. From (3.4) and (3.6), we have

$$\omega^{\Delta}(t) \le -r(t)q(t)x^{\alpha_3-1}(\tau(t)) + \frac{r^{\Delta}(t)}{r(\sigma(t))}\omega(\sigma(t))$$

$$-M\frac{r(t)\tau^{\Delta}(t)}{x(\tau(t))x(\tau(\sigma(t)))}\left(\frac{\delta(\tau(t),t_1)}{a(\tau(t))}\right)^{\frac{1}{\alpha_1}} b^2(\sigma(t))\left((a(\sigma(t))(x^{\Delta}(\sigma(t)))^{\alpha_1})^{\Delta}\right)^{2\alpha_2}.$$

Noting that $x^{\Delta}(t) > 0$, $t \in [t_1,\infty)_{\mathbb{T}}$ and from (3.3), we get

$$\omega^{\Delta}(t) \le -r(t)q(t)x^{\alpha_3-1}(\tau(t)) + \frac{r^{\Delta}(t)}{r(\sigma(t))}\omega(\sigma(t)) - \frac{MQ(t)\tau^{\Delta}(t)}{r^2(\sigma(t))}\omega^2(\sigma(t)). \tag{3.7}$$

Next, we consider the following three cases:
Case (i). Let $\alpha_3 > 1$. From $x^{\Delta}(t) > 0$, there exist constants c_1 and $t_2 \ge t_1$, such that

$$x(t) \ge x(t_2) = c_1.$$

Hence

$$x^{\alpha_3-1}(\tau(t)) \ge m_1, \quad t \ge t_2, \tag{3.8}$$

where $m_1 = c_1^{\alpha_3-1}$.
Case (ii). Let $\alpha_3 = 1$. Then

$$x^{\alpha_3-1}(\tau(t)) = 1, \quad t \ge t_1. \tag{3.9}$$

Case (iii). Let $\alpha_3 < 1$. From (3.5), we obtain

$$(a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} \le b_1^{\frac{1}{\alpha_2}} b^{-\frac{1}{\alpha_2}}(t), \quad t \ge t_1. \tag{3.10}$$

Integrating (3.10) from t_1 to t, we have

$$a(t)(x^{\Delta}(t))^{\alpha_1} \le a(t_1)(x^{\Delta}(t_1))^{\alpha_1} + b_1^{\frac{1}{\alpha_2}}\delta(t,t_1).$$

Thus there exist constants $b_2 > 0$ and $t_2 \ge t_1$, such that

$$a(t)(x^{\Delta}(t))^{\alpha_1} \le b_2\delta(t,t_1).$$

Dividing by $a(t)$ and integrating from t_2 to t, we get

$$x(t) \le x(t_2) + b_2^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\left(\frac{\delta(s,t_1)}{a(s)}\right)^{\frac{1}{\alpha_1}}\Delta s.$$

Then there exists a constant $b_3 > 0$ such that

$$x(t) \le b_3 \int_{t_2}^{t} \left(\frac{\delta(s,t_1)}{a(s)} \right)^{\frac{1}{\alpha_1}} \Delta s,$$

that is

$$x^{\alpha_3 - 1}(\tau(t)) \ge m_2 \eta^{\alpha_3 - 1}(t, t_2), \quad t \ge t_2, \tag{3.11}$$

where $m_2 = b_3^{\alpha_3 - 1}$ and $\eta(t, t_2) = \int_{t_2}^{t} (\delta(s,t_1)/a(s))^{1/\alpha_1} \Delta s$.

Combining (3.7) with (3.8), (3.9) and (3.11), we have

$$\begin{aligned} \omega^{\Delta}(t) &\le -r(t)q(t)\xi(t) + \frac{r^{\Delta}(t)}{r(\sigma(t))}\omega(\sigma(t)) - \frac{MQ(t)\tau^{\Delta}(t)}{r^2(\sigma(t))}\omega^2(\sigma(t)) \qquad (3.12) \\ &= -r(t)q(t)\xi(t) - \left[\frac{\sqrt{MQ(t)\tau^{\Delta}(t)}}{r(\sigma(t))}\omega(\sigma(t)) - \frac{r^{\Delta}(t)}{2\sqrt{MQ(t)\tau^{\Delta}(t)}} \right]^2 + \frac{(r^{\Delta}(t))^2}{4MQ(t)\tau^{\Delta}(t)} \\ &\le -r(t)q(t)\xi(t) + \frac{(r^{\Delta}(t))^2}{4MQ(t)\tau^{\Delta}(t)}, \end{aligned}$$

that is

$$\omega^{\Delta}(t) \le -\left(r(t)q(t)\xi(t) - \frac{(r^{\Delta}(t))^2}{4MQ(t)\tau^{\Delta}(t)} \right). \tag{3.13}$$

Integrating (3.13) from t_2 to t, we obtain

$$-\omega(t_2) < \omega(t) - \omega(t_2) \le -\int_{t_2}^{t} \left(r(s)q(s)\xi(s) - \frac{(r^{\Delta}(s))^2}{4MQ(s)\tau^{\Delta}(s)} \right) \Delta s,$$

which yields

$$\int_{t_2}^{t} \left(r(s)q(s)\xi(s) - \frac{(r^{\Delta}(s))^2}{4MQ(s)\tau^{\Delta}(s)} \right) \Delta s < \omega(t_2)$$

for all large t and this leads to a contradiction with (3.1).

Case (II): $x^{\Delta}(t) < 0, \ \ (a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} > 0, \ \ t \in [t_1, \infty)_{\mathbb{T}}$.

Since $x(t) > 0$ and $x^{\Delta}(t) < 0$, $\lim_{t\to\infty} x(t)$ exists and $\lim_{t\to\infty} x(t) = l \ge 0$. We claim that $l = 0$. Otherwise, $\lim_{t\to\infty} x(t) = l > 0$. Then $x(t) \ge l$, for $t \ge t_1$. Integrating (1.1) from t to ∞, we get

$$b(t) \left((a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} \right)^{\alpha_2} \ge \int_{t}^{\infty} q(s) x^{\alpha_3}(s) \Delta s,$$

which yields

$$(a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} \ge \left[\frac{1}{b(t)} \int_{t}^{\infty} q(s) x^{\alpha_3}(s) \Delta s \right]^{\frac{1}{\alpha_2}}.$$

Integrating again from t to ∞, we obtain

$$-a(t)(x^{\Delta}(t))^{\alpha_1} \ge \int_{t}^{\infty} \left[\frac{1}{b(v)} \int_{v}^{\infty} q(s) x^{\alpha_3}(s) \Delta s \right]^{\frac{1}{\alpha_2}} \Delta v,$$

that is,

$$-x^{\Delta}(t) \ge \left[\frac{1}{a(t)} \int_{t}^{\infty} \left[\frac{1}{b(v)} \int_{v}^{\infty} q(s) x^{\alpha_3}(s) \Delta s \right]^{\frac{1}{\alpha_2}} \Delta v \right]^{\frac{1}{\alpha_1}}.$$

Integrating from t_0 to ∞, we have

$$x(t_0) \ge \int_{t_0}^{\infty} \left[\frac{1}{a(u)} \int_{u}^{\infty} \left[\frac{1}{b(v)} \int_{v}^{\infty} q(s) x^{\alpha_3}(s) \Delta s \right]^{\frac{1}{\alpha_2}} \Delta v \right]^{\frac{1}{\alpha_1}} \Delta u.$$

Since $x(t) \geq l$, we see that

$$x(t_0) \geq l^{\frac{\alpha_3}{\alpha_1\alpha_2}} \int_{t_0}^{\infty} \left[\frac{1}{a(u)} \int_u^{\infty} \left[\frac{1}{b(v)} \int_v^{\infty} q(s)\Delta s \right]^{\frac{1}{\alpha_2}} \Delta v \right]^{\frac{1}{\alpha_1}} \Delta u,$$

which is a contradiction with the condition (3.2). Therefore, $l = 0$, that is, $\lim_{t\to\infty} x(t) = 0$. This completes the proof.

Remark 3.1 *It is easy to see that when* $\alpha_3 = 1$, (1.1) *can be transformed into a similar form with* (1.6)*, where* $f(x(t)) = x(t)$*. In this paper, replacing* $\tau(t)$ *with* t*, we use the same Riccati transformation with [11], i.e.,*

$$\omega(t) = r(t)\frac{b(t)\left((a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta}\right)^{\alpha_2}}{x(t)},$$

and Theorem 2.1 extends and improves Theorem 2.1 in [11]. Similarly, (1.1) *can be simplified to* (1.3) *and Theorem 2.1 complements Theorem 1 in [7].*

Remark 3.2 *In [7] and [11], Yu and Wang, Erbe, Peterson and Saker proved that every solution converges to zero if* (1.7) *holds, respectively. But one can easily see that this result can't be applied if*

$$\int_{t_0}^{\infty} q(s)\Delta s < \infty,$$

so our results extend and improve the results in [11].

Remark 3.3 *If the assumption* (3.2) *is not satisfied, we have some sufficient conditions which ensure that every solution of* (1.1) *oscillates or* $\lim_{t\to\infty} x(t)$ *exists (finite).*

Remark 3.4 *From Theorem 3.1, we can obtain different conditions for oscillation of all solutions of* (1.1) *with different choices of* r.

Taking $r(t) = 1$ and $r(t) = t$ in Theorem 2.1 respectively, we have the following two results.

Corollary 3.1 *Assume that* (1.2), (3.2) *and* $\alpha_1\alpha_2 \geq 1$ *hold. Furthermore, assume that*

$$\int_{t_0}^{\infty} q(s)\xi(s)\Delta s = \infty, \tag{3.14}$$

where ξ *is defined as in Theorem 3.1. Then every solution of* (1.1) *is either oscillatory or converges to zero.*

Corollary 3.2 *Assume that* (1.2), (3.2) *and* $\alpha_1\alpha_2 \geq 1$ *hold. Furthermore, assume that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[sq(s)\xi(s) - \frac{1}{4Ms\tau^{\Delta}(s)} \left(\frac{\delta(\tau(t), t_1)}{a(\tau(t))} \right)^{-\frac{1}{\alpha_1}} \right] \Delta s = \infty,$$

where M *and* ξ *are defined as in Theorem 3.1. Then every solution of* (1.1) *is either oscillatory or converges to zero.*

Theorem 3.2 *Assume that* (1.2), (3.2) *and* $\alpha_1\alpha_2 \geq 1$ *hold. Furthermore, assume that there exist* $m \geq 1$ *and a positive function* $r \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ *such that for sufficiently large* t_1, t_2 *with* $t_2 \geq t_1 \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} (t-s)^m \left[r(s)q(s)\xi(s) - \frac{(r^{\Delta}(s))^2}{4MQ(s)\tau^{\Delta}(s)} \right] \Delta s = \infty, \tag{3.15}$$

where M, Q *and* ξ *are defined as in Theorem 3.1. Then every solution of* (1.1) *is either oscillatory or converges to zero.*

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.1). We may assume without loss of generality that there exists a number $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that $x(t) > 0,\ x(\tau(t)) > 0$ and the conclusions of Lemmas 2.1 and 2.2 hold for all $t \in [t_1, \infty)_{\mathbb{T}}$. We only consider the case when x is eventually positive, since the case when x is eventually negative is similar. Since (1.2) holds, in view of Lemma 2.1, there are two possible cases.

Case (I): $x^{\Delta}(t) > 0, \quad (a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}$.
We define the function ω by (3.3) again and proceeding as in the proof of Theorem 3.1, we have $\omega(t) > 0$ and (3.13).

Multiplying (3.13) by $(t-s)^m$ and integrating from t_1 to t, we get

$$\int_{t_1}^{t} (t-s)^m \left(r(s)q(s)\xi(s) - \frac{(r^{\Delta}(s))^2}{4MQ(s)\tau^{\Delta}(s)} \right) \Delta s \leq -\int_{t_1}^{t} (t-s)^m \omega^{\Delta}(s) \Delta s. \tag{3.16}$$

Using the integration by parts formula, we obtain

$$\int_{t_1}^{t} (t-s)^m \omega^{\Delta}(s) \Delta s = -\omega(t_1)(t-t_1)^m - \int_{t_1}^{t} Q(t,s)\omega(\sigma(s))\Delta s, \tag{3.17}$$

where $Q(t,s) = ((t-s)^m)^{\Delta_s}$. From (3.16), (3.17) and multiplying $1/t^m$, we have

$$\frac{1}{t^m} \int_{t_1}^{t} (t-s)^m \left[r(s)q(s)\xi(s) - \frac{(r^{\Delta}(s))^2}{4MQ(s)\tau^{\Delta}(s)} \right] \Delta s \leq \omega(t_1) \left(\frac{t-t_1}{t} \right)^m + \frac{1}{t^m} \int_{t_1}^{t} Q(t,s)\omega(\sigma(s))\Delta s.$$

Since

$$Q(t,s) = \begin{cases} -m(t-s)^{m-1}, & \text{if } \mu(s) = 0, \\ \frac{(t-\sigma(s))^m - (t-s)^m}{\mu(s)}, & \text{if } \mu(s) > 0, \end{cases}$$

and noting that $m \geq 1$, $Q(t,s) \leq 0$ for $t \geq \sigma(s)$, we obtain

$$\frac{1}{t^m} \int_{t_1}^{t} (t-s)^m \left[r(s)q(s)\xi(s) - \frac{(r^{\Delta}(s))^2}{4MQ(s)\tau^{\Delta}(s)} \right] \Delta s \leq \omega(t_1) \left(\frac{t-t_1}{t} \right)^m,$$

that is

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m \left[r(s)q(s)\xi(s) - \frac{(r^{\Delta}(s))^2}{4MQ(s)\tau^{\Delta}(s)} \right] \Delta s \leq \omega(t_1),$$

which is a contradiction with (3.15).

Case (II): $x^{\Delta}(t) < 0, \quad (a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}$.
Proceeding as in the proof of Theorem 3.1, we get a contradiction with (3.2). This completes the proof.

Remark 3.5 *From Theorem 3.2, we can obtain different conditions for oscillation of all solutions of* (1.1) *with different choices of* r.

Theorem 3.3 *Assume that* (1.2), (3.2) *and* $\alpha_1\alpha_2 \geq 1$ *hold. Furthermore, assume that there exist* $m \geq 1$ *and a positive function* $r \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ *such that for sufficiently large* $t_1,\ t_2$ *with* $t_2 \geq t_1 \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} \left[(t-s)^m r(s)q(s)\xi(s) - \frac{r^2(\sigma(s))}{4M(t-s)^m Q(s)\tau^{\Delta}(s)} P^2(t,s) \right] \Delta s = \infty, \tag{3.18}$$

where $M,\ Q$ *and* ξ *are defined as in Theorem 3.1 and*

$$P(t,s) = (t-s)^m \frac{r^{\Delta}(s)}{r(\sigma(s))} + Q(t,s), \quad t \geq s \geq t_0.$$

Then every solution of (1.1) *is either oscillatory or converges to zero.*

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.1). We may assume without loss of generality that there exists a number $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that $x(t) > 0,\ x(\tau(t)) > 0$ and the conclusions of Lemmas 2.1 and 2.2 hold for all $t \in [t_1, \infty)_{\mathbb{T}}$. We only consider the case when x is eventually positive, since the case when x is eventually negative is similar. Since (1.2) holds, in view of Lemma 2.1, there are two possible cases.

Case (I): $x^{\Delta}(t) > 0, \quad (a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}$.
We define the function ω by (3.3) again and proceeding as in the proof of Theorem 3.1, we have $\omega(t) > 0$ and (3.12).

Multiplying (3.12) by $(t-s)^m$ and integrating from t_1 to t, we have

$$\int_{t_1}^{t} (t-s)^m \omega^{\Delta}(s)\Delta s \le -\int_{t_1}^{t} (t-s)^m r(s)q(s)\xi(s)\Delta s + \int_{t_1}^{t} (t-s)^m \frac{r^{\Delta}(s)}{r(s)}\omega(\sigma(s))\Delta s - \int_{t_1}^{t} M(t-s)^m \frac{Q(s)\tau^{\Delta}(s)}{r^2(\sigma(s))}\omega^2(\sigma(s))\Delta s. \tag{3.19}$$

From (3.17) and (3.19), it follows that

$$\begin{aligned}\int_{t_1}^{t} (t-s)^m r(s)q(s)\xi(s)\Delta s &\le \omega(t_1)(t-t_1)^m + \int_{t_1}^{t} \left[(t-s)^m \frac{r^{\Delta}(s)}{r(\sigma(s))}\omega(\sigma(s)) + Q(t,s)\omega(\sigma(s))\right]\Delta s \\ &\quad - \int_{t_1}^{t} M(t-s)^m \frac{Q(s)\tau^{\Delta}(s)}{r^2(\sigma(s))}\omega^2(\sigma(s))\Delta s \\ &\le \omega(t_1)(t-t_1)^m + \int_{t_1}^{t} \frac{r^2(\sigma(s))}{4M(t-s)^m Q(s)\tau^{\Delta}(s)} P^2(t,s)\Delta s,\end{aligned}$$

and this implies that

$$\limsup_{t\to\infty} \frac{1}{t^m}\int_{t_1}^{t} \left[(t-s)^m r(s)q(s)\xi(s) - \frac{r^2(\sigma(s))}{4M(t-s)^m Q(s)\tau^{\Delta}(s)}P^2(t,s)\right]\Delta s \le \omega(t_1),$$

which contradicts (3.18).

Case (II): $x^{\Delta}(t) < 0, \quad (a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}$.
The remainder of the proof is similar to that of Theorem 3.1, so we omit the details. This completes the proof.

Theorem 3.4 *Assume that* (1.2), (3.2) *and* $\alpha_1\alpha_2 \ge 1$ *hold. Furthermore, assume that there exist functions* $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv \{(t,s) : t \ge s \ge t_0\}$ *such that*

$$H(t,t) = 0, \quad t \ge t_0, \quad H(t,s) > 0, \quad t > s \ge t_0, \tag{3.20}$$

H has a nonpositive continuous $\Delta-$partial derivation $H^{\Delta_s}(t,s)$ with respect to the second variable and satisfies

$$H^{\Delta_s}(t,s) + H(t,s)\frac{r^{\Delta}(s)}{r(\sigma(s))} = -\frac{h(t,s)}{r(\sigma(s))}H^{\frac{1}{2}}(t,s) \tag{3.21}$$

and for sufficiently large $t_1,\ t_2$ with $t_2 \ge t_1 \ge t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,s)}\int_{t_0}^{t} \left[H(t,s)r(s)q(s)\xi(s) - \frac{(h_-(t,s))^2}{4MQ(s)\tau^{\Delta}(s)}\right]\Delta s = \infty, \tag{3.22}$$

where $r \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ is a positive function, M, Q and ξ are defined as in Theorem 3.1 and $h_-(t,s) = \max\{0, -h(t,s)\}$. Then every solution of (1.1) *is either oscillatory or converges to zero.*

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.1). We may assume without loss of generality that there exists a number $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that $x(t) > 0,\ x(\tau(t)) > 0$ and the conclusions of Lemmas 2.1 and 2.2 hold for all $t \in [t_1, \infty)_{\mathbb{T}}$. We only consider the case when x is eventually positive, since the case when x is eventually negative is similar. Since (1.2) holds, in view of Lemma 2.1, there are two possible cases.

Case (I): $x^{\Delta}(t) > 0, \quad (a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}$.
We define the function ω by (3.3) again and proceeding as in the proof of Theorem 3.1, we have $\omega(t) > 0$ and (3.12).

Multiplying (3.12) by $H(t,s)$ and integrating from t_1 to t, we see that

$$\int_{t_1}^{t} H(t,s)r(s)q(s)\xi(s)\Delta s \le -\int_{t_1}^{t} H(t,s)\omega^{\Delta}(s)\Delta s + \int_{t_1}^{t} H(t,s)\frac{r^{\Delta}(s)}{r(\sigma(s))}\omega(\sigma(s))\Delta s - \int_{t_1}^{t} MH(t,s)\frac{Q(s)\tau^{\Delta}(s)}{r^2(\sigma(s))}\omega^2(\sigma(s))\Delta s.$$

From (3.20), (3.21) and the above inequality, we have

$$\begin{aligned}\int_{t_1}^{t} H(t,s)r(s)q(s)\xi(s)\Delta s &\le H(t,t_1)\omega(t_1) + \int_{t_1}^{t} H^{\Delta_s}(t,s)\omega(\sigma(s))\Delta s \\ &\quad + \int_{t_1}^{t} H(t,s)\frac{r^{\Delta}(s)}{r(\sigma(s))}\omega(\sigma(s))\Delta s - \int_{t_1}^{t} MH(t,s)\frac{Q(s)\tau^{\Delta}(s)}{r^2(\sigma(s))}\omega^2(\sigma(s))\Delta s \\ &= H(t,t_1)\omega(t_1) + \int_{t_1}^{t}\left[-\frac{h(t,s)}{r(\sigma(s))}H^{\frac{1}{2}}(t,s)\omega(\sigma(s)) - MH(t,s)\frac{Q(s)\tau^{\Delta}(s)}{r^2(\sigma(s))}\omega^2(\sigma(s))\right]\Delta s \\ &\le H(t,t_1)\omega(t_1) + \int_{t_1}^{t}\left[\frac{h_-(t,s)}{r(\sigma(s))}H^{\frac{1}{2}}(t,s)\omega(\sigma(s)) - MH(t,s)\frac{Q(s)\tau^{\Delta}(s)}{r^2(\sigma(s))}\omega^2(\sigma(s))\right]\Delta s.\end{aligned}$$

This implies that

$$\begin{aligned}\int_{t_1}^{t} H(t,s)r(s)q(s)\xi(s)\Delta s &\le H(t,t_1)\omega(t_1) + \int_{t_1}^{t}\frac{(h_-(t,s))^2}{4MQ(s)\tau^{\Delta}(s)}\Delta s \\ &\quad - \int_{t_1}^{t}\left[\frac{\sqrt{MH(t,s)Q(s)\tau^{\Delta}(s)}}{r(\sigma(s))}\omega(\sigma(s)) - \frac{h_-(t,s)}{2\sqrt{MQ(s)\tau^{\Delta}(s)}}\right]^2\Delta s \\ &\le H(t,t_1)\omega(t_1) + \int_{t_1}^{t}\frac{(h_-(t,s))^2}{4MQ(s)\tau^{\Delta}(s)}\Delta s,\end{aligned}$$

that is

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)r(s)q(s)\xi(s) - \frac{(h_-(t,s))^2}{4MQ(s)\tau^{\Delta}(s)}\right]\Delta s \le \omega(t_1),$$

which contradicts (3.22).

Case (II): $x^{\Delta}(t) < 0, \quad (a(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}$.
Again the same arguments as in the proof of Theorem 3.1, we get a contradiction with (3.2). This completes the proof.

4 Examples

In this section, we will show the applications of our oscillation criteria in two examples. Firstly, we will give an example to illustrate Corollary 3.1.

Example 4.1 Consider the third order nonlinear dynamic equation

$$\left(\frac{1}{t}\left(\left(\frac{1}{t}(x^{\Delta}(t))^5\right)^{\Delta}\right)^{\frac{1}{3}}\right)^{\Delta} + \frac{1}{t^{\lambda}}x^{\alpha_3}(t-1) = 0, \quad t \ge 1, \tag{4.1}$$

where $\alpha_3 > 1$ is a ratio of odd positive integers and $0 < \lambda \le 1$. Set

$$b(t) = a(t) = 1/t,\ q(t) = 1/t^{\lambda},\ \alpha_1 = 5,\ \alpha_2 = 1/3,\ \tau(t) = t-1.$$

For any $t \geq 1$ we have

$$\delta(t,t_1) = \int_1^t b^{-\frac{1}{\alpha_2}}(s)\Delta s = \int_1^t s^3 \Delta s.$$

It is clear that the conditions (C_1), (C_2) and (1.2) are satisfied. Applying Corollary 3.1, it remains to satisfy the conditions (3.2) and (3.14).

$$\int_1^\infty \left[\frac{1}{a(u)}\int_u^\infty \left[\frac{1}{b(v)}\int_v^\infty q(s)\Delta s\right]^{\frac{1}{\alpha_2}} \Delta v\right]^{\frac{1}{\alpha_1}} \Delta u$$

$$= \int_1^\infty \left[u\int_u^\infty \left[v\int_v^\infty \frac{1}{s^\lambda}\Delta s\right]^3 \Delta v\right]^{\frac{1}{5}} \Delta u = \infty.$$

Noting that $\alpha_3 > 1$, we get $\xi(t) = m_1$. Letting $r(t) = 1$ and from $\alpha_1\alpha_2 = 5/3 > 1$, we obtain

$$\limsup_{t\to\infty} \int_1^t \left[r(s)q(s)\xi(s) - \frac{(r^\Delta(s))^2}{4MQ(s)\tau^\Delta(s)}\right]\Delta s = \limsup_{t\to\infty}\int_1^t m_1\frac{1}{s^\lambda}\Delta s = \infty.$$

We can see that (3.2) and (3.14) hold. Hence, by Corollary 3.1, every solution of (4.1) oscillates or converges to zero.

The next example illustrates Theorem 3.2.

Example 4.2 Examine the third order nonlinear dynamic equation

$$\left(\left(\left(t^{\alpha_1}(x^\Delta(t))^{\alpha_1}\right)^\Delta\right)^{\frac{1}{\alpha_1}}\right)^\Delta + \left(\int_1^t \frac{(s-1)^{\frac{1}{\alpha_1}}}{s}\Delta s\right)^{1-\alpha_3} x^{\alpha_3}(\tau(t)) = 0, \quad t \geq 1, \tag{4.2}$$

where $\alpha_1,\ \alpha_2 = 1/\alpha_1$ and $\alpha_3 < 1$ are ratios of positive odd integers. Let

$$b(t) = 1,\ a(t) = t^{\alpha_1},\ q(t) = \left(\int_1^t \frac{(s-1)^{\frac{1}{\alpha_1}}}{s}\Delta s\right)^{1-\alpha_3}.$$

For any $t \geq 1$ we have

$$\delta(t,t_1) = \int_1^t b^{-\frac{1}{\alpha_2}}(s)\Delta s = t-1, \quad \eta(t,t_2) = \int_1^t \frac{(s-1)^{\frac{1}{\alpha_1}}}{s}\Delta s.$$

It is clear that the conditions (C_1), (C_2), (1.2) and (3.2) are satisfied. Applying Theorem 3.2, it remains to satisfy the condition (3.15). Taking $m = 2$, $r(t) = 1$ for any $t \geq s \geq 1$ and from $\alpha_1\alpha_2 = 1$, $\alpha_3 < 1$, we get

$$\xi(t) = m_2\eta^{\alpha_3-1}(t,t_2) = m_2\left(\int_1^t \frac{(s-1)^{\frac{1}{\alpha_1}}}{s}\Delta s\right)^{\alpha_3-1}$$

and

$$\limsup_{t\to\infty}\frac{1}{t^2}\int_1^t (t-s)^2 q(s)\xi(s)\Delta s = \limsup_{t\to\infty}\frac{1}{t^2}\int_1^t m_2(t-s)^2\Delta s = \infty.$$

Hence, by Theorem 3.2, every solution of (4.2) oscillates or converges to zero.

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