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Positive solutions of two-point boundary value problems of nonlinear fractional differential equation at resonance*

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Abstract: This paper is concerned with a kind of nonlinear fractional differential boundary value problem at resonance with Caputo's fractional derivative. Our main approach is the recent Leggett-Williams norm-type theorem for coincidences due to O'Regan and Zima. The most interesting point is the acquisition of positive solutions for fractional differential boundary value problem at resonance. Moreover, an example is constructed to show that our result here is valid.

Keywords: Fractional differential equation; Resonance; Cone; Positive solution.

MSC: 26A33; 34B15

1. Introduction

This paper deals with positive solutions to the following boundary value problem:

$$ {}^cD^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \qquad 0 < t < 1 \tag{1.1}$$

$$ u(0) = 0, \quad u'(0) = u'(1), \tag{1.2}$$

where ${}^cD^{\alpha}_{0+}$ is the Caputo's fractional derivative of order α, $1 < \alpha \leq 2$ is a real number, and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function.

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Due to the fact that the fractional differential operator ${}^cD^\alpha_{0+}$ is not inventible under Caputo's derivative, boundary value problems (in short:BVPs) of this type are referred to as problems at resonance.

Recently, fractional differential equations (in short:FDE) have been studied extensively. For an extensive collection of such results, we refer the readers to the monographs [1-4] and the reference therein.

Some basic theory for the initial value problems of FDE involving Riemann-Liouville differential operator has been discussed [5-10]. Also, there are some papers which deal with the existence of positive solutions for BVPs of nonlinear FDE by using techniques of topological degree theory [11-16]. For example, the existence and multiplicity of positive solutions for the equation

$$D^\alpha_{0+}u(t) = f(t, u(t)), \quad 0 < t < 1, \ \ 1 < \alpha \le 2, \tag{1.3}$$

subject to the Dirichlet boundary condition

$$u(0) = u(1) = 0 \tag{1.4}$$

have been studied by Bai and L$\ddot{u}$ [13] by means of the well-known Krasnosel'skii fixed point theorem and Leggett-Williams fixed point theorem. D^α_{0+} is the standard Riemann-Liouville fractional derivative there.

In [14] and [15], Zhang also studied the existence of positive solutions of Eq.(1.3) under the boundary conditions

$$u(0) = \nu \ne 0, \ \ u(1) = \rho \ne 0 \tag{1.5}$$

and

$$u(0) + u'(0) = 0, \ \ u(1) + u'(1) = 0, \tag{1.6}$$

respectively. Due to the fact that the BVPs based on Riemann-Liouville derivative with non-zero boundary conditions can't be converted into an equivalent integral equation, while the Caputo's derivative is to meet the requirements. The conditions (1.5) and (1.6) are not zero boundary value, so the author investigated the BVPs (1.3)-(1.5) and (1.3)-(1.6) by involving the Caputo's fractional derivative.

M. El-Shahed [16] established the existence of positive solutions to BVP

$$D_{0+}^{\alpha} u(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1, \ \ 2 < \alpha \leq 3, \tag{1.7}$$

$$u(0) = u'(0) = u'(1) = 0 \tag{1.8}$$

by applying Krasnosel'skii fixed point theorem.

From above works, we can see a fact, although the BVPs of nonlinear FDE have been studied by some authors, to the best of our knowledge, all of existing works are limited to non-resonance boundary conditions. For the resonance case, as far as we know, no contributions exist. The aim of this paper is to fill the gap in the relevant literature. Our main tool is the recent Leggett-Williams norm-type theorem for coincidences due to O'Regan and Zima [17].

2. Preliminaries

For the convenience of the reader, we demonstrate and study the definitions and some fundamental facts of Caputo's fractional derivative.

Definition 2.1. The Riemann-Liouville fractional integral of order α is defined by

$$(I_{0+}^{\alpha} y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{y(s)}{(t-s)^{1-\alpha}} ds, \quad (t > 0, \alpha > 0) \tag{2.1}$$

where $\Gamma(\alpha)$ is the Euler gamma function defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad (z > 0) \tag{2.2}$$

for which, the reduction formula

$$\Gamma(z+1) = z\Gamma(z), \quad (z > 0), \quad \Gamma(1) = 1, \ \Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{2.3}$$

and formula

$$\int_0^1 t^{z-1} (1-t)^{\omega - 1} dt = \frac{\Gamma(z)\Gamma(\omega)}{\Gamma(z+\omega)}, \quad (z, \omega \notin \mathbb{Z}_0^-) \tag{2.4}$$

hold.

Definition 2.2. Caputo's derivative of order α for a function $y \in AC^n[0,1]$ can be represented by

$$({}^cD_{0+}^{\alpha}y)(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds =: (I_{0+}^{n-\alpha}D^n y)(t), \quad (t>0, \alpha>0) \tag{2.5}$$

where $D^n = \frac{d^n}{dt^n}$ and $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of α, and $AC^n[0,1] = \{f : [0,1] \to \mathbb{R} |\ D^{n-1}f \in AC[0,1]\}$.

Remark 2.1. Under natural conditions on the function $y(t)$, Caputo's derivative becomes a conventional m-th derivative of the function $y(t)$ as $\alpha \to m$(see [2]).

From definitions 2.1 and 2.2, we can deduce the following statement.

Lemma 2.1[4]**.** *The fractional differential equation*

$${}^cD_{0+}^{\alpha}y(t) = 0$$

has solutions $y(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, \cdots, n-1$, $n = [\alpha]+1$. *Furthermore, for* $y \in AC^n[0,1]$,

$$(I_{0+}^{\alpha}\,{}^cD_{0+}^{\alpha}y)(t) = y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}t^k \tag{2.6}$$

and

$$({}^cD_{0+}^{\alpha}I_{0+}^{\alpha}y)(t) = y(t). \tag{2.7}$$

In the following, we review some standard facts on Fredholm operators and cones in Banach spaces. Let X, Y be real Banach spaces. Consider a linear mapping $L : \mathrm{dom}L \subset X \to Y$ and a nonlinear mapping $N : X \to Y$.

Definition 2.3. Suppose that $X_1 \subset X$ is a subspace. A mapping $P : X \to X_1$ is a projector provided that

(i) $P^2x = Px$ for all $x \in X$,

(ii) $P(\lambda x + \mu y) = \lambda Px + \mu Py$ for all $x, y \in X$, $\lambda, \mu \in \mathbb{R}$.

Throughout we assume

1° L is a Fredholm operator of index zero, i.e. $\mathrm{Im}L$ is closed and $\mathrm{dimKer}L = \mathrm{codimIm}L < \infty$.

The assumption 1° implies that there exist continuous projections $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Ker}Q = \mathrm{Im}L$ with $X = \mathrm{Ker}L \oplus \mathrm{Ker}P$, $Y = \mathrm{Im}L \oplus \mathrm{Im}Q$ and $\mathrm{dimIm}Q = \mathrm{dimKer}L < \infty$. And we can define an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$. Denote by L_p the restriction of L to $\mathrm{Ker}P \cap \mathrm{dom}L$. Clearly, L_p is an isomorphism from $\mathrm{Ker}P \cap \mathrm{dom}L$ to $\mathrm{Im}L$, we denote its inverse by $K_p : \mathrm{Im}L \to \mathrm{Ker}P \cap \mathrm{dom}L$. It is known (see [19]) that the coincidence equation $Lx = Nx$ is equivalent to

$$x = (P + JQN)x + K_P(I - Q)Nx.$$

A nonempty closed convex set $C \subset X$ is said to be a cone in X provided that:

(i) $\mu x \in C$ for all $x \in C$ and $\mu \geq 0$,

(ii) $x, -x \in C$ implies $x = \theta$.

It is well known that C induces a partial order in X by

$$x \preceq y \quad \text{if and only if} \quad y - x \in C.$$

We will write $x \npreceq y$ for $y - x \notin C$. Moreover, for every $u \in C \setminus \{0\}$ there exists a positive number $\sigma(u)$ such that

$$||x + u|| \geq \sigma(u)||x||$$

for all $x \in C$. It is clear that if $\sigma(u) > 0$ is such that $||x + u|| \geq \sigma(u)||x||$ for all $x \in C$, then for every $\lambda > 0$,

$$||x + \lambda u|| \geq \sigma(u)||x|| \ \text{ for all } \ x \in C.$$

Let $\gamma : X \to C$ be a retraction, that is, a continuous mapping such that $\gamma(x) = x$ for all $x \in C$. Set

$$\Psi := P + JQN + K_p(I - Q)N \qquad \text{and} \qquad \Psi_\gamma := \Psi \circ \gamma.$$

We make use of the following result due to O'Regan and Zima [17].

Theorem 2.1. *Let C be a cone in X and let Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that the following conditions hold.*

2° *$QN : X \to Y$ is continuous and bounded and $K_p(I - Q)N : X \to X$ is compact on every bounded subset of X,*

3° *$Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap \mathrm{Im}L$ and $\lambda \in (0, 1)$,* 4° γ *maps subsets of* $\overline{\Omega}_2$ *into bounded subsets of* C,

5° $\deg\{[I-(P+JQN)\gamma]|_{\mathrm{Ker}L}, \mathrm{Ker}L\cap\Omega_2, 0\} \neq 0$,

6° *there exists* $u_0 \in C\setminus\{0\}$ *such that* $||x|| \leq \sigma(u_0)||\Psi x||$ *for* $x \in C(u_0)\cap\partial\Omega_1$, *where* $C(u_0) = \{x \in C : \mu u_0 \preceq x \ for\ some\ \mu > 0\}$ *and* $\sigma(u_0)$ *such that* $||x+u_0|| \geq \sigma(u_0)||x||$ *for every* $x \in C$,

7° $(P+JQN)\gamma(\partial\Omega_2) \subset C$,

8° $\Psi_\gamma(\overline{\Omega}_2\setminus\Omega_1) \subset C$.

Then the equation $Lx = Nx$ *has a solution in the set* $C\cap(\overline{\Omega}_2\setminus\Omega_1)$.

For simplicity of notation, we set

$$G(t,s) = \begin{cases} 1-\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}+\frac{t^\alpha}{\Gamma(\alpha+2)}+\frac{(1-s)^{2-\alpha}}{\Gamma(\alpha)}\int_s^1\frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}}d\tau-\frac{(1-s)^{2-\alpha}(t-s)^{\alpha-1}}{t(\alpha-1)\Gamma(\alpha+1)} & 0\leq s\leq t\leq 1, \\ 1-\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}+\frac{t^\alpha}{\Gamma(\alpha+2)}+\frac{(1-s)^{2-\alpha}}{\Gamma(\alpha)}\int_s^1\frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}}d\tau, & 0\leq t\leq s\leq 1. \end{cases}$$

Note that $G(t,s)\geq 0$ for $t,s\in[0,1]$. Set $\quad 0<\kappa\leq\min\{1, \frac{1}{\max\limits_{t,s\in[0,1]} G(t,s)}\}$.

Remark 2.2. The computation of the function $G(t,s)$ is shown in the proof of Theorem 3.1.

3. Main results

In order to prove the existence result, we present here a definition.

Definition 3.1. We say that the function $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ satisfies the L^1-Carath*é*odory conditions, if

(A1) for each $u\in\mathbb{R}$, the mapping $t\mapsto f(t,u)$ is Lebesgue measurable on [0,1],

(A2) for a.e. $t\in[0,1]$, the mapping $u\mapsto f(t,u)$ is continuous on $\mathbb{R}$,

(A3) for each $r>0$, there exists $\alpha_r\in L^1[0,1]$ satisfying $\alpha_r(t)>0$ on $[0,1]$ such that

$$|u|\leq r \ \text{ implies } \quad |f(t,u)|\leq\alpha_r(t).$$

In this paper, we consider the Banach spaces $X = C[0,1]$ and $Y = L^1[0,1]$ with the supper norm $||x|| = \max\limits_{t\in[0,1]}|x(t)|$ and Lebesgue absolutely integrable norm $||y|| = \int_0^1|y(t)|dt$, respectively. Define $L:\mathrm{dom}L\to Y$ by $Lx(t) = -{}^cD_{0+}^\alpha x(t)$ with

$$\mathrm{dom}L = \{x\in X : x\in AC^n[0,1], x(0)=0, x'(0)=x'(1),\ {}^cD_{0+}^\alpha x\in L^1[0,1]\}$$

and $N : X \to Y$ by $Nx(t) = f(t, x(t))$.

In order to obtain our main results, we firstly present and prove the following lemma.

Lemma 3.1. *$L : \mathrm{dom}L \subset X \to Y$ is a Fredholm operator of index zero, and the linear operator $K_p : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ can be written as*

$$(K_p y)(t) = \int_0^1 \frac{k(t,s)y(s)}{(1-s)^{2-\alpha}} ds,$$

where

$$k(t,s) = \frac{(1-s)^{2-\alpha}}{\Gamma(\alpha)} \begin{cases} \alpha(\alpha-1)t\int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}}d\tau - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1 \\ \alpha(\alpha-1)t\int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}}d\tau, & 0 \le t \le s \le 1. \end{cases}$$

Proof. It is clear that

$$\mathrm{Ker}L = \{x \in \mathrm{dom}L : x(t) = ct \text{ on } [0,1]\}.$$

We will show that

$$\mathrm{Im}L = \{y \in Y : \int_0^1 \frac{y(s)}{(1-s)^{2-\alpha}} ds = 0\}. \tag{3.1}$$

Since the problem

$$-{}^cD_{0+}^{\alpha} x(t) = y(t) \tag{3.2}$$

has solution $x(t)$ satisfies boundary conditions (1.2) if and only if

$$\int_0^1 \frac{y(s)}{(1-s)^{2-\alpha}} ds = 0. \tag{3.3}$$

In fact, if (3.2) has solution $x(t)$ satisfies (1.2), then from (3.2) we have

$$x'(t) = -\frac{1}{\Gamma(\alpha-1)} \int_0^t \frac{y(s)}{(t-s)^{2-\alpha}} ds + x'(0).$$

In view of $x'(0) = x'(1)$, we can obtain that

$$\int_0^1 \frac{y(s)}{(1-s)^{2-\alpha}} ds = 0.$$

On the other hand, if (3.3) holds, setting

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t \frac{y(s)}{(t-s)^{1-\alpha}} ds + Ct,$$

where C is arbitrary constant, then $x(t)$ is a solution of (3.2), and $x(0) = 0$, $x'(0) = x'(1)$. Hence (3.1) holds.

Next, we define $P : X \to X$ by $(Px)(t) = \alpha(\alpha-1)t\int_0^1 \frac{x(s)}{(1-s)^{2-\alpha}}ds$ and $Q : Y \to Y$ by

$$(Qy)(t) = \alpha(\alpha-1)t\int_0^1 \frac{y(s)}{(1-s)^{2-\alpha}}ds, \quad t \in [0,1].$$

It is easy to see that the operators P and Q are all projections. In fact, for $t \in [0,1]$,

$$\begin{aligned}
(P^2x)(t) &= P(Px)(t) = \alpha(\alpha-1)t\int_0^1 \frac{(Px)(s)}{(1-s)^{2-\alpha}}ds \\
&= \alpha^2(\alpha-1)^2 t\int_0^1 \frac{x(s)}{(1-s)^{2-\alpha}}ds \cdot \int_0^1 \frac{s}{(1-s)^{2-\alpha}}ds \\
&= \alpha(\alpha-1)t\int_0^1 \frac{x(s)}{(1-s)^{2-\alpha}}ds \\
&= (Px)(t).
\end{aligned}$$

The same to the operator Q.

In the sense of isomorphism, $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Ker}Q = \mathrm{Im}L$. So $\mathrm{dimKer}L = 1 = \mathrm{dimIm}Q = \mathrm{codimIm}L$. Notice that $\mathrm{Im}L$ is closed, L is a Fredholm operator of index zero.

For $y \in \mathrm{Im}L$, the inverse $K_p : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ of L_p can be given by

$$(K_p y)(t) = \int_0^1 k(t,s)\frac{y(s)}{(1-s)^{2-\alpha}}ds$$

where

$$k(t,s) = \frac{(1-s)^{2-\alpha}}{\Gamma(\alpha)}\begin{cases} \alpha(\alpha-1)t\int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}}d\tau - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1 \\ \alpha(\alpha-1)t\int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}}d\tau, & 0 \le t \le s \le 1. \end{cases} \tag{3.4}$$

In fact, for $x \in \mathrm{dom}L \cap \mathrm{Ker}P$, we have $y(t) = -{}^cD_{0+}^\alpha x(t) \in \mathrm{Im}L$ and $\int_0^1 \frac{x(s)}{(1-s)^{2-\alpha}}ds = 0$. Then

$$(K_p y)(t) = x(t) = -I_{0+}^\alpha y(t) + Ct = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + Ct,$$

and

$$\begin{aligned}
0 &= \int_0^1 \frac{x(\tau)}{(1-\tau)^{2-\alpha}}d\tau \\
&= -\frac{1}{\Gamma(\alpha)}\int_0^1 \frac{1}{(1-\tau)^{2-\alpha}}\int_0^\tau (\tau-s)^{\alpha-1}y(s)dsd\tau + C\int_0^1 \frac{\tau}{(1-\tau)^{2-\alpha}}d\tau
\end{aligned}$$

$$= -\frac{1}{\Gamma(\alpha)}\int_0^1 y(s)\int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}}d\tau ds + \frac{C}{\alpha(\alpha-1)}.$$

We can solve that

$$C = \frac{\alpha(\alpha-1)}{\Gamma(\alpha)}\int_0^1 y(s)\int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}}d\tau ds.$$

Therefore,

$$\begin{aligned}(K_p y)(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{\alpha(\alpha-1)t}{\Gamma(\alpha)}\int_0^1 y(s)\int_s^1 \frac{(\tau-s)^{\alpha-1}}{(1-\tau)^{2-\alpha}}d\tau ds \\ &= \int_0^1 k(t,s)\frac{y(s)}{(1-s)^{2-\alpha}}ds,\end{aligned}$$

where $k(t,s)$ is given by (3.4). □

Remark 3.1. It is not difficult to see that $|k(t,s)| \le 3$ for $t, s \in [0,1]$.

Now we state our main result on the existence of a positive solution for BVP (1.1)-(1.2).

Theorem 3.1. *Assume that*

(H1) $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ *satisfies the* L^1*-Carathéodory conditions,* $f(t,0)\not\equiv 0$ *for* $t\in[0,1]$,

(H2) *there exist positive constants* b_1, b_2, b_3, c_1, c_2 *and* B *with* $B > \frac{c_2}{c_1}\alpha + \frac{3b_2c_2}{b_1c_1(\alpha-1)} + \frac{3b_3}{b_1(\alpha-1)}$ *such that*

$$\begin{aligned}&-\kappa x \le f(t,x),\\ &f(t,x) \le -c_1 x + c_2,\\ &f(t,x) \le -b_1|f(t,x)| + b_2 x + b_3\end{aligned}$$

for $t\in[0,1]$, $x\in[0,B]$,

(H3) *there exist* $b\in(0,B)$, $\rho\in(0,1]$, $\delta\in(0,1)$ *and* $q\in L^1[0,1]$, $q(t)\ge 0$ *on* $[0,1]$, $h\in C((0,b],\mathbb{R}^+)$ *such that* $f(t,x)\ge q(t)h(x)$ *for* $t\in[0,1]$ *and* $x\in(0,b]$. $\frac{h(x)}{x^\rho}$ *is non-increasing on* $x\in(0,b]$ *with*

$$h(b)\int_0^1 G(\frac{1}{\alpha},s)\frac{q(s)}{(1-s)^{2-\alpha}}ds \ge \frac{b(1-\delta)}{(\alpha-1)\delta^\rho}. \tag{3.5}$$

Then the BVP (1.1)-(1.2) *has at least one positive solution on* $[0,1]$.

Proof. Consider the cone

$$C = \{x\in X : x(t)\ge 0 \text{ on } [0,1]\}.$$

Let

$$\Omega_1 = \{x \in X : \delta||x|| < |x(t)| < b \ \text{ on } [0,1]\}$$

and

$$\Omega_2 = \{x \in X : ||x|| < B\}.$$

Clearly, Ω_1 and Ω_2 are bounded and open sets, and

$$\overline{\Omega}_1 = \{x \in X : \delta||x|| \le |x(t)| \le b \ \text{ on } [0,1]\} \subset \Omega_2$$

(see [17]). Moreover, $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Let $J = I$ and $(\gamma x)(t) = |x(t)|$ for $x \in X$. Then γ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C, which means that 4° holds.

In order to prove 3°, suppose that there exist $x_0 \in \partial\Omega_2 \cap C \cap \operatorname{dom} L$ and $\lambda_0 \in (0,1)$ such that $Lx_0 = \lambda_0 N x_0$, then ${}^cD^{\alpha}_{0+}x_0(t) + \lambda_0 f(t, x_0(t)) = 0$ for all $t \in [0,1]$. In view of (H2), we have

$$-\frac{1}{\lambda_0}\, {}^cD^{\alpha}_{0+}x_0(t) = f(t, x_0(t)) \le -\frac{1}{\lambda_0} b_1 |{}^cD^{\alpha}_{0+}x_0(t)| + b_2 x_0(t) + b_3.$$

Hence,

$$\begin{aligned} 0 &= -x_0'(1) + x_0'(0) \\ &= -(I^{\alpha-1}_{0+}\, {}^cD^{\alpha-1}_{0+} D x_0)(1) \\ &\le -\frac{b_1}{\Gamma(\alpha-1)} \int_0^1 \frac{|{}^cD^{\alpha}_{0+}x_0(s)|}{(1-s)^{2-\alpha}} ds + \frac{\lambda_0 b_2}{\Gamma(\alpha-1)} \int_0^1 \frac{x_0(s)}{(1-s)^{2-\alpha}} ds \\ &\quad + \frac{\lambda_0 b_3}{\Gamma(\alpha-1)} \int_0^1 \frac{1}{(1-s)^{2-\alpha}} ds, \end{aligned}$$

which gives

$$\int_0^1 \frac{|{}^cD^{\alpha}_{0+}x_0(s)|}{(1-s)^{2-\alpha}} ds \le \frac{b_2}{b_1} \int_0^1 \frac{x_0(s)}{(1-s)^{2-\alpha}} ds + \frac{b_3}{b_1(\alpha-1)}. \tag{3.6}$$

Similarly, from (H2), we also obtain

$$\int_0^1 \frac{x_0(s)}{(1-s)^{2-\alpha}} ds \le \frac{c_2}{c_1(\alpha-1)}. \tag{3.7}$$

On the other hand,

$$x_0(t) \;=\; \alpha(\alpha-1)t \int_0^1 \frac{x_0(s)}{(1-s)^{2-\alpha}} ds - \int_0^1 k(t,s) \frac{{}^cD^{\alpha}_{0+}x_0(s)}{(1-s)^{2-\alpha}} ds$$

$$
\begin{aligned}
&\le \frac{c_2}{c_1}\alpha + \int_0^1 |k(t,s)| \cdot \frac{|{}^cD^\alpha_{0+}x_0(s)|}{(1-s)^{2-\alpha}}ds \\
&\le \frac{c_2}{c_1}\alpha + \frac{3b_2c_2}{b_1c_1(\alpha-1)} + \frac{3b_3}{b_1(\alpha-1)}. \qquad (3.8)
\end{aligned}
$$

(3.6), (3.7) and (3.8) yield

$$
B = ||x_0|| \le \frac{c_2}{c_1}\alpha + \frac{3b_2c_2}{b_1c_1(\alpha-1)} + \frac{3b_3}{b_1(\alpha-1)},
$$

which contradicts (H2).

To prove 5°, consider $x \in \mathrm{Ker}L \cap \overline{\Omega}_2$. Then $x(t) = ct$ on $[0,1]$. Let

$$
H(ct,\lambda) = ct - \lambda\alpha(\alpha-1)t\int_0^1 \frac{|cs|}{(1-s)^{2-\alpha}}ds - \lambda\alpha(\alpha-1)t\int_0^1 \frac{f(s,|cs|)}{(1-s)^{\alpha-2}}ds
$$

for $c \in [-B,B]$ and $\lambda \in [0,1]$. Define homeomorphism $M : \mathrm{Ker}L \cap \overline{\Omega}_2 \to \mathbb{R}$ by $M(ct) = c$, then

$$
\deg\{H(ct,\lambda), \mathrm{Ker}L \cap \Omega_2, 0\} = \deg\{MH(M^{-1}c,\lambda), M(\mathrm{Ker}L \cap \Omega_2), M(0)\}
$$

and $M(0) = 0$. It is easy to show that $0 = MH(M^{-1}c,\lambda)$ implies $c \ge 0$. Suppose $0 = MH(M^{-1}B,\lambda)$ for some $\lambda \in (0,1]$, we would have

$$
0 \le B(1-\lambda) = \lambda\alpha(\alpha-1)\int_0^1 \frac{f(s,Bs)}{(1-s)^{\alpha-2}}ds \le \lambda\alpha(\alpha-1)\int_0^1 \frac{-c_1Bs+c_2}{(1-s)^{\alpha-2}}ds = \lambda(-c_1B+\alpha c_2) < 0,
$$

which is a contradiction. In addition, if $\lambda = 0$, then $B = 0$, which is impossible. Thus, $MH(M^{-1}x,\lambda) \ne 0$ for $x \in M(\mathrm{Ker}L \cap \partial\Omega_2)$, $\lambda \in [0,1]$. As a result,

$$
\deg\{MH(M^{-1}c,1), M(\mathrm{Ker}L \cap \Omega_2), 0\} = \deg\{MH(M^{-1}c,0), M(\mathrm{Ker}L \cap \Omega_2), 0\}.
$$

However,

$$
\deg\{MH(M^{-1}c,0), M(\mathrm{Ker}L \cap \Omega_2), 0\} = \deg\{I, M(\mathrm{Ker}L \cap \Omega_2), 0\} = 1.
$$

Then

$$
\begin{aligned}
&\deg\{[I-(P+JQN)\gamma]_{\mathrm{Ker}L}, \mathrm{Ker}L \cap \Omega_2, 0\} \\
= &\deg\{H(\cdot,1), \mathrm{Ker}L \cap \Omega_2, 0\} \\
= &\deg\{MH(M^{-1}c,1), M(\mathrm{Ker}L \cap \Omega_2), 0\} \ne 0.
\end{aligned}
$$

Next, we prove 8°. Let $x \in \overline{\Omega}_2 \setminus \Omega_1$ and $t \in [0,1]$,

$$\begin{aligned}
(\Psi_\gamma x)(t) &= \alpha(\alpha-1)t\int_0^1 \frac{|x(s)|}{(1-s)^{2-\alpha}}ds + \alpha(\alpha-1)t\int_0^1 \frac{f(s,|x(s)|)}{(1-s)^{2-\alpha}}ds \\
&\quad + \int_0^1 \frac{k(t,s)}{(1-s)^{2-\alpha}}[f(s,|x(s)|) - \alpha(\alpha-1)s\int_0^1 \frac{f(\tau,|x(\tau)|)d\tau}{(1-\tau)^{2-\alpha}}]ds \\
&= \alpha(\alpha-1)t\int_0^1 \frac{|x(s)|}{(1-s)^{2-\alpha}}ds + \alpha(\alpha-1)t\int_0^1 G(t,s)\frac{f(s,|x(s)|)}{(1-s)^{2-\alpha}}ds \\
&\geq \alpha(\alpha-1)t\int_0^1 (1-\kappa G(t,s))\frac{|x(s)|}{(1-s)^{2-\alpha}}ds \geq 0.
\end{aligned}$$

Hence, $\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$, i.e. 8° holds.

Since for $x \in \partial\Omega_2$,

$$\begin{aligned}
(P+JQN)\gamma x &= \alpha(\alpha-1)t\int_0^1 \frac{|x(s)|}{(1-s)^{2-\alpha}}ds + \alpha(\alpha-1)t\int_0^1 \frac{f(s,|x(s)|)}{(1-s)^{2-\alpha}}ds \\
&\geq \alpha(\alpha-1)t\int_0^1 \frac{1-\kappa}{(1-s)^{2-\alpha}}|x(s)|ds \geq 0.
\end{aligned}$$

Thus, $(P+JQN)\gamma x \subset C$ for $x \in \partial\Omega_2$, 7° holds.

Next, we verify 6°. Let $u_0(t) \equiv 1$ on $[0,1]$. Then $u_0 \in C \setminus \{0\}$, $C(u_0) = \{x \in C : x(t) > 0 \text{ on } [0,1]\}$ and we can take $\sigma(u_0) = 1$. Let $x \in C(u_0) \cap \partial\Omega_1$. Then $x(t) > 0$ on $[0,1]$, $0 < ||x|| \leq b$ and $x(t) \geq \delta||x||$ on $[0,1]$. For every $x \in C(u_0) \cap \partial\Omega_1$, by (H3), we have

$$\begin{aligned}
(\Psi x)(\frac{1}{\alpha}) &= \alpha(\alpha-1)\frac{1}{\alpha}\int_0^1 \frac{x(s)}{(1-s)^{2-\alpha}}ds + \alpha(\alpha-1)\frac{1}{\alpha}\int_0^1 G(\frac{1}{\alpha},s)\frac{f(s,x(s))}{(1-s)^{2-\alpha}}ds \\
&\geq \delta||x|| + (\alpha-1)\int_0^1 G(\frac{1}{\alpha},s)\frac{q(s)h(x(s))}{(1-s)^{2-\alpha}}ds \\
&= \delta||x|| + (\alpha-1)\int_0^1 \frac{G(\frac{1}{\alpha},s)q(s)}{(1-s)^{2-\alpha}} \cdot \frac{h(x(s))}{x^\rho(s)}x^\rho(s)ds \\
&\geq \delta||x|| + (\alpha-1)\delta^\rho||x||^\rho\int_0^1 \frac{G(t_0,s)q(s)}{(1-s)^{2-\alpha}} \cdot \frac{h(b)}{b^\rho}ds \\
&= \delta||x|| + (\alpha-1)\delta^\rho||x|| \cdot \frac{h(b)}{b} \cdot \frac{b^{1-\rho}}{||x||^{1-\rho}}\int_0^1 \frac{G(\frac{1}{\alpha},s)q(s)}{(1-s)^{2-\alpha}}ds \\
&\geq ||x||.
\end{aligned}$$

Thus, $||x|| \leq \sigma(u_0)||\Psi x||$ for all $x \in C(u_0) \cap \partial\Omega_1$.

In view of Lemma 3.1, the condition 1° is satisfied. Since f is a L^1-Carathéodory function, 2° holds.

By Theorem 2.1, the BVP (1.1)-(1.2) has a positive solution x^* on $[0,1]$ with $||x^*|| \leq B$. $x^*(t)$ is not a trivial solution due to the fact that $f(t,0) \not\equiv 0$ for $t \in [0,1]$. This completes the proof of Theorem 3.1. $\square$

Remark 3.2. Note that with the projection $P(x) = x(0)$, conditions 7° and 8° of Theorem 2.1 are no longer satisfied.

To illustrate how our main result can be used in practice, we present here an example.

Example 3.1. Consider

$$\begin{cases} {}^cD_{0^+}^{1.5}x(t) + \frac{1}{500}(1+t-t^2)(x^2-8x+12)(x-1) = 0, & t \in (0,1), \\ x(0) = 0, \ x'(0) = x'(1). \end{cases} \tag{3.9}$$

Corresponding to Eq. (1.1), here we take $\alpha = 1.5$ and $f(t,x) = \frac{1}{500}(1+t-t^2)(x^2-8x+12)(x-1)$. And we can obtain that

$$G(t,s) = \begin{cases} 1 - \frac{\sqrt{\pi}}{8} + \frac{8}{15\sqrt{\pi}}t^{\frac{3}{2}} + \frac{2\sqrt{1-s}}{\sqrt{\pi}}\int_s^1 \sqrt{\frac{\tau-s}{1-\tau}}d\tau - \frac{8\sqrt{(1-s)(t-s)}}{3\sqrt{\pi}t}, & 0 \leq s \leq t \leq 1, \\ 1 - \frac{\sqrt{\pi}}{8} + \frac{8}{15\sqrt{\pi}}t^{\frac{3}{2}} + \frac{2\sqrt{1-s}}{\sqrt{\pi}}\int_s^1 \sqrt{\frac{\tau-s}{1-\tau}}d\tau, & 0 \leq t \leq s \leq 1. \end{cases}$$

Obviously, $G(t,s) \geq 0$ for $t,s \in [0,1]$.

Let $\kappa = \frac{1}{5}$, $B = 6$ and $b = \frac{1}{2}$, we may choose $b_1 = 4$, $b_2 = \frac{3}{20}$, $b_3 = \frac{1}{3}$, $c_1 = \frac{1}{50}$, $c_2 = \frac{1}{25}$ such that (H2) holds, and take $\rho = 1$, $\delta = 0.995$, $q(t) = 1 + t(1-t)$, $h(x) = \frac{1}{50}x$ for $t \in [0,1]$, $x \in (0, \frac{1}{2}]$ such that (H3) holds.

In addition, it is easy to check that (H1) is satisfied by the definition of f. Therefore, the BVP (3.9) has at least one positive solution on $[0,1]$ according to Theorem 3.1.

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