Existence of positive solutions for singular impulsive differential equations with integral boundary conditions on an infinite interval in Banach spaces*

Xu Chen[1], Xingqiu Zhang[1,2†]

[1] School of Mathematics, Liaocheng University, Liaocheng, 252059, Shandong, China

[2] School of Mathematics, Huazhong University of Science and Technology, Wuhan, 430074, Hubei, China

Email: woshchxu@163.com, zhxq197508@163.com

Abstract In this paper, the Mönch fixed point theorem is used to investigate the existence of positive solutions for the second-order boundary value problem with integral boundary conditions of nonlinear impulsive differential equations on an infinite interval in a Banach space.

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1 Introduction

The theory of boundary-value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering,underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. In recent years, the theory of ordinary differential equations in Banach space has become a new important branch of investigation (see, for example, [1-4] and references therein). In a recent paper [7], using the cone theory and monotone iterative technique, Zhang et al investigated the existence of minimal nonnegative solution of the following nonlocal boundary value problems for second-order nonlinear impulsive differential equations on an infinite interval with an infinite number of impulsive times

$$\begin{cases} -x''(t) = f(t, x(t), x'(t)), \quad t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \cdots, \\ \Delta x'|_{t=t_k} = \overline{I}_k(x(t_k)), \quad k = 1, 2, \cdots, \\ x(0) = \displaystyle\int_0^\infty g(t)x(t)\mathrm{d}t, \quad x'(\infty) = 0, \end{cases}$$

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†Corresponding author.

where $J=[0,+\infty)$, $f\in C(J\times R^+\times R^+,R^+)$, $R^+=[0,+\infty]$, $0<t_1<t_2<\ldots<t_k<\ldots$, $t_k\to\infty$, $I_k\in C[R^+,R^+],\overline{I}_k\in C[R^+,R^+]$, $g(t)\in C[R^+,R^+)$, with $\int_0^\infty g(t)\mathrm{d}t<1$.

Very recently, by using Schauder fixed point theorem, Guo [6] obtained the existence of positive solutions for a class of nth-order nonlinear impulsive singular integro-differential equations in a Banach space. Motivated by Guo's work, in this paper, we shall use the cone theory and the Mönch fixed point theorem to investigate the positive solutions for a class of second-order nonlinear impulsive integro-differential equations in a Banach space.

Consider the following boundary value problem with integral boundary conditions for second-order nonlinear impulsive integro-differential equation of mixed type in a real Banach space E:

$$\begin{cases} -x''(t)=f(t,x(t),x'(t)), \quad t\in J, t\neq t_k,\\ \Delta x|_{t=t_k}=I_{0k}(x(t_k),x'(t_k)),\\ \Delta x'|_{t=t_k}=I_{1k}(x(t_k),x'(t_k)), \quad k=1,2,\cdots,\\ x(0)=\displaystyle\int_0^\infty g(t)x(t)\mathrm{d}t, \quad x'(\infty)=x_\infty, \end{cases} \tag{1}$$

where $J=[0,\infty)$, $J_+=(0,\infty)$, $0<t_1<t_2<\ldots<t_k<\ldots,t_k\to\infty$, $J_k=(t_k,t_{k+1}]$ $(k=1,2,\cdots)$, $J'_+=J_+\backslash\{t_1\ldots,t_k\ldots\}$, f may be singular at $t=0$ and $x=\theta$ or $x'=\theta$. I_{0k} and I_{1k} may be singular at $x=\theta$ or $x'=\theta$, θ is the zero element of E, $g(t)\in L[0,\infty)$ with $\int_0^\infty g(t)\mathrm{d}t<1$, $\int_0^\infty tg(t)\mathrm{d}t<\infty, x(\infty)=\lim_{t\to\infty}x'(t), x_\infty\geq x_0^*$, $x_0^*\in P_+$, P_+ is the same as that defined in Section 2. $\Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t=t_k$, i.e., $\Delta x|_{t=t_k}=x(t_k^+)-x(t_k^-)$, where $x(t_k^+)$, $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t=t_k$ respectively. $\Delta x'|_{t=t_k}$ has a similar meaning for $x'(t)$.

The main features of the present paper are as follows: Firstly, compared with [7], the second-order boundary value problem we discussed here is in Banach spaces and nonlinear term permits singularity not only at $t=0$ but also at $x,x'=\theta$. Secondly, compared with [6], the relative compact conditions we used are weaker.

2 Preliminaries and several lemmas

Let $PC[J,E]=\{x|x(t):J\to E,\ x$ is continuous at $t\neq t_k$ and left continuous at $t=t_k$, $x(t_k^+)$ exists, $k=1,2,\cdots\}$. $PC^1[J,E]=\{x|x\in PC[J,E],\ x'(t)$ exists at $t\neq t_k$ and $x'(t_k^+)$, $x'(t_k^-)$ exist $k=1,2,\cdots\}$.

$$FPC[J,E]=\Big\{x\in PC[J,E]:\sup_{t\in J}\frac{\|x(t)\|}{t+1}<+\infty\Big\},$$

$$DPC^1[J,E]=\Big\{x\in PC^1[J,E]:\sup_{t\in J}\frac{\|x(t)\|}{t+1}<+\infty,\ \text{and}\ \sup_{t\in J}\|x'(t)\|<+\infty\Big\}.$$

Obviously, $FPC[J,E]$ is a Banach space with norm

$$\|x\|_F=\sup_{t\in J}\frac{\|x(t)\|}{t+1}.$$

and $DPC^1[J,E]$ is also a Banach space with norm

$$\|x\|_D=\max\{\|x\|_F,\|x'\|_1\},$$

where

$$\|x'\|_1 = \sup_{t\in J} \|x'(t)\|.$$

The basic space using in this paper is $DPC^1[J,E]$.

Let P be a normal cone in E with normal constant N which defines a partial ordering in E by $x \le y$. If $x \le y$ and $x \ne y$, we write $x < y$. Let $P_+ = P\backslash\{\theta\}$. So, $x \in P_+$ if and only if $x > \theta$. For details on cone theory, see [4].

Let $P_{0\lambda} = \{x \in P : x \ge \lambda x_0^*\}$, $(\lambda > 0)$. Obviously, $P_{0\lambda} \subset P_+$ for any $\lambda > 0$. When $\lambda = 1$, we write $P_0 = P_{01}$, i.e. $P_0 = \{x \in P : x \ge x_0^*\}$. Let $P(F) = \{x \in FPC[J,E] : x(t) \ge \theta,\ \forall\ t \in J\}$, and $P(D) = \{x \in DPC^1[J,E] : x(t) \ge \theta,\ x'(t) \ge \theta,\ \forall\ t \in J\}$. It is clear, $P(F)$, $P(D)$ are cones in $FPC[J,E]$ and $DPC^1[J,E]$, respectively. A map $x \in DPC^1[J,E] \cap C^2[J'_+,E]$ is called a positive solution of BVP (1) if $x \in P(D)$ and $x(t)$ satisfies BVP (1).

Let α, α_F, α_D denote the Kuratowski measure of non-compactness in E, $FPC[J,E]$, $DPC^1[J,E]$. For details on the definition and properties of the measure of non-compactness, the reader is referred to references [1-4].

Denote

$$\lambda^* = \min\Big\{\frac{\int_0^\infty tg(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}, 1\Big\}.$$

Let us list the following assumptions, which will stand throughout this paper.

(H$_1$) $f \in C[J_+\times P_{0\lambda}\times P_{0\lambda}, P]$ for any $\lambda > 0$ and there exist $a, b, c \in L[J_+, J]$ and $z \in C[J_+\times J_+, J]$ such that

$$\|f(t,x,y)\| \le a(t) + b(t)z(\|x\|,\|y\|),\ \forall\ t \in J_+,\ x \in P_{0\lambda^*},\ y \in P_{0\lambda^*}$$

and

$$\frac{\|f(t,x,y)\|}{c(t)(\|x\|+\|y\|)} \to 0, \qquad \text{as } x \in P_{0\lambda^*},\ y \in P_{0\lambda^*},\ \ \|x\|+\|y\| \to \infty,$$

uniformly for $t \in J_+$, and

$$\int_0^\infty a(t)\mathrm{d}t = a^* < \infty,\ \int_0^\infty b(t)\mathrm{d}t = b^* < \infty,\ \int_0^\infty c(t)(1+t)\mathrm{d}t = c^* < \infty.$$

(H$_2$) $I_{ik} \in C[P_{0\lambda}\times P_{0\lambda}, P]$ for any $\lambda > 0$ and there exist $F_i \in L[J_+\times J_+, J_+]$ and constants η_{ik}, γ_{ik}, $(i = 0,1, k = 1,2\cdots)$ such that

$$\|I_{ik}(x,y)\| \le \eta_{ik}F_i(\|x\|,\|y\|),\quad x \in P_{0\lambda^*},\ y \in P_{0\lambda^*}\ (i = 0,1),$$

and

$$\frac{\|I_{ik}(t,x,y)\|}{\gamma_{ik}(\|x\|+\|y\|)} \to 0, \qquad \text{as } x \in P_{0\lambda^*},\ y \in P_{0\lambda^*},\ \|x\|+\|y\| \to \infty,$$

uniformly for $(i = 0,1, k = 1,2\cdots)$, here

$$0 < \eta_i^* = \sum_{k=1}^\infty \eta_{ik} < \infty,\quad 0 < \gamma_i^* = \sum_{k=1}^\infty \gamma_{ik}(1+t_k) < \infty.$$

(H$_3$) For any $t \in J_+, R > 0$ and countable bounded set $V_i \subset DPC^1[J, P_{0\lambda^* R}]$ $(i = 0, 1)$, there exist $h_i(t) \in L[J, J]$ $(i = 0, 1)$ and positive constants m_{ikj} $(i, j = 0, 1, k = 1, 2 \cdots)$ such that

$$\alpha(f(t, V_0(t), V_1(t))) \leq \sum_{i=0}^{1} h_i(t)\alpha(V_i(t)), \ \alpha(I_{ik}(V_0(t), V_1(t))) \leq \sum_{j=0}^{1} m_{ikj}\alpha(V_j(t)),$$

$$h^* = \int_0^\infty h_0(t)(1+t) + h_1(t)\mathrm{d}t < \infty, \quad m^* = \sum_{k=1}^{\infty}\sum_{i=0}^{1}(m_{ik0}(1+t_k) + m_{ik1}) < \infty,$$

where

$$P_{0\lambda^* R} = \{x \in P : x \geq \lambda^* x_0^*, \ \|x\| < R\}.$$

(H$_4$) $t \in J_+, \ \lambda^* x_0^* \leq x_i \leq \overline{x}_i$ $(i = 0, 1)$, imply $f(t, x_0, x_1) \leq f(t, \overline{x}_0, \overline{x}_1)$.

In what follows, we write $Q = \{x \in DPC^1[J, P] : x^{(i)}(t) \geq \lambda^* x_0^*, \ \forall t \in J, \ i = 0, 1\}$. Evidently, Q is a closed convex set in $DPC^1[J, E]$. We shall reduce BVP (1) to an impulsive integral equations in E. To this end, we first consider operator A defined by

$$\begin{aligned}(Ax)(t) \ &= \ \frac{1}{1 - \int_0^\infty g(t)\mathrm{d}t}\Big\{x_\infty \int_0^\infty tg(t)\mathrm{d}t + \int_0^\infty g(t)\Big[\int_0^\infty G(t,s)f(s, x(s), x'(s))\mathrm{d}s \\ &\quad + \sum_{k=1}^{\infty} G(t, t_k)I_{1k}(x(t_k), x'(t_k)) + \sum_{k=1}^{\infty} G'_s(t, t_k)I_{0k}(x(t_k), x'(t_k))\Big]\mathrm{d}t\Big\} + tx_\infty \\ &\quad + \int_0^\infty G(t,s)f(s, x(s), x'(s))\mathrm{d}s + \sum_{k=1}^{\infty} G(t, t_k)I_{1k}(x(t_k), x'(t_k)) \\ &\quad + \sum_{k=1}^{\infty} G'_s(t, t_k)I_{0k}(x(t_k), x'(t_k)),\end{aligned} \tag{2}$$

where

$$G(t,s) = \begin{cases} t, & 0 \leq t \leq s < +\infty, \\ s, & 0 \leq s \leq t < +\infty, \end{cases} \qquad G'_s(t,s) = \begin{cases} 0, & 0 \leq t \leq s < +\infty, \\ 1, & 0 \leq s \leq t < +\infty. \end{cases}$$

Lemma 1 *If conditions* (H$_1$) $-$ (H$_2$) *are satisfied, then operator A defined by* (2) *is a continuous operator from Q into Q.*

Proof. Let

$$\varepsilon_0 = \frac{1}{8c^*\Big(1 + \frac{\int_0^\infty g(t)\mathrm{d}t}{1 - \int_0^\infty g(t)\mathrm{d}t}\Big)}, \tag{3}$$

and

$$r = \frac{\lambda^* \|x_0^*\|}{N} > 0. \tag{4}$$

By (H$_1$), there exists a $R > r$ such that

$$\|f(t, x, y)\| \leq \varepsilon_0 c(t)(\|x\| + \|y\|), \ \forall \ t \in J_+, \ x \in P_{0\lambda^*}, \ y \in P_{0\lambda^*}, \ \|x\| + \|y\| > R,$$

and

$$\|f(t,x,y)\| \le a(t) + Mb(t),\ \forall\ t \in J_+,\ x \in P_{0\lambda^*},\ y \in P_{0\lambda^*},\ \|x\| + \|y\| \le R,$$

where

$$M = \max\{z(u_0, u_1) : r \le u_i \le R\ (i = 0, 1)\}.$$

Hence

$$\|f(t,x,y)\| \le \varepsilon_0 c(t)(\|x\| + \|y\|) + a(t) + Mb(t),\ \forall\ t \in J_+,\ x \in P_{0\lambda^*},\ y \in P_{0\lambda^*}. \tag{5}$$

On the other hand, let

$$\overline{\varepsilon}_i = \frac{1}{8\gamma_i^*\left(1 + \frac{\int_0^\infty g(t)\mathrm{d}t}{1 - \int_0^\infty g(t)\mathrm{d}t}\right)}\ (i = 0, 1). \tag{6}$$

We see that, by condition (H_2), there exists a $R_1 > r$ such that

$$\|I_{ik}(x,y)\| \le \overline{\varepsilon}_i \gamma_{ik}(\|x\| + \|y\|),\ \forall\ x \in P_{0\lambda^*},\ y \in P_{0\lambda^*},\ \|x\| + \|y\| > R_1\ (i = 0, 1, k = 1, 2 \cdots),$$

and

$$\|I_{ik}(x,y)\| \le \eta_{ik} M_1,\ \forall\ x \in P_{0\lambda^*},\ y \in P_{0\lambda^*},\ \|x\| + \|y\| \le R_1\ (i = 0, 1, k = 1, 2 \cdots),$$

where

$$M_1 = \max\{F_i(u_0, u_1) : r \le u_i \le R\ (i = 0, 1)\}.$$

Hence

$$\|I_{ik}(x,y)\| \le \overline{\varepsilon}_i \gamma_{ik}(\|x\| + \|y\|) + \eta_{ik} M_1,\ \forall\ x \in P_{0\lambda^*},\ y \in P_{0\lambda^*},\ i = 0, 1,\ k = 1, 2 \cdots. \tag{7}$$

Let $x \in Q$, by (5), we can get

$$\begin{aligned} \|f(t, x(t), x'(t)\| &\le \varepsilon_0 c(t)(1+t)\Big(\frac{\|x(t)\|}{t+1} + \frac{\|x'(t)\|}{t+1}\Big) + a(t) + Mb(t) \\ &\le \varepsilon_0 c(t)(1+t)(\|x\|_F + \|x'\|_1) + a(t) + Mb(t) \\ &\le 2\varepsilon_0 c(t)(1+t)\|x\|_D + a(t) + Mb(t), \forall\ t \in J_+, \end{aligned} \tag{8}$$

which together with condition (H_1) implies the convergence of the infinite integral

$$\int_0^\infty \|f(s, x(s), x'(s))\| \mathrm{d}s. \tag{9}$$

On the other hand, by (7), we have

$$\begin{aligned} \|I_{ik}(x(t_k), x'(t_k))\| &\le \overline{\varepsilon}_i \gamma_{ik}(1+t_k)\Big(\frac{\|x(t_k)\|}{t_k+1} + \frac{\|x'(t_k)\|}{t_k+1}\Big) + \eta_{ik} M_1 \\ &\le \overline{\varepsilon}_i \gamma_{ik}(1+t_k)(\|x\|_F + \|x'\|_1) + \eta_{ik} M_1 \\ &\le 2\overline{\varepsilon}_i \gamma_{ik}(1+t_k)\|x\|_D + \eta_{ik} M_1\ (i = 0, 1), \end{aligned} \tag{10}$$

which together with (2), (H_1) and (H_2) implies that

$$\begin{aligned}\|(Ax)(t)\| &\leq \frac{1}{1-\int_0^\infty g(t)\mathrm{d}t}\Big\{\|x_\infty\|\int_0^\infty tg(t)\mathrm{d}t+\int_0^\infty g(t)\Big[\int_0^\infty G(t,s)\|f(s,x(s),x'(s))\|\mathrm{d}s\\ &\quad+\sum_{k=1}^\infty G(t,t_k)\|I_{1k}(x(t_k),x'(t_k))\|+\sum_{k=1}^\infty G'_s(t,t_k)\|I_{0k}(x(t_k),x'(t_k))\|\Big]\mathrm{d}t\Big\}\\ &\quad+t\|x_\infty\|+\int_0^\infty G(t,s)\|f(s,x(s),x'(s))\|\mathrm{d}s+\sum_{k=1}^\infty G(t,t_k)\|I_{1k}(x(t_k),x'(t_k))\| \quad (11)\\ &\quad+\sum_{k=1}^\infty G'_s(t,t_k)\|I_{0k}(x(t_k),x'(t_k))\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\|(Ax)(t)\|}{1+t} &\leq \frac{1}{1-\int_0^\infty g(t)\mathrm{d}t}\Big\{\|x_\infty\|\int_0^\infty tg(t)\mathrm{d}t+\int_0^\infty g(t)\Big[\int_0^\infty \|f(s,x(s),x'(s))\|\mathrm{d}s\\ &\quad+\sum_{k=1}^\infty \|I_{1k}(x(t_k),x'(t_k))\|+\sum_{k=1}^\infty \|I_{0k}(x(t_k),x'(t_k))\|\Big]\mathrm{d}t\Big\}+\|x_\infty\|\\ &\quad+\int_0^\infty \|f(s,x(s),x'(s))\|\mathrm{d}s+\sum_{k=1}^\infty \|I_{1k}(x(t_k),x'(t_k))\|\\ &\quad+\sum_{k=1}^\infty \|I_{0k}(x(t_k),x'(t_k))\| \qquad (12)\\ &\leq \Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)[2\varepsilon_0 c^*\|x\|_D+a^*+Mb^*]+\Big(1+\frac{\int_0^\infty tg(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\|x_\infty\|\\ &\quad+\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\sum_{i=0}^{1}\sum_{k=1}^\infty \|I_{ik}(x(t_k),x'(t_k))\|\\ &\leq \frac{1}{2}\|x\|_D+\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)(a^*+Mb^*+\eta_0^*M_1+\eta_1^*M_1)\\ &\quad+\Big(1+\frac{\int_0^\infty tg(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\|x_\infty\|.\end{aligned}$$

Differentiating (2), we get

$$(A'x)(t)=\int_t^\infty f(s,x(s),x'(s))\mathrm{d}s+\sum_{t_k\geq t} I_{1k}(x(t_k),x'(t_k))+x_\infty. \qquad (13)$$

Hence,

$$\begin{aligned}\|(A'x)(t)\| &\leq \int_0^\infty \|f(s,x(s),x'(s))\|\mathrm{d}s+\|x_\infty\|+\sum_{k=1}^\infty \|I_{1k}(x(t_k),x'(t_k))\|\\ &\leq 2\varepsilon_0 c^*\|x\|_D+a^*+Mb^*+\|x_\infty\|+2\overline{\varepsilon}_1\gamma_1^*\|x\|_D+\eta_1^*M_1 \qquad (14)\\ &\leq \frac{1}{2}\|x\|_D+a^*+Mb^*+\|x_\infty\|+\eta_1^*M_1,\quad \forall\ t\in J.\end{aligned}$$

It follows from (12) and (14) that

$$\begin{aligned}\|Ax\|_D &\leq \frac{1}{2}\|x\|_D+\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)(a^*+Mb^*+\eta_0^*M_1+\eta_1^*M_1)\\ &\quad+\Big(1+\frac{\int_0^\infty tg(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\|x_\infty\|. \qquad (15)\end{aligned}$$

So, $Ax \in DPC^1[J, E]$. On the other hand, it can be easily seen that

$$(Ax)(t) \geq \Big(\frac{\int_0^\infty g(t)\mathrm{d}t}{1 - \int_0^\infty g(t)\mathrm{d}t}\Big)x_\infty \geq \lambda^* x_\infty \geq \lambda^* x_0^*, \ \forall\, t \in J,$$

$$(A'x)(t) \geq x_\infty \geq \lambda^* x_\infty \geq \lambda^* x_0^*, \ \forall\, t \in J.$$

Hence, $Ax \in Q$. Thus, we have proved that A maps Q into Q and (15) holds.

Finally, we show that A is continuous. Let $(x_m, \overline{x}) \in Q$, $\|x_m - \overline{x}\|_D \to 0$ $(m \to \infty)$. Then $\{x_m\}$ is a bounded subset of Q. Thus, there exists $r > 0$ such that $\|x_m\|_D < r$ for $m \geq 1$ and $\|\overline{x}\|_D \leq r$. Similar to (12) and (14), it is easy to get

$$\begin{aligned}\|Ax_m - A\overline{x}\|_D \leq \Big(1 + \frac{\int_0^\infty g(t)\mathrm{d}t}{1 - \int_0^\infty g(t)\mathrm{d}t}\Big)\int_0^\infty \|f(s, x_m(s), x_m'(s)) - f(s, \overline{x}(s), \overline{x'}(s))\|\mathrm{d}s \\ + \Big(1 + \frac{\int_0^\infty g(t)\mathrm{d}t}{1 - \int_0^\infty g(t)\mathrm{d}t}\Big)\Big(\sum_{i=0}^{1}\sum_{k=1}^{\infty} \|I_{ik}(x_m(t_k), x_m'(t_k)) - I_{ik}(\overline{x}(t_k), \overline{x'}(t_k))\|\Big).\end{aligned} \tag{16}$$

It is clear that,

$$f(t, x_m(t), x_m'(t)) \to f(t, \overline{x}(t), \overline{x}'(t)) \text{ as } m \to \infty, \ \forall\, t \in J_+. \tag{17}$$

By (8), we get

$$\begin{aligned}\|f(t, x_m(t), x_m'(t)) - f(t, \overline{x}(t), \overline{x}'(t))\| &\leq 4\varepsilon_0 c(t)(1+t)r + 2a(t) + 2Mb(t) \\ &= \sigma(t) \in L[J, J], \ m = 1, 2, 3, \cdots, \ \forall\, t \in J_+.\end{aligned} \tag{18}$$

It follows from (17), (18) and the dominated convergence theorem that

$$\lim_{m\to\infty} \int_0^\infty \|f(s, x_m(s), x_m'(s)) - f(s, \overline{x}(s), \overline{x}'(s))\|\mathrm{d}s = 0. \tag{19}$$

It is clear that,

$$I_{ik}(x_m(t_k), x_m'(t_k)) \to I_{ik}(\overline{x}(t_k), \overline{x'}(t_k)), \text{ as } m \to \infty, \ i = 0, 1, \ k = 1, 2\cdots. \tag{20}$$

So,

$$\lim_{m\to\infty} \Big(\sum_{i=0}^{1}\sum_{k=1}^{\infty} \|I_{ik}(x_m(t_k), x_m'(t_k)) - I_{ik}(\overline{x}(t_k), \overline{x'}(t_k))\|\Big) = 0. \tag{21}$$

It follows from (16), (19) and (21) that $\|Ax_m - A\overline{x}\|_D \to 0$ as $m \to \infty$. Therefore, the continuity of A is proved.

Lemma 2 *If condition* (H$_1$) *and* (H$_2$) *are satisfied, then $x \in Q \cap C^2[J_+', E]$ is a solution of BVP (1) if and only if $x \in Q$ is a solution of the following impulsive integral equation:*

$$\begin{aligned} x(t) &= \frac{1}{1-\int_0^\infty g(t)\mathrm{d}t}\Big\{x_\infty\int_0^\infty tg(t)\mathrm{d}t+\int_0^\infty g(t)\Big[\int_0^\infty G(t,s)f(s,x(s),x'(s))\mathrm{d}s \\ &\quad +\sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k))+\sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k))\Big]\mathrm{d}t\Big\}+tx_\infty \qquad (22) \\ &\quad +\int_0^\infty G(t,s)f(s,x(s),x'(s))\mathrm{d}s+\sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) \\ &\quad +\sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)). \end{aligned}$$

Proof. Suppose that $x\in Q\cap C^2[J'_+,E]$ is a solution of BVP (1). For $t\in J$, integrating (1) from 0 to t, we have

$$-x'(t)+x'(0)=\int_0^t f(s,x(s),x'(s))\mathrm{d}s+\sum_{t_k<t}I_{1k}(x(t_k),x'(t_k)). \qquad (23)$$

Taking limit for $t\to\infty$, we get

$$-x_\infty+x'(0)=\int_0^\infty f(s,x(s),x'(s))\mathrm{d}s+\sum_{k=1}^\infty I_{1k}(x(t_k),x'(t_k)). \qquad (24)$$

Thus,

$$x'(0)=x_\infty+\int_0^\infty f(s,x(s),x'(s))\mathrm{d}s+\sum_{k=1}^\infty I_{1k}(x(t_k),x'(t_k)). \qquad (25)$$

$$\begin{aligned} x'(t) &= x_\infty+\int_0^\infty f(s,x(s),x'(s))\mathrm{d}s+\sum_{k=1}^\infty I_{1k}(x(t_k),x'(t_k))-\int_0^t f(s,x(s),x'(s))\mathrm{d}s \\ &\quad -\sum_{t_k<t}I_{1k}(x(t_k)x'(t_k)). \qquad (26) \end{aligned}$$

$$x'(t)=x_\infty+\int_t^\infty f(s,x(s),x'(s))\mathrm{d}s+\sum_{k=1}^\infty I_{1k}(x(t_k),x'(t_k))-\sum_{t_k<t}I_{1k}(x(t_k),x'(t_k)). \qquad (27)$$

Integrating (27) from 0 to t, we obtain

$$\begin{aligned} x(t) &= x(0)+tx_\infty+\int_0^\infty G(t,s)f(s,x(s),x'(s))\mathrm{d}s+\sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) \\ &\quad +\sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)), \qquad (28) \end{aligned}$$

which together with the boundary value condition implies that

$$\begin{aligned} x(0) &= \int_0^\infty g(t)x(t)\mathrm{d}t=x(0)\int_0^\infty g(t)\mathrm{d}t+x_\infty\int_0^\infty tg(t)\mathrm{d}t+\sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) \\ &\quad +\int_0^\infty g(t)\Big[\int_0^\infty G(t,s)f(s,x(s),x'(s))\mathrm{d}s+\sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k))\Big]\mathrm{d}t. \qquad (29) \end{aligned}$$

Thus,

$$x(0) = \frac{1}{1-\int_0^\infty g(t)\mathrm{d}t}\Big\{x_\infty \int_0^\infty tg(t)\mathrm{d}t + \int_0^\infty g(t)\Big[\int_0^\infty G(t,s)f(s,x(s),x'(s))\mathrm{d}s + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) + \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k))\Big]\mathrm{d}t\Big\}. \tag{30}$$

Substituting (30) into (28), we have

$$\begin{aligned} x(t) = {} & \frac{1}{1-\int_0^\infty g(t)\mathrm{d}t}\Big\{x_\infty \int_0^\infty tg(t)\mathrm{d}t + \int_0^\infty g(t)\Big[\int_0^\infty G(t,s)f(s,x(s),x'(s))\mathrm{d}s \\ & + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) + \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k))\Big]\mathrm{d}t\Big\} + tx_\infty \\ & + \int_0^\infty G(t,s)f(s,x(s),x'(s))\mathrm{d}s + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) \\ & + \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)). \end{aligned} \tag{31}$$

Obviously, integral $\int_0^t \int_s^\infty f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\mathrm{d}s$ is convergent.

Conversely, if x a solution of integral equation, then direct differentiation gives the proof.

Lemma 3 *Let* (H_1) *be satisfied,* $V \subset Q$ *be a bounded set. Then* $\frac{(AV)(t)}{1+t}$ *and* $(A'V)(t)$ *are equicontinuous on any finite subinterval* J_k *of* J *and for any* $\varepsilon > 0$, *there exists* $N > 0$ *such that*

$$\Big\|\frac{(Ax)(t')}{1+t'} - \frac{(Ax)(t'')}{1+t''}\Big\| < \varepsilon, \quad \|(A'x)(t') - (A'x)(t'')\| < \varepsilon \tag{32}$$

uniformly with respect to $x \in V$ *as* $t',\ t'' \geq N$.

Proof. For $x \in V,\ t'' > t',\ t'',t' \in J_k$, we have

$$\begin{aligned} & \Big\|\frac{(Ax)(t')}{1+t'} - \frac{(Ax)(t'')}{1+t''}\Big\| \\ & \leq |t'-t''|\cdot\Big(1+\frac{\int_0^\infty tg(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\|x_\infty\| + \Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\cdot \\ & \quad \Big\{\Big\|\frac{t'}{1+t'}\int_{t'}^\infty f(s,x(s),x'(s))\mathrm{d}s - \frac{t''}{1+t''}\cdot\int_{t''}^\infty f(s,x(s),x'(s))\mathrm{d}s\Big\| \\ & \quad + \Big\|\int_0^{t'} \frac{s}{1+t'}f(s,x(s),x'(s))\mathrm{d}s - \int_0^{t''}\frac{s}{1+t''}f(s,x(s),x'(s))\mathrm{d}s\Big\|\Big\} \\ & \quad + \Big|\frac{1}{1+t'} - \frac{1}{1+t''}\Big|\int_0^\infty g(t)\Big[\sum_{k=1}^\infty G(t,t_k)\|I_{1k}(x(t_k),x'(t_k))\| \\ & \quad + \sum_{k=1}^\infty G'_s(t,t_k)\|I_{0k}(x(t_k),x'(t_k))\|\Big]\mathrm{d}t + \Big|\frac{1}{1+t'} - \frac{1}{1+t''}\Big|\sum_{k=1}^\infty G(t',t_k)\|I_{1k}(x(t_k),x'(t_k))\| \\ & \quad + \Big|\frac{1}{1+t'} - \frac{1}{1+t''}\Big|\sum_{k=1}^\infty G'_s(t',t_k)\|I_{0k}(x(t_k),x'(t_k))\| \end{aligned} \tag{33}$$

$$\leq |t'-t''|\cdot\Big(1+\frac{\int_0^\infty tg(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\|x_\infty\|+\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\cdot$$

$$\Big\{\Big|\frac{t'}{1+t'}-\frac{t''}{1+t''}\Big|\cdot\Big\|\int_0^\infty f(s,x(s),x'(s))\mathrm{d}s\Big\|+\Big\|\int_{t'}^{t''} sf(s,x(s),x'(s))\mathrm{d}s\Big\|$$

$$+\frac{t''}{1+t''}\Big\|\int_{t'}^{t''} f(s,x(s),x'(s))\mathrm{d}s\Big\|+\Big|\frac{1}{1+t'}-\frac{1}{1+t''}\Big|\cdot\Big\|\int_0^{t'} sf(s,x(s),x'(s))\mathrm{d}s\Big\|$$

$$+\Big|\frac{t'}{1+t'}-\frac{t''}{1+t''}\Big|\cdot\Big\|\int_0^{t'} f(s,x(s),x'(s))\mathrm{d}s\Big\|\Big\}$$

$$+\Big|\frac{1}{1+t'}-\frac{1}{1+t''}\Big|\int_0^\infty g(t)\Big[\sum_{k=1}^\infty G(t,t_k)\|I_{1k}(x(t_k),x'(t_k))\|+\sum_{k=1}^\infty G'_s(t,t_k)\|I_{0k}(x(t_k),x'(t_k))\|\Big]\mathrm{d}t$$

$$+|t'-t''|\Big[\sum_{k=1}^\infty G(t',t_k)\|I_{1k}(x(t_k),x'(t_k))\|+\sum_{k=1}^\infty G'_s(t',t_k)\|I_{0k}(x(t_k),x'(t_k))\|\Big],$$

which implies that $\{\frac{AV(t)}{1+t}:x\in V\}$ is equicontinuous on any finite subinterval J_k of J.

Since $V\subset Q$ is bounded, there exists $r>0$ such that for any $x\in V$, $\|x\|_D\leq r$. By (13), $t'',t'\in J_k$, we get

$$\begin{aligned}\|(A'x)(t')-(A'x)(t'')\| &= \Big\|\int_{t'}^{t''} f(s,x(s),x'(s))\mathrm{d}s+\sum_{t_k\geq t'} I_{1k}(x(t_k),x'(t_k))+x_\infty\\ &\quad-\sum_{t_k\geq t''} I_{1k}(x(t_k),x'(t_k))-x_\infty\Big\|\\ &\leq \int_{t'}^{t''}[2\varepsilon_0 rc(s)(1+s)+a(s)+Mb(s)]\mathrm{d}s.\end{aligned} \tag{34}$$

In the following, we are in position to show that for any $\varepsilon>0$, there exists $N>0$ such that

$$\Big\|\frac{(Ax)(t')}{1+t'}-\frac{(Ax)(t'')}{1+t''}\Big\|<\varepsilon,\quad \|(A'x)(t')-(A'x)(t'')\|<\varepsilon$$

uniformly with respect to $x\in V$ as $t',t''\geq N$.

Combining with (33), we need only to show that for any $\varepsilon>0$, there exists sufficiently large $N>0$ such that

$$\Big\|\int_0^{t'}\frac{s}{1+t'}f(s,x(s),x'(s))\mathrm{d}s-\int_0^{t''}\frac{s}{1+t''}f(s,x(s),x'(s))\mathrm{d}s\Big\|<\varepsilon$$

for all $x\in V$ as $t',t''\geq N$. The rest part of the proof is very similar to Lemma 2.3 in [5], we omit the details.

Lemma 4 *Let* (H_1) *and* (H_2) *are satisfied,* V *be a bounded set in* $DPC^1[J,E]$*. Then*

$$\alpha_D(AV)=\max\Big\{\sup_{t\in J}\alpha\Big(\frac{(AV)(t)}{1+t}\Big),\ \sup_{t\in J}\alpha((AV)'(t))\Big\}.$$

Proof. The proof is similar to that of Lemma 2.4 in [5], we omit it.

Lemma 5 ([1,2])*Mönch Fixed-Point Theorem. Let* Q *be a closed convex set of* E *and* $u\in Q$*. Assume that the continuous operator* $F:Q\to Q$ *has the following property:* $V\subset Q$ *countable,* $V\subset\overline{co}(\{u\}\cup F(V))\Longrightarrow V$ *is relatively compact. Then* F *has a fixed point in* Q*.*

Lemma 6 *If* (H_4) *is satisfied, then for* $x,\ y \in Q,\ x^{(i)} \leq y^{(i)},\ t \in J\ (i = 0, 1)$ *imply that* $(Ax)^{(i)} \leq (Ay)^{(i)},\ t \in J\ (i = 0, 1)$.

Proof. It is easy to see that this lemma follows from (2), (13) and condition (H_4). The proof is obvious.

3 Main results

Theorem 1 *Assume conditions* (H_1), (H_2) *and* (H_3) *are satisfied. If* $\Big(1 + \frac{\int_0^\infty g(t)\mathrm{d}t}{1 - \int_0^\infty g(t)\mathrm{d}t}\Big) \cdot (2h^* + m^*) < 1$, *then BVP* (1) *has a positive solution* $\overline{x} \in DPC^1[J, E] \cap C^2[J'_+, E]$ *satisfying* $(\overline{x})^{(i)}(t) \geq \lambda^* x_0^*$ *for* $t \in J\ (i = 0, 1)$.

Proof. By Lemma 1, operator A defined by (2) is a continuous operator from Q into Q, and by Lemma 2, we need only to show that A has a fixed point $\overline{x}$ in Q. Choose

$$R > 2\Big\{\Big(1 + \frac{\int_0^\infty g(t)\mathrm{d}t}{1 - \int_0^\infty g(t)\mathrm{d}t}\Big)(a^* + Mb^* + \eta_0^* M_1 + \eta_1^* M_1) + \Big(1 + \frac{\int_0^\infty tg(t)\mathrm{d}t}{1 - \int_0^\infty g(t)\mathrm{d}t}\Big)\|x_\infty\|\Big\}, \tag{35}$$

and let $Q_1 = \{x \in Q : \|x\|_D \leq R\}$. Obviously, Q_1 is a bounded closed convex set in space $DPC^1[J, E]$. It is easy to see that Q_1 is not empty since $\lambda^*(1+t)x_\infty \in Q_1$. It follows from (15) and (35) that $x \in Q_1$ implies that $Ax \in Q_1$, i.e., A maps Q_1 into Q_1. Now, we are in position to show that $A(Q_1)$ is relatively compact. Let $V = \{x_m : m = 1, 2, \cdots\} \subset Q_1$ satisfying $V \subset \overline{co}\{\{u\} \cup AV\}$ for some $u \in Q_1$. Then $\|x_m\|_D \leq R$. We have, by (2) and (13)

$$\begin{aligned}
(Ax_m)(t) \quad = \quad & \frac{1}{1 - \int_0^\infty g(t)\mathrm{d}t}\Big\{x_\infty \int_0^\infty tg(t)\mathrm{d}t + \int_0^\infty g(t)\Big[\int_0^\infty G(t,s)f(s, x_m(s), x'_m(s))\mathrm{d}s \\
& + \sum_{k=1}^{\infty} G(t, t_k)I_{1k}(x_m(t_k), x'_m(t_k)) + \sum_{k=1}^{\infty} G'_s(t, t_k)I_{0k}(x_m(t_k), x'_m(t_k))\Big]\mathrm{d}t\Big\} + tx_\infty \\
& + \int_0^\infty G(t,s)f(s, x_m(s), x'_m(s))\mathrm{d}s + \sum_{k=1}^{\infty} G(t, t_k)I_{1k}(x_m(t_k), x'_m(t_k)) \\
& + \sum_{k=1}^{\infty} G'_s(t, t_k)I_{0k}(x_m(t_k), x'_m(t_k)),
\end{aligned} \tag{36}$$

and

$$(A'x_m)(t) = \int_t^\infty f(s, x_m(s), x'_m(s))\mathrm{d}s + \sum_{t_k \geq t} I_{1k}(x_m(t_k)) + x_\infty. \tag{37}$$

By Lemma 4, we have

$$\alpha_D(AV) = \max\Big\{\sup_{t\in J} \alpha((AV)'(t)),\ \ \sup_{t\in J} \alpha\Big(\frac{(AV)(t)}{1+t}\Big)\Big\}, \tag{38}$$

where $(AV)(t) = \{(Ax_m)(t) : m = 1, 2, 3, \cdots\}$, and $(AV)'(t) = \{(A'x_m)(t) : m = 1, 2, 3, \cdots\}$.

By (9), we know that the infinite integral $\int_0^\infty \|f(t, x(t), x'(t))\|\mathrm{d}t$ is convergent uniformly for $m = 1, 2, 3, \cdots$. So, for any $\varepsilon > 0$, we can choose a sufficiently large $T > 0$ such that

$$\int_T^\infty \|f(t, x(t), x'(t))\|\mathrm{d}t < \varepsilon. \tag{39}$$

Then, by Guo et al. [1, Theorem 1.2.3], (2), (36), (37), (39) and (H_3), we obtain

$$\begin{aligned}\alpha\Big(\frac{(AV)(t)}{1+t}\Big) &\le \frac{1}{1+t}\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\Big\{2\int_0^T \alpha(f(t,V(t),V'(t))\mathrm{d}t+2\varepsilon\Big\}\\ &\quad+\frac{1}{1+t}\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\sum_{k=1}^{\infty}\sum_{i=0}^{1}\alpha(I_{ik}(V(t_k),V'(t_k)))\\ &\le 2\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\int_0^\infty \alpha(f(t,V(t),V'(t))\mathrm{d}t+2\varepsilon\\ &\quad+\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\sum_{i=0}^{1}\sum_{k=1}^{\infty}\alpha(I_{ik}(V(t_k),V'(t_k)))\\ &\le 2\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\alpha_D(V)\int_0^\infty h_0(t)(1+t)+h_1(t)\mathrm{d}t\\ &\quad+\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\alpha_D(V)\sum_{k=1}^{\infty}\sum_{i=0}^{1}(m_{ik0}(1+t_k)+m_{ik1})+2\varepsilon.\\ &\le 2\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)h^*\alpha_D(V)+m^*\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\alpha_D(V)+2\varepsilon,\end{aligned} \tag{40}$$

and

$$\alpha((AV)'(t))\le 2\int_0^\infty \alpha(f(s,V(s),V'(s))\mathrm{d}s+2\varepsilon\le 2\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)h^*\alpha_D(V)+2\varepsilon. \tag{41}$$

By (38), (40) and (41) that

$$\alpha_D(AV)\le\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)(2h^*+m^*)\alpha_D(V). \tag{42}$$

On the other hand, $\alpha_D(V)\le\alpha_D\{\overline{co}(\{u\}\cup(AV))\}=\alpha_D(AV)$. Then, (42) implies $\alpha_D(V)=0$, i.e., V is relatively compact in $DPC^1[J,E]$. Hence, the Mönch fixed point theorem guarantees that A has a fixed point $\overline{x}$ in Q_1. Thus, Theorem 1 is proved.

Theorem 2 *Let cone P be normal and conditions $(H_1)-(H_4)$ be satisfied. Then BVP (1) has a positive solution $y\in Q\cap[J'_+,E]$ which is minimal in the sense that $x^{(i)}(t)\ge y^{(i)}(t),\ t\in J\ (i=0,1)$ for any positive solution $x\in Q\cap[J'_+,E]$ of BVP (1). Moreover, $\|y\|_D\le 2\gamma+\|x_0\|_D$, where*

$$\gamma=\Big\{\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)(a^*+Mb^*+\eta_0^*M_1+\eta_1^*M_1)+\Big(1+\frac{\int_0^\infty tg(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\|x_\infty\|\Big\},$$

and there exists a monotone iterative sequence $\{x_m(t)\}$ such that $x_m^{(i)}(t)\to y^{(i)}(t)$ as $m\to\infty$ ($i=0,1$) uniformly on J and $x''_m(t)\to y''(t)$ as $m\to\infty$ for any $t\in J_+$, where

$$\begin{aligned}x_0(t) &= \frac{1}{1-\int_0^\infty g(t)\mathrm{d}t}\Big\{x_\infty\int_0^\infty tg(t)\mathrm{d}t+\int_0^\infty g(t)\Big[\int_0^\infty G(t,s)f(s,\lambda^*x_0^*,\lambda^*x_0^*)\mathrm{d}s\\ &\quad+\sum_{k=1}^{\infty}G(t,t_k)I_{1k}(\lambda^*x_0^*,\lambda^*x_0^*)+\sum_{k=1}^{\infty}G'_s(t,t_k)I_{0k}(\lambda^*x_0^*,\lambda^*x_0^*)\Big]\mathrm{d}t\Big\}+tx_\infty\\ &\quad+\int_0^\infty G(t,s)f(s,\lambda^*x_0^*,\lambda^*x_0^*)\mathrm{d}s+\sum_{k=1}^{\infty}G(t,t_k)I_{1k}(\lambda^*x_0^*,\lambda^*x_0^*)\\ &\quad+\sum_{k=1}^{\infty}G'_s(t,t_k)I_{0k}(\lambda^*x_0^*,\lambda^*x_0^*),\end{aligned} \tag{43}$$

and

$$\begin{aligned}x_m(t) &= \frac{1}{1-\int_0^\infty g(t)\mathrm{d}t}\Big\{x_\infty\int_0^\infty tg(t)\mathrm{d}t+\int_0^\infty g(t)\Big[\int_0^\infty G(t,s)f(s,x_{m-1}(s),x'_{m-1}(s))\mathrm{d}s\\&+\sum_{k=1}^\infty G(t,t_k)I_{1k}(x_{m-1}(t_k),x'_{m-1}(t_k))+\sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x_{m-1}(t_k),x'_{m-1}(t_k))\Big]\mathrm{d}t\Big\}+tx_\infty\\&+\int_0^\infty G(t,s)f(s,x_{m-1}(s),x'_{m-1}(s))\mathrm{d}s+\sum_{k=1}^\infty G(t,t_k)I_{1k}(x_{m-1}(t_k),x'_{m-1}(t_k))\\&+\sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x_{m-1}(t_k),x'_{m-1}(t_k)),\ \forall\ t\in J\ (m=1,2,3,\cdots).\end{aligned}\tag{44}$$

Proof. From (43), we can see that $x_0\in C[J,E]$ and

$$x'_0(t)=\int_t^\infty f(s,\lambda^*x_0^*,\lambda^*x_0^*)\mathrm{d}s+\sum_{t_k\geq t}I_{1k}(\lambda^*x_0^*,\lambda^*x_0^*)+x_\infty.\tag{45}$$

By (43) and (45), we have that $x_0^{(i)}\geq\lambda^*x_\infty\geq\lambda^*x_0^*$ $(i=0,1)$ and

$$\begin{aligned}\|x_0(t)\| &\leq \frac{1}{1-\int_0^\infty g(t)\mathrm{d}t}\Big\{\|x_\infty\|\int_0^\infty tg(t)\mathrm{d}t+\int_0^\infty g(t)\Big[\int_0^\infty\|f(s,\lambda^*x_0^*,\lambda^*x_0^*)\|\mathrm{d}s\\&+\sum_{k=1}^\infty\|I_{1k}(\lambda^*x_0^*,\lambda^*x_0^*)\|+\sum_{k=1}^\infty\|I_{0k}(\lambda^*x_0^*,\lambda^*x_0^*)\|\Big]\mathrm{d}t\Big\}+\|x_\infty\|\\&+\int_0^\infty\|f(s,\lambda^*x_0^*,\lambda^*x_0^*)\|\mathrm{d}s+\sum_{k=1}^\infty\|I_{1k}(\lambda^*x_0^*,\lambda^*x_0^*)\|\\&+\sum_{k=1}^\infty\|I_{0k}(\lambda^*x_0^*,\lambda^*x_0^*)\|\\&\leq\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\int_0^\infty a(s)+b(s)z(\|\lambda^*x_0^*\|,\|\lambda^*x_0^*\|)\mathrm{d}s\\&+\Big(1+\frac{\int_0^\infty g(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\sum_{i=0}^1\sum_{k=1}^\infty\eta_{ik}F_i(\|\lambda^*x_0^*\|,\|\lambda^*x_0^*\|)\\&+\Big(1+\frac{\int_0^\infty tg(t)\mathrm{d}t}{1-\int_0^\infty g(t)\mathrm{d}t}\Big)\|x_\infty\|,\end{aligned}$$

and

$$\begin{aligned}\|x'_0(t)\| &\leq \int_t^\infty\|f(s,\lambda^*x_0^*,\lambda^*x_0^*)\|\mathrm{d}s+\sum_{t_k\geq t}\|I_{1k}(\lambda^*x_0^*,\lambda^*x_0^*)\|+\|x_\infty\|\\&\leq\int_0^\infty a(s)+b(s)z(\|\lambda^*x_0^*\|,\|\lambda^*x_0^*\|)\mathrm{d}s+\sum_{k=1}^\infty\eta_{ik}F_i(\|\lambda^*x_0^*\|,\|\lambda^*x_0^*\|)+\|x_\infty\|,\end{aligned}$$

which imply that $\|x_0\|_D<\infty$. Thus, $x_0\in DPC^1[J,E]$. It follows from (2) and (44) that

$$x_m(t)=(Ax_{m-1})(t),\ \forall\ t\in J,\ m=1,2,3,\cdots.\tag{46}$$

By Lemma 1, we have $x_m\in Q$ and

$$\|x_m\|_D=\|Ax_{m-1}\|_D\leq\frac{1}{2}\|x_{m-1}\|_D+\gamma.\tag{47}$$

By Lemma 6 and (46), we get

$$\lambda^* x_0^* \leq x_0^{(i)}(t) \leq x_1^{(i)}(t) \leq \cdots \leq x_m^{(i)}(t) \leq \cdots, \ \forall\, t \in J \ (i = 0, 1). \tag{48}$$

It follows from (47), by induction, that

$$\begin{aligned} \|x_m\|_D &\leq \gamma + \frac{1}{2}\gamma + \cdots + \Big(\frac{1}{2}\Big)^{m-1}\gamma + \Big(\frac{1}{2}\Big)^m \|x_0\|_D \leq \frac{\gamma[1 - (\frac{1}{2})^m]}{1 - \frac{1}{2}} + \|x_0\|_D \\ &\leq 2\gamma + \|x_0\|_D \ (m = 1, 2, 3, \cdots). \end{aligned} \tag{49}$$

Let $K = \{x \in Q : \|x\|_D \leq 2\gamma + \|x_0\|_D\}$. Then, K is a bounded closed convex set in space $DPC^1[J, E]$ and operator A maps K into K. Clearly, K is not empty since $x_0 \in K$. Let $W = \{x_m : m = 0, 1, 2, \cdots\}, AW = \{Ax_m : m = 0, 1, 2, \cdots\}$. Obviously, $W \subset K$ and $W = \{x_0\} \cup A(W)$. Similar to above proof of Theorem 1, we can obtain $\alpha_D(AW) = 0$, i.e., W is relatively compact in $DPC^1[J, E]$. So, there exists a $y \in DPC^1[J, E]$ and a subsequence $\{x_{m_j} : j = 1, 2, 3, \cdots\} \subset W$ such that $\{x_{m_j}^{(i)}(t) : j = 1, 2, 3, \cdots\}$ converges to $y^{(i)}(t)$ uniformly on J $(i = 0, 1)$. Since that P is normal and $\{x_m^{(i)}(t) : m = 1, 2, 3, \cdots\}$ is nondecreasing, it is easy to see that the entire sequence $\{x_m^{(i)}(t) : m = 1, 2, 3, \cdots\}$ converges to $y^{(i)}(t)$ uniformly on J $(i = 0, 1)$. Since $x_m \in K$ and K is a closed convex set in space $DPC^1[J, E]$, we have $y \in K$. It is clear,

$$f(s, x_m(s), x'_m(s)) \to f(s, y(s), y'(s)), \quad \text{as } m \to \infty, \ \forall\, s \in J_+. \tag{50}$$

By (H_1) and (49), we have

$$\begin{aligned} \|f(s, x_m(s), x'_m(s)) - f(s, y(s), y'(s))\| &\leq 4\varepsilon_0 c(s)(1 + s)\|x_m\|_D + 2a(s) + 2Mb(s) \\ &\leq 4\varepsilon_0 c(s)(1 + s)(2\gamma + \|x_0\|_D) + 2a(s) + 2Mb(s). \end{aligned} \tag{51}$$

Noticing (50) and (51) and taking limit as $m \to \infty$ in (44), we obtain

$$\begin{aligned} y(t) &= \frac{1}{1 - \int_0^\infty g(t)\mathrm{d}t}\Big\{x_\infty \int_0^\infty tg(t)\mathrm{d}t + \int_0^\infty g(t)\Big[\int_0^\infty G(t, s)f(s, y(s), y'(s))\mathrm{d}s \\ &\quad + \sum_{k=1}^\infty G(t, t_k)I_{1k}(y(t_k), y'(t_k)) + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(y(t_k), y'(t_k))\Big]\mathrm{d}t\Big\} + tx_\infty \\ &\quad + \int_0^\infty G(t, s)f(s, y(s), y'(s))\mathrm{d}s + \sum_{k=1}^\infty G(t, t_k)I_{1k}(y(t_k), y'(t_k)) \\ &\quad + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(y(t_k), y'(t_k)), \end{aligned} \tag{52}$$

which implies by Lemma 2 that $y \in K \cap C^2[J_+, E]$ and $y(t)$ is a positive solution of BVP (1). Differentiating (44) twice, we get

$$x''_m(t) = -f(t, x_{m-1}(t), x'_{m-1}(t)), \quad \forall\, t \in J_+, \ m = 1, 2, 3, \cdots.$$

Hence, by (50), we obtain

$$\lim_{m\to\infty} x''_m(t) = -f(t, y(t), y'(t)) = y''(t), \ \forall\, t \in J_+.$$

Let $u(t)$ be any positive solution of BVP (1). By Lemma 2, we have $u \in Q$ and $u(t) = (Au)(t)$, for $t \in J$. It is clear that $u^{(i)}(t) \geq \lambda^* x_0^* > \theta$ for any $t \in J$ $(i = 0, 1)$. So, by Lemma 6, we have

$u^{(i)}(t) \geq x_0^{(i)}(t)$ for any $t \in J$ $(i = 0, 1)$. Assume that $u^{(i)}(t) \geq x_{m-1}^{(i)}(t)$ for $t \in J, m \geq 1$ $(i = 0, 1)$. Then, it follows from Lemma 6 that $(Au)^{(i)}(t) \geq Ax_{m-1}^{(i)}(t)$ for $t \in J$ $(i = 0, 1)$, i.e. $u^{(i)}(t) \geq x_m^{(i)}(t)$ for $t \in J$ $(i = 0, 1)$. Hence, by induction, we get

$$u^{(i)}(t) \geq x_m^{(i)}(t) \ \forall\ t \in J\ (i = 0, 1; m = 0, 1, 2, \cdots). \tag{53}$$

Now, taking limits in (53), we get $u^{(i)}(t) \geq y^{(i)}(t)$ for $t \in J$ $(i = 0, 1)$. The proof is proved.

Theorem 3 *Let cone P be fully regular and conditions* (H_1), (H_2) *and* (H_4) *be satisfied. Then the conclusion of Theorem 2 holds.*

Proof. The proof is almost the same as that of Theorem 2. The only difference is that, instead of using condition (H_3), the conclusion $\alpha_D(W) = 0$ is implied directly by (48) and (49), the full regularity of P and Lemma 4.

4 An example

Consider the infinite system of scalar second order impulsive singular integro-differential equations

$$\begin{cases} -x_n''(t) = \dfrac{1}{4n^3\sqrt[3]{e^{2t}}(2+5t)^9}\Big(5 + x_n(t) + x_{2n}'(t) + \dfrac{2}{3n^2 x_n(t)} + \dfrac{4}{7n^5 x_{2n}'(t)}\Big)^{\frac{1}{2}} \\ \qquad + \dfrac{1}{4\sqrt[6]{t}(1+3t)^2}\ln\left[(1+3t)x_n(t)\right], \\ \Delta x|_{t=t_k} = \dfrac{1}{n^3}\cdot\dfrac{k}{2^{k+1}}\Big(\dfrac{1}{x_n(t_k)} + x_{2n}'(t_k)\Big)^{\frac{1}{5}}, \quad k = 1, 2, \cdots, \\ \Delta x'|_{t=t_k} = \dfrac{1}{n^4}\cdot\dfrac{1}{(k+1)^3}\Big(\dfrac{1}{x_{n+1}(t_k)} + x_{n+2}'(t_k)\Big)^{\frac{1}{7}}, \quad k = 1, 2, \cdots, \\ x_n(0) = \displaystyle\int_0^\infty e^{-t^2} x_n(t)\mathrm{d}t, \quad x_n'(\infty) = \dfrac{1}{n}, \quad n = 1, 2, \cdots. \end{cases} \tag{54}$$

Proposition 1 *Infinite system (54) has a minimal positive solution $x_n(t)$ satisfying $x_n(t), x_n'(t) \geq \frac{1}{n}$ for $0 \leq t < +\infty$ $(n = 1, 2, 3, \cdots)$, and this minimal solution can be obtained by taking limits from some iterative sequences.*

Proof. Let $E = c_0 = \{x = (x_1, \cdots, x_n, \cdots) : x_n \to 0\}$ with the norm $\|x\| = \sup_n |x_n|$. Obviously, $(E, \|\cdot\|)$ is a real Banach space. Choose $P = \{x = (x_n) \in c_0 : x_n \geq 0, n = 1, 2, 3, \cdots\}$. It is easy to verify that P is a normal cone in E with normal constant 1. Now we consider infinite system (54), which can be regarded as a BVP of form (1) in E with $x_\infty = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$. In this situation, $x = (x_1, \cdots, x_n, \cdots), y = (y_1, \cdots, y_n, \cdots), f = (f_1, \cdots, f_n, \cdots)$, and $I_{ik} = (I_{ik1}, \cdots, I_{ikn}, \cdots)$ $(i = 0, 1)$, in which

$$\begin{aligned} f_n(t, x, y) = &\frac{1}{4n^3\sqrt[3]{e^{2t}}(2+5t)^9}\Big(5 + x_n + y_{2n} + \frac{2}{3n^2 x_n} + \frac{4}{7n^5 y_{2n}}\Big)^{\frac{1}{2}} \\ &+ \frac{1}{4\sqrt[6]{t}(1+3t)^2}\ln\left[(1+3t)x_n\right], \end{aligned} \tag{55}$$

and

$$I_{0kn} = \frac{1}{n^3}\cdot\frac{k}{2^{k+1}}\Big(\frac{1}{x_n} + y_{2n}\Big)^{\frac{1}{5}}, \quad I_{1kn} = \frac{1}{n^4}\cdot\frac{1}{(k+1)^3}\Big(x_{n+1} + \frac{1}{y_{n+2}}\Big)^{\frac{1}{7}}. \tag{56}$$

Let $x_0^* = x_\infty = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$. Then $P_{0\lambda} = \{x = (x_1, x_2, \cdots, x_n, \cdots) : x_n \geq \frac{\lambda}{n},\ n = 1, 2, 3, \cdots\}$, for $\lambda > 0$. It is clear, $f \in C[J_+ \times P_{0\lambda} \times P_{0\lambda}, P]$ for any $\lambda > 0$. Noticing that $\sqrt[3]{e^{2t}} > \sqrt[6]{t}$ for $t > 0$, by (55), we get

$$\|f(t, x, y)\| \leq \frac{1}{4\sqrt[6]{t}(1+3t)^2}\Big\{\Big(\frac{143}{21} + \|x\| + \|y\|\Big)^{\frac{1}{2}} + \ln\left[(1+3t)\|x\|\right]\Big\}, \tag{57}$$

which imply (H_1) is satisfied for $a(t) = 0,\ b(t) = c(t) = \dfrac{1}{4\sqrt[6]{t}(1+3t)^2}$, and

$$z(u_0, u_1) = \Big(\frac{143}{21} + u_0 + u_1\Big)^{\frac{1}{2}} + \ln[(1+3t)u_0].$$

On the other hand, for $x \in P_{0\lambda^*}, y \in P_{0\lambda^*}$, we have, by (56)

$$\|I_{0k}(x, y)\| \leq \frac{k}{2^{k+1}}\Big(1 + \|y\|\Big)^{\frac{1}{5}}, \quad \|I_{1k}(x, y)\| \leq \frac{1}{(k+1)^3}\Big(\|x\| + 1\Big)^{\frac{1}{7}},$$

which imply (H_2) is satisfied for

$$F_0(u_0, u_1) = (1 + u_1)^{\frac{1}{5}},\ F_1(u_0, u_1) = (1 + u_0)^{\frac{1}{7}},$$

and

$$\eta_{0k} = \frac{k}{2^{k+1}},\ \eta_{1k} = \frac{1}{(k+1)^3},\ \gamma_{0k} = \frac{k}{2^{k+1}(1+t_k)},\ \gamma_{1k} = \frac{1}{(k+1)^3(1+t_k)}.$$

Let $f^1 = \{f_1^1, f_2^1, \cdots, f_n^1, \cdots\}, \quad f^2 = \{f_1^2, f_2^2, \cdots, f_n^2, \cdots\}$, where

$$f_n^1(t, x, y) = \frac{1}{4n^3\sqrt[3]{e^{2t}}(2+5t)^9}\Big(5 + x_n + y_{2n} + \frac{2}{3n^2 x_n} + \frac{4}{7n^5 y_{2n}}\Big)^{\frac{1}{2}}, \tag{58}$$

$$f_n^2(t, x, y) = \frac{1}{4\sqrt[6]{t}(1+3t)^2}\ln[(1+3t)x_n]. \tag{59}$$

Let $t \in J_+$, and $R > 0$ be given and $\{z^{(m)}\}$ be any sequence in $f^1(t \times P_{0\lambda^* R}, P_{0\lambda^* R})$, where $z^{(m)} = (z_1^{(m)}, \cdots, z_n^{(m)}, \cdots)$. By (58), we have

$$0 \leq z_n^{(m)} \leq \frac{1}{4n^3\sqrt[3]{e^{2t}}(2+5t)^9}\Big(\frac{143}{21} + 2R\Big)^{\frac{1}{2}}\ (n, m = 1, 2, 3, \cdots). \tag{60}$$

So, $\{z_n^{(m)}\}$ is bounded and by the diagonal method together with the method of constructing subsequence, we can choose a subsequence $\{m_i\} \subset \{m\}$ such that

$$\{z_n^{(m)}\} \to \overline{z}_n \ \text{ as } i \to \infty\ (n = 1, 2, 3, \cdots), \tag{61}$$

which implies by (60)

$$0 \leq \overline{z}_n \leq \frac{1}{4n^3\sqrt[3]{e^{2t}}(2+5t)^9}\Big(\frac{143}{21} + 2R\Big)^{\frac{1}{2}}\ (n = 1, 2, 3, \cdots). \tag{62}$$

Hence $\overline{z} = (\overline{z}_1, \cdots, \overline{z}_n, \cdots) \in c_0$. It is easy to see from (60)-(62) that

$$\|z^{(m_i)} - \overline{z}\| = \sup_n |z_n^{(m_i)} - \overline{z}_n| \to 0 \text{ as } i \to \infty.$$

Thus, we have proved that $f^1(t \times P_{0\lambda^* R}, P_{0\lambda^* R})$ is relatively compact in c_0.
For any $t \in J_+,\ R > 0,\ x,\ y,\ \overline{x},\ \overline{y} \in D \subset P_{0\lambda^* R}$, we have by (59)

$$\begin{aligned} |f_n^2(t,x,y) - f_n^2(t,\overline{x},\overline{y})| &= \frac{1}{4\sqrt[6]{t}(1+3t)^2} |\ln[(1+3t)x_n] - \ln[(1+3t)\overline{x}_n]| \\ &\leq \frac{1}{4\sqrt[6]{t}(1+3t)} \frac{|x_n - \overline{x}_n|}{(1+3t)\xi_n}, \end{aligned} \tag{63}$$

where ξ_n is between x_n and $\overline{x}_n$. By (63), we get

$$\|f^2(t,x,y) - f^2(t,\overline{x},\overline{y})\| \leq \frac{1}{4\sqrt[6]{t}(1+3t)} \|x - \overline{x}\|,\ x,\ y,\ \overline{x},\ \overline{y} \in D. \tag{64}$$

Thus, by (64), it is easy to see that (H_3) holds for $h_0(t) = \frac{1}{4\sqrt[6]{t}(1+3t)}$. Our conclusion follows from Theorem 2. This completes the proof.

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