Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **56**, 1-14; **http://www.math.u-szeged.hu/ejqtde/**

Multiple solutions of nonlocal boundary value problems for fractional differential equations on the half-line [*]

Xiping Liu, Mei Jia[†]

College of Science, University of Shanghai for Science and Technology, Shanghai, 200093, China

Abstract: In this paper, we study the existence of multiple solutions of nonlocal boundary value problems for fractional differential equations with integral boundary conditions on the half-line. Applying the fixed point theory and the upper and lower solutions method, some new results on the existence of at least three nonnegative solutions are obtained. An example is presented to illustrate the application of our main results.

Keywords: Fractional differential equations; Caputo derivative; Integral boundary condition; Lower and upper solutions; Half-line.

MSC: 34B15, 26A33.

1 Introduction

In this paper, we consider the following nonlocal boundary value problem for fractional differential equations with integral boundary condition on the half-line

$$\begin{cases} {}^{C}D^{\alpha}(p(t)u'(t)) + q(t)f(t,u(t)) = 0, \ t > 0, \\ p(0)u'(0) = 0, \\ \lim\limits_{t\to+\infty} u(t) = \int_0^{+\infty} g(t)u(t)\mathrm{d}t, \end{cases} \tag{1.1}$$

where ${}^{C}D^{\alpha}$ is the standard Caputo derivative, $0 < \alpha < 1$ is a constant, f, g, p and q are given functions.

Boundary value problems (BVPs) of differential equation have received much attention in recent years due to their broad applications in applied mathematics and physics. There are many papers

---

[*] Supported by Natural Science Foundation of China (No. 11071164) and the Innovation Program of Shanghai Municipal Education Commission (No. 10ZZ93).

[†] Corresponding author E-mail address: jiamei-usst@163.com (M. Jia).

concerning the existence of solutions, positive solutions or multiple solutions of two point BVPs, three point BVPs, m-point even nonlocal boundary conditions such as integral boundary conditions about the integer order differential equation. For details we can refer to [6, 10, 13, 16–21, 24, 26]. Boundary value problems on the half-line have been applied in unsteady flow of gas through a semi-infinite porous medium, the theory of drain flows, etc. In the paper [1], Agarwal and O'Regan gave infinite interval problems modeling phenomena which arise in the theory of plasma and electrical potential theory. In [6, 10, 11, 13, 19, 26], authors studied two-point or multipoint boundary value problems on the half-line by using different method. The papers [20, 21] studied the existence of positive solutions for second-order boundary value problems of differential equations system with integral boundary condition on the half-line.

It is well known that fractional order differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, and they also have been of great interests, see [9, 15]. Recently, there are some papers which deal with the existence of the solutions of the boundary values problems for fractional differential equations on finite intervals. For details, see [2, 4, 5, 7, 8, 12, 14, 23, 25, 27, 28] and the references therein.

In [9] and [15], the basic theories for the fractional calculus and the fractional differential equations were discussed. In [5], Benchohra, Hamania and Ntouyas investigated the existence and uniqueness of solutions for problem:

$$\begin{cases} {}^{C}D^{\alpha}y(t) = f(t, y(t)), \quad t \in [0, T],\ 1 < \alpha \le 2, \\ y(0) = g(y),\ y(T) = y_T. \end{cases}$$

By using Schauder's fixed point theorem combined with the diagonalization method, Arara and co-authors (see [4]) studied the existence of solutions for boundary value problems for fractional order differential equation of the form

$$\begin{cases} {}^{C}D^{\alpha}y(t) + f(t, y(t)) = 0,\ t \in [0, \infty),\ 1 < \alpha \le 2, \\ y(0) = y_0,\ y \text{ is bounded on } [0, \infty). \end{cases}$$

In [2], Ahmad and Nieto stuided some existence results for a boundary value problem involving a nonlinear integrodifferential equation of fractional order $1 < q \le 2$ with integral boundary conditions by using contraction mapping principle and Krasnoselskiı́'s fixed point theorem.

However, researches for the multiple solutions of the fractional differential equations with nonlocal boundary condition on infinite intervals are few. In this paper, we aim to discuss the multiple solutions for fractional differential equations with integral boundary condition on the half-line. Applying the well-known Amann theorem and the upper and lower solutions method, we obtain a new result on the existence of at least three distinct nonnegative solutions under some conditions. An example is presented to illustrate the application of our main result.

2 Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper. We denote that $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$.

Definition 2.1 (See [9, 15]) Let $\alpha > 0$. The fractional (arbitrary) order integral of the function $y : \mathbb{R}^+ \to \mathbb{R}$ of order α is defined by

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \mathrm{d}s.$$

provided the integral exists, where Γ is the Gamma function.

Definition 2.2 (See [9, 15]) The Caputo fractional order derivative of the function y of order α is defined by

$${}^C D^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds,$$

provided the right side is pointwise defined on $(0, +\infty)$, where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Throughout the paper, we suppose that the following hypotheses are satisfied:

(H1) $g \in L^1(\mathbb{R}^+)$, $g(t) \geq 0$, $t \in \mathbb{R}^+$, and $0 \leq \int_0^{+\infty} g(t)\mathrm{d}t := ||g||_1 < 1$.

(H2) $p(t) > 0$ for all $t \in \mathbb{R}^+$, $\int_0^{+\infty} \frac{1}{p(r)}\mathrm{d}r$ exists and the function $k(s) = \int_s^{+\infty} \frac{(r-s)^{\alpha-1}}{p(r)}\mathrm{d}r < +\infty$ is continuous on $\mathbb{R}^+$.

It is obvious that $0 < 1 - ||g||_1 \leq 1$ if (H1) holds. From (H2), we can get that $k(s) \geq 0$ and $\lim_{s\to+\infty} k(s) = 0$. So $k(s)$ is bounded, which implies that there exists a constant $K_0 > 0$ such that

$$0 \leq k(s) \leq K_0 = \sup_{s\in\mathbb{R}^+} k(s), \text{ for } s \in \mathbb{R}^+.$$

We define that

$$K(t,s) = \begin{cases} \int_t^{+\infty} \frac{(r-s)^{\alpha-1}}{p(r)}\mathrm{d}r, & 0 \leq s < t, \\ \int_s^{+\infty} \frac{(r-s)^{\alpha-1}}{p(r)}\mathrm{d}r, & 0 \leq t \leq s, \end{cases} \tag{2.1}$$

and

$$G(t,s) = \frac{1}{(1-||g||_1)\Gamma(\alpha)} \left((1-||g||_1)K(t,s) + \int_0^{+\infty} g(r)K(r,s)\mathrm{d}r \right). \tag{2.2}$$

By (H1), (H2), (2.1) and (2.2), we can easily get that K and G satisfy the following lemma.

Lemma 2.1 *Suppose (H1) and (H2) hold. Then*

(1) $K(t,s)$ *is well defined and continuous, for all* $(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+$;

(2) $0 \leq K(t,s) \leq K(s,s) = k(s) \leq K_0$, *for all* $(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+$;

(3) $G(t,s)$ *is well defined, continuous, and* $0 \leq G(t,s) \leq G(s,s) \leq \frac{1}{(1-||g||_1)\Gamma(\alpha)} K_0$, *for all* $(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+$;

(4) *For any* $s\in\mathbb{R}^+$, $\lim\limits_{t\to+\infty}K(t,s)=0$, *and denote*

$$G_\infty(s):=\lim_{t\to+\infty}G(t,s)=\frac{1}{(1-||g||_1)\Gamma(\alpha)}\int_0^{+\infty}g(r)K(r,s)\mathrm{d}r,$$

then $G_\infty(s)$ *is continuous, and*

$$G_\infty(s)\le\frac{1}{(1-||g||_1)\Gamma(\alpha)}K_0.$$

□

Lemma 2.2 *Suppose that (H1) and (H2) hold, and* $h\in L^1(\mathbb{R}^+)$. *Then the following boundary value problem*

$$\begin{cases}{}^CD^\alpha(p(t)u'(t))+h(t)=0,\ t>0,\\ p(0)u'(0)=0,\\ u(\infty)=\int_0^{+\infty}g(t)u(t)\mathrm{d}t,\end{cases}\tag{2.3}$$

has a unique solution

$$u(t)=\int_0^{+\infty}G(t,s)h(s)\mathrm{d}s,$$

where $u(\infty):=\lim\limits_{t\to+\infty}u(t)$.

Proof. By (2.3), we have

$$p(t)u'(t)=p(0)u'(0)-I^\alpha h(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)\mathrm{d}s.$$

We get $u'(t)=-\frac{1}{\Gamma(\alpha)p(t)}\int_0^t(t-s)^{\alpha-1}h(s)\mathrm{d}s$, and

$$\begin{aligned}u(t)=&u(0)-\int_0^t\left(\frac{1}{\Gamma(\alpha)p(r)}\int_0^r(r-s)^{\alpha-1}h(s)\mathrm{d}s\right)\mathrm{d}r\\=&u(0)-\frac{1}{\Gamma(\alpha)}\int_0^t\mathrm{d}r\int_0^r\frac{(r-s)^{\alpha-1}h(s)}{p(r)}\mathrm{d}s\\=&u(0)-\frac{1}{\Gamma(\alpha)}\int_0^t\mathrm{d}s\int_s^t\frac{(r-s)^{\alpha-1}h(s)}{p(r)}\mathrm{d}r.\end{aligned}\tag{2.4}$$

So

$$u(\infty)=u(0)-\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\mathrm{d}s\int_s^{+\infty}\frac{(r-s)^{\alpha-1}h(s)}{p(r)}\mathrm{d}r=\int_0^{+\infty}g(t)u(t)\mathrm{d}t.$$

Then

$$u(0)=\int_0^{+\infty}g(r)u(r)\mathrm{d}r+\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\mathrm{d}s\int_s^{+\infty}\frac{(r-s)^{\alpha-1}h(s)}{p(r)}\mathrm{d}r.\tag{2.5}$$

Substituting (2.5) into (2.4), we have

$$\begin{aligned}u(t)=&\int_0^{+\infty}g(r)u(r)\mathrm{d}r+\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\mathrm{d}s\int_s^{+\infty}\frac{(r-s)^{\alpha-1}h(s)}{p(r)}\mathrm{d}r-\frac{1}{\Gamma(\alpha)}\int_0^t\mathrm{d}s\int_s^t\frac{(r-s)^{\alpha-1}h(s)}{p(r)}\mathrm{d}r\\=&\int_0^{+\infty}g(r)u(r)\mathrm{d}r+\frac{1}{\Gamma(\alpha)}\int_0^t\mathrm{d}s\int_t^{+\infty}\frac{(r-s)^{\alpha-1}h(s)}{p(r)}\mathrm{d}r+\frac{1}{\Gamma(\alpha)}\int_t^{+\infty}\mathrm{d}s\int_s^{+\infty}\frac{(r-s)^{\alpha-1}h(s)}{p(r)}\mathrm{d}r\\=&\int_0^{+\infty}g(r)u(r)\mathrm{d}r+\frac{1}{\Gamma(\alpha)}\int_0^t\Big(\int_t^{+\infty}\frac{(r-s)^{\alpha-1}}{p(r)}\mathrm{d}r\Big)h(s)\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_t^{+\infty}\Big(\int_s^{+\infty}\frac{(r-s)^{\alpha-1}}{p(r)}\mathrm{d}r\Big)h(s)\mathrm{d}s.\end{aligned}$$

By (2.1), we get that

$$u(t)=\int_0^{+\infty} g(r)u(r)\mathrm{d}r+\frac{1}{\Gamma(\alpha)}\int_0^{+\infty} K(t,s)h(s)\mathrm{d}s. \tag{2.6}$$

So

$$g(t)u(t)=g(t)\int_0^{+\infty} g(r)u(r)\mathrm{d}r+\frac{g(t)}{\Gamma(\alpha)}\int_0^{+\infty} K(t,s)h(s)\mathrm{d}s.$$

$$\begin{aligned}\int_0^{+\infty} g(t)u(t)\mathrm{d}t&=\int_0^{+\infty} g(t)\mathrm{d}t\cdot\int_0^{+\infty} g(r)u(r)\mathrm{d}r+\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\Big(g(r)\int_0^{+\infty} K(t,s)h(s)\mathrm{d}s\Big)\mathrm{d}r\\&=\int_0^{+\infty} g(t)\mathrm{d}t\cdot\int_0^{+\infty} g(r)u(r)\mathrm{d}r+\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\mathrm{d}s\int_0^{+\infty} g(r)K(r,s)h(s)\mathrm{d}r\\&=\int_0^{+\infty} g(t)\mathrm{d}t\cdot\int_0^{+\infty} g(r)u(r)\mathrm{d}r+\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\Big(\int_0^{+\infty} K(r,s)g(r)\mathrm{d}r\Big)h(s)\mathrm{d}s.\end{aligned}$$

Noting that $0<1-||g||_1=1-\int_0^{+\infty} g(t)\mathrm{d}t\le 1$, we have

$$\int_0^{+\infty} g(r)u(r)\mathrm{d}r=\frac{1}{(1-||g||_1)\Gamma(\alpha)}\int_0^{+\infty}\Big(\int_0^{+\infty} K(r,s)g(r)\mathrm{d}r\Big)h(s)\mathrm{d}s. \tag{2.7}$$

Substituting (2.7) into (2.6), we have

$$\begin{aligned}u(t)=&\frac{1}{(1-||g||_1)\Gamma(\alpha)}\int_0^{+\infty}\Big(\int_0^{+\infty} K(r,s)g(r)\mathrm{d}r\Big)h(s)\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_0^{+\infty} K(t,s)h(s)\mathrm{d}s\\=&\frac{1}{(1-||g||_1)\Gamma(\alpha)}\int_0^{+\infty}\Big(\int_0^{+\infty} K(r,s)g(r)\mathrm{d}r+(1-||g||_1)K(t,s)\Big)h(s)\mathrm{d}s\\=&\int_0^{+\infty} G(t,s)h(s)\mathrm{d}s,\end{aligned}$$

where

$$G(t,s)=\frac{1}{(1-||g||_1)\Gamma(\alpha)}\Big(\int_0^{+\infty} K(r,s)g(r)\mathrm{d}r+(1-||g||_1)K(t,s)\Big).$$

□

Lemma 2.3 *Suppose (H1) and (H2) hold, if $u=u(t)$ satisfies*

$$\begin{cases} {}^{C}D^{\alpha}(p(t)u'(t))\le 0,\ t\in(0,+\infty),\\ p(0)u'(0)\le 0,\\ u(\infty)-\int_0^{+\infty} g(r)u(r)\mathrm{d}r\ge 0.\end{cases} \tag{2.8}$$

Then $u(t)\ge 0$ for $t\in\mathbb{R}^+$.

Proof. Let ${}^{C}D^{\alpha}(p(t)u'(t))=-h(t)\le 0$, $p(0)u'(0)=k_0\le 0$ and $u(\infty)-\int_0^{+\infty} g(r)u(r)\mathrm{d}r=k_1\ge 0$. We consider the following boundary value problem

$$\begin{cases} {}^{C}D^{\alpha}(p(t)u'(t))=-h(t),\ t\in(0,+\infty),\\ p(0)u'(0)=k_0,\\ u(\infty)-\int_0^{+\infty} g(r)u(r)\mathrm{d}r=k_1.\end{cases} \tag{2.9}$$

Similar to the proof of Lemma 2.2, we can obtain that the BVP (2.9) has a unique solution.

$$u(t) = \frac{k_1}{(1-||g||_1)} - \frac{k_0}{(1-||g||_1)}\left(\int_0^{+\infty} \mathrm{d}s \int_s^{+\infty} \frac{g(s)}{p(r)}\mathrm{d}r + (1-||g||_1)\int_t^{+\infty} \frac{\mathrm{d}r}{p(r)}\right) + \int_0^{+\infty} G(t,s)h(s)\mathrm{d}s. \tag{2.10}$$

Since $k_0 \leq 0$, $k_1 \geq 0$ and $h(t) \geq 0$ for $t > 0$, it is easy to show

$$u(t) \geq 0, \ \text{ for } \ t \in \mathbb{R}^+$$

from (2.10), (H1) and (H2). □

Let E be a Banach space, $P \subset E$ be a cone in E. A cone P is called solid if it contains interior points, i.e., $\overset{\circ}{P} \neq \emptyset$. Every cone P in E defines a partial ordering in E given by $x \preceq y$ iff $y - x \in P$. If $x \preceq y$ and $x \neq y$, we write $x \precneqq y$; if a cone P is solid and $y - x \in \overset{\circ}{P}$, we write $x \ll y$. A cone P is said to be normal if there exists a constant $N > 0$ such that $0 \preceq x \preceq y$ implies $||x|| \leq N||y||$. If P is normal, then every order interval $[x, y] = \{z \in E | x \preceq z \preceq y\}$ is bounded.

The following Lemma 2.4 is the well-known Amann three-solution theorem (see [3, 22]), which will be used in the later proof of our main results about the multiple solutions of the boundary value problem.

Lemma 2.4 *Let E be a Banach space, and P be a normal solid cone. Suppose that there exist α_1, β_1, α_2, $\beta_2 \in E$ with*

$$\alpha_1 \precneqq \beta_1 \precneqq \alpha_2 \precneqq \beta_2,$$

and $T : [\alpha_1, \beta_2] \longrightarrow E$ is a completely continuous strongly increasing operator such that

$$\alpha_1 \preceq T\alpha_1, \ \ T\beta_1 \precneqq \beta_1, \ \ \alpha_2 \precneqq T\alpha_2, \ \ T\beta_2 \preceq \beta_2.$$

Then the operator T has at least three fixed points x_1, x_2, x_3 such that

$$\alpha_1 \preceq x_1 \ll \beta_1, \ \ \alpha_2 \ll x_2 \preceq \beta_2, \ \ \alpha_2 \npreceq x_3 \npreceq \beta_1.$$

3 Multiple solutions of the boundary value problem

Definition 3.1. $u = u(t)$ is called an upper (lower) solution of boundary value problem (1.1), if it satisfies

$$\begin{cases} {}^{C}D^{\alpha}(p(t)u'(t)) + q(t)f(t, u(t)) \leq 0 \ (\geq 0), \ t > 0, \\ p(0)u'(0) \leq 0 \ (\geq 0), \\ u(\infty) - \int_0^{+\infty} g(t)u(t)\mathrm{d}t \geq 0 \ (\leq 0). \end{cases}$$

In order to obtain the results, we suppose the following conditions hold:

(H3) $q \in L^1(\mathbb{R}^+)$, $q(t)$ is nonnegative on $\mathbb{R}^+$ and $q > 0$ a.e. on $\mathbb{R}^+$.

(H4) $f : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a Carathéodory function, that is to say, $f(\cdot, u)$ is measurable for any $u \in \mathbb{R}^+$ and $f(t, \cdot)$ is continuous for almost every $t \in \mathbb{R}^+$. $f(t, u)$ is bounded for $t \in \mathbb{R}^+$ when u is bounded, and

$$f(t, u_1) < f(t, u_2) \ \text{ with } \ u_1 < u_2 \in \mathbb{R}^+, \ \text{ for almost every } \ t \in \mathbb{R}^+.$$

Let $E = \{u \in C(\mathbb{R}^+) \mid \lim\limits_{t\to+\infty} u(t) \ \text{exists}\}$ be endowed with the norm $\| u \| := \sup\limits_{t\in\mathbb{R}^+} |u(t)|$, then E is a Banach space. We define the cone $P \subset E$ by

$$P := \{u \in E \mid u(t) \geq 0, \ \ t \in \mathbb{R}^+\}.$$

Obviously, P is a normal solid cone in E, and $u \preceq v \in E$ if and only if $u(t) \leq v(t)$ for $t \in \mathbb{R}^+$. $u \precneqq v \in E$ if and only if $u(t) \leq v(t)$ and $u(t) \not\equiv v(t)$, which implies that there exists an interval $[a_0, b_0] \subset \mathbb{R}^+$ such that $u(t) < v(t)$ for $t \in [a_0, b_0]$.

Lemma 3.1 *(See [11]) Let E be defined as before and $D \subset E$. Then D is relatively compact in E if the following conditions hold:*
(a) D is uniformly bounded in E;
(b) the functions from D are equicontinuous on any compact interval of $[0, +\infty)$;
(c) the functions from D are equiconvergent, that is, for any given $\varepsilon > 0$, there exists a $R_0 = R(\varepsilon) > 0$ such that $|u(t) - u(+\infty)| < \varepsilon$, for any $t > R_0$, $u \in D$.

Now, we define an operator $T : \ P \longrightarrow E$ by

$$(Tu)(t) = \int_0^{+\infty} G(t, s)q(s)f(s, u(s))\mathrm{d}s.$$

Lemma 3.2 *Suppose that (H1)–(H4) hold. Then the operator $T : P \longrightarrow P$, and it is completely continuous.*

Proof. First of all, let us show the operator T is well defined, and $T : P \longrightarrow P$.

For any fixed $u \in P$, it implies that u is bounded, by Lemma 2.1, (H3) and (H4), and we can get $(Tu)(t) \geq 0$ for $t \in \mathbb{R}^+$. And there exists a constant $f_{M_0} > 0$ such that $0 \leq f(t, u(t)) \leq f_{M_0}$ for any $t \in \mathbb{R}^+$. Then

$$0 \leq (Tu)(t) = \int_0^{+\infty} G(t, s)q(s)f(s, u(s))\mathrm{d}s \leq \frac{K_0 f_{M_0}}{(1 - ||g||_1)\Gamma(\alpha)} \int_0^{+\infty} q(s)\mathrm{d}s < +\infty.$$

And

$$\lim_{t\to+\infty} (Tu)(t) = \lim_{t\to+\infty} \int_0^{+\infty} G(t, s)q(s)f(s, u(s))\mathrm{d}s = \frac{1}{(1 - ||g||_1)\Gamma(\alpha)} \int_0^{+\infty} G_\infty(s)q(s)f(s, u(s))\mathrm{d}s < +\infty.$$

Thus, $T : P \longrightarrow P$ is well defined.

Secondly, we show that T is continuous.

Let $\{u_n\} \subset P$, $u \in P$, and $u_n \to u_0$ as $n \to \infty$. So, there exists a constant $f_{M_1} > 0$, such that $0 \le f(t, u_n(t))$, $f(t, u_0(t)) \le f_{M_1}$ for any $t \in \mathbb{R}^+$. By (H3), (H4) and Lemma 2.1, we can see

$$||Tu_n - Tu_0|| = \sup_{t\in\mathbb{R}^+} |(Tu_n)(t) - (Tu_0)(t)| \le \int_0^{+\infty} G(s,s)q(s)|f(s,u_n(s)) - f(s,u_0(s))|\mathrm{d}s,$$

$$G(s,s)q(s)|f(s,u_n(s)) - f(s,u_0(s))| \le \frac{2K_0 f_{M_1}}{(1-||g||_1)\Gamma(\alpha)}q(s)$$

and

$$\lim_{n\to\infty} \big(f(s,u_n(s)) - f(s,u_0(s))\big) = 0, \text{ a.e. } s \in \mathbb{R}^+.$$

According to the Lebesgue's dominated convergence theorem, we can show

$$\lim_{n\to\infty} \int_0^{+\infty} G(s,s)q(s)|f(s,u_n(s)) - f(s,u_0(s))|\mathrm{d}s = 0.$$

Therefore, the operator T is continuous.

Finally, we will prove that the operator T maps bounded sets into relatively compact sets.

For the bounded set $\Omega \subset P$, there exists a constant $M_2 > 0$, such that $||u|| \le M_2$ for any $u \in \Omega$. Thus there exists a constant $f_{M_2} > 0$, such that $0 \le f(t, u(t)) \le f_{M_2}$ for any $t \in \mathbb{R}^+$. And

$$0 \le (Tu)(t) = \int_0^{+\infty} G(t,s)q(s)f(s,u(s))\mathrm{d}s \le \frac{K_0 f_{M_2}}{(1-||g||_1)\Gamma(\alpha)} \int_0^{+\infty} q(s)\mathrm{d}s < +\infty.$$

Thus, the set $T(\Omega)$ is uniformly bounded.

For any $[a,b] \subset [0,+\infty)$ and any t_1, $t_2 \in [a,b]$, by Lemma 2.1, we have $G(t_1,s) - G(t_2,s) \to 0$, as $t_1 \to t_2$ for any $s \in \mathbb{R}^+$. And

$$0 \le |G(t_1,s) - G(t_2,s)|q(s)f(s,u(s)) \le 2G(s,s)q(s)f_{M_2} \le \frac{2K_0 f_{M_2}}{(1-||g||_1)\Gamma(\alpha)}q(s).$$

Then

$$\begin{aligned} |(Tu)(t_1) - (Tu)(t_2)| =& \Big|\int_0^{+\infty} G(t_1,s)q(s)f(s,u(s))\mathrm{d}s - \int_0^{+\infty} G(t_2,s)q(s)f(s,u(s))\mathrm{d}s\Big| \\ \le& \int_0^{+\infty} |G(t_1,s) - G(t_2,s)|q(s)f(s,u(s))\mathrm{d}s \\ \le& \int_0^{+\infty} |G(t_1,s) - G(t_2,s)|q(s)f_{M_2}\mathrm{d}s \\ \to& 0, \quad \text{as } \ t_1 \to t_2. \end{aligned} \tag{3.1}$$

That is, Tu from $T(\Omega)$ is equicontinuous on any compact interval of $[0,+\infty)$.

By Lemma 2.1, we have

$$\begin{aligned} |(Tu)(t) - (Tu)(\infty)| =& \Big|\int_0^{+\infty} \big(G(t,s) - G_\infty(s)\big)q(s)f(s,u(s))\mathrm{d}s\Big| \\ \le& \int_0^{+\infty} |G(t,s) - G_\infty(s)|q(s)f_{M_2}\mathrm{d}s \\ \to& 0, \quad \text{as } \ t \to +\infty. \end{aligned}$$

Then Tu from $T(\Omega)$ is equiconvergent.

Using Lemma 3.1, we can obtain that the set $T(\Omega)$ is a relatively compact set. Hence, the operator T maps bounded sets into relatively compact sets.

Therefore, we can get that the operator T is completely continuous.

$\square$

Theorem 3.3 *Suppose that (H1)–(H4) hold, and there exist two lower solutions* x_1, x_2 *and two upper solutions* y_1, y_2 *of boundary value problem* (*1.1*) *such that* $x_2,\ y_1$ *are not the solutions of the boundary value problem* (*1.1*) *with*

$$x_1 \lneqq y_1 \lneqq x_2 \lneqq y_2.$$

Then the boundary value problem (*1.1*) *has at least three distinct nonnegative solutions* $u_1,\ u_2,\ u_3$ *which satisfy that for* $t \in \mathbb{R}^+$

$$x_1(t) \leq u_1(t) < y_1(t),\ \ x_2(t) < u_2(t) \leq y_2(t),\ \ x_2(t) \nleq u_3(t) \nleq y_1(t).$$

Proof. It is obvious that the boundary value problem (1.1) has nonnegative solutions if and only if the operator T has fixed points on P.

It follows from Lemma 3.2 that $T:\ [x_1, y_2] \to P$ is completely continuous.

Let us prove that T is a strongly increasing operator.

For any $w_1, w_2 \in P$, with $w_1 \lneqq w_2$, that is to say that $w_1(t) \leq w_2(t)$ for all $t \in \mathbb{R}^+$, and there exists $[a_0, b_0] \subset \mathbb{R}^+$ such that $w_1(t) < w_2(t)$ for any $t \in [a_0, b_0]$.

Hence, for any $t \in \mathbb{R}^+$, using the conditions (H3) and (H4) we have

$$\begin{aligned}(Tw_2)(t) - (Tw_1)(t) &= \int_0^{+\infty} G(t,s)q(s)\big(f(s,w_2(s)) - f(s,w_1(s))\big)\mathrm{d}s \\ &\geq \int_{a_0}^{b_0} G(t,s)q(s)\big(f(s,w_2(s)) - f(s,w_1(s))\big)\mathrm{d}s \\ &>0.\end{aligned}$$

We can get that

$$0 < (Tw_2)(t) - (Tw_1)(t) \in \mathring{P}.$$

Hence, we conclude that T is a strongly increasing operator.

Let us now prove that $x_1 \lneqq Tx_1$.

We denote $x = Tx_1 - x_1$.

Noting that x_1 is the lower solution of boundary value problems (1.1) and applying the definition of the operator T, we have

$$\begin{aligned}{}^C D^\alpha(p(t)x'(t)) =& {}^C D^\alpha\Big(p(t)\big((Tx_1)'(t)-x_1'(t)\big)\Big)\\ =& {}^C D^\alpha(p(t)(Tx_1)'(t)) - {}^C D^\alpha(p(t)x_1'(t))\\ =& -q(t)f(t,x_1(t)) - {}^C D^\alpha(p(t)x_1'(t))\\ \le& 0,\end{aligned}$$

$$x'(0)=(Tx_1)'(0)-x_1'(0)\le 0,$$

and

$$x(\infty)-\int_0^{+\infty} g(t)x(t)\mathrm{d}t=(Tx_1)(\infty)-x_1(\infty)-\int_0^{+\infty} g(t)\big((Tx_1)(t)-x_1(t)\big)\mathrm{d}t\ge 0.$$

It follows from Lemma 2.3 that

$$x(t)=(Tx_1)(t)-x_1(t)\ge 0,\ \text{ for } \ t\in\mathbb{R}^+.$$

Then

$$x_1\preceq Tx_1.$$

Similarly, we can get that

$$x_2\preceq Tx_2.$$

Since x_2 is an lower solution of (1.1) and not a solution of (1.1), we have $(Tx_2)\ne x_2$. Thus

$$x_2 \precneqq Tx_2.$$

Using the same method, we can also get that

$$Ty_1\precneqq y_1, \quad Ty_2\preceq y_2.$$

Using the Lemma 2.4, we obtain T has at least three fixed points $u_1,\ u_2,\ u_3\in[x_1,\ y_2]$. Moreover, $u_1,\ u_2,\ u_3\in P$ and

$$x_1\preceq u_1\ll y_1,\ \ x_2\ll u_2\preceq y_2,\ \ x_2\not\preceq u_3\not\preceq y_1.$$

Hence, the boundary value problem (1.1) has at least three distinct nonnegative solutions $u_1,\ u_2,\ u_3\in[x_1,\,y_2]$ and we see for $t\in\mathbb{R}^+$

$$x_1(t)\le u_1(t)<y_1(t),\ \ x_2(t)<u_2(t)\le y_2(t),\ \ x_2(t)\not\le u_3(t)\not\le y_1(t).$$

□

4 Illustration

To illustrate our main results, we present an example.

Example 4.1. Consider the following integral boundary value problem

$$\begin{cases} {}^{C}D^{\frac{1}{2}}(p(t)u'(t)) + q(t)f(t,u) = 0,\ t > 0, \\ p(0)u'(0) = 0, \\ u(\infty) = \int_0^{+\infty} e^{-2t}u(t)\mathrm{d}t, \end{cases} \tag{4.1}$$

where

$$p(t) = e^t,\ f(t,u) = \frac{200}{\sqrt{\pi}} \begin{cases} u^2,\ 0 \le u < 1, \\ \sqrt{u}\ ,\ u \ge 1. \end{cases}$$

We take $\alpha = \frac{1}{2}$, $q(t) = te^{-t}$, $g(t) = e^{-2t}$. It is easy to show that

$$g \in L^1(\mathbb{R}^+),\ 0 \le ||g||_1 = \int_0^{+\infty} g(t)\mathrm{d}t = \frac{1}{2} < 1, \text{ and } 1 - ||g||_1 = 1 - \int_0^{+\infty} g(t)\mathrm{d}t = \frac{1}{2},$$

then (H1) holds.

$$p(t) = e^t > 0,\ \ k(s) = \int_s^{+\infty} \frac{(r-s)^{\alpha-1}}{p(r)}\mathrm{d}r = \sqrt{\pi}e^{-s},\ \ s \in \mathbb{R}^+,$$

that is (H2) holds.

$$q(t) = te^{-t},\ \ \int_0^{+\infty} q(t)\mathrm{d}t = \int_0^{+\infty} te^{-t}\mathrm{d}t = 1 < +\infty,$$

it implies that (H3) holds.

We can easily verify the condition (H4) holds.

In view of Lemma 2.2, for any $h \in L^1(\mathbb{R}^+)$, $u(t) = \int_0^{+\infty} G(t,s)h(s)\mathrm{d}s$ satisfies the boundary conditions of (4.1).

Now, let $h(t) = \frac{200t}{\sqrt{\pi}}e^{-t}$. It is obvious $h \in L^1(\mathbb{R}^+)$.

For $t \in \mathbb{R}^+$, we take

$$x_1(t) = 0,\ x_2(t) = \frac{1}{24^2}\int_0^{+\infty} G(t,s)h(s)\mathrm{d}s,$$

and

$$y_1(t) = \frac{1}{53^2}\int_0^{+\infty} G(t,s)h(s)\mathrm{d}s,\ y_2(t) = 53\int_0^{+\infty} G(t,s)h(s)\mathrm{d}s.$$

Then x_1, x_2, y_1, $y_2 \in P$.

It is easy to see $0 = x_1(t) < y_1(t) < x_2(t) < y_2(t)$ for $t \in \mathbb{R}^+$, that is $x_1 \lneqq y_1 \lneqq x_2 \lneqq y_2$.

Moreover, we have

$$p(0)x_i'(0) = 0,\ p(0)y_i'(0) = 0,\ x_i(\infty) = \int_0^{+\infty} e^{-2t}x_i(t)\mathrm{d}t, y_i(\infty) = \int_0^{+\infty} e^{-2t}y_i(t)\mathrm{d}t,\ i = 1, 2.$$

Through calculation, we can get that $24 < \int_0^{+\infty} G(t,s)h(s)\mathrm{d}s < 53$, and we can easily verify

$$^{C}D^{\frac{1}{2}}(p(t)x_1'(t)) + q(t)f(t, x_1(t)) = 0,$$

$$^{C}D^{\frac{1}{2}}(p(t)x_2'(t)) + q(t)f(t, x_2(t)) > {}^{C}D^{\frac{1}{2}}(p(t)x_2'(t)) + \frac{1}{24^2}h(t) = 0,$$

$$^{C}D^{\frac{1}{2}}(p(t)y_1'(t)) + q(t)f(t, y_1(t)) < {}^{C}D^{\frac{1}{2}}(p(t)y_1'(t)) + \frac{1}{53^2}h(t) = 0$$

and

$$^{C}D^{\frac{1}{2}}(p(t)y_2'(t)) + q(t)f(t, y_2(t)) \leq {}^{C}D^{\frac{1}{2}}(p(t)y_2'(t)) + 53h(t) = 0.$$

Therefore, $x_1(t), x_2(t)$ are lower solutions of BVP (4.1), and $y_1(t),\ y_2(t)$ are upper solutions of BVP (4.1).

It follows from Theorem 3.3 that the boundary value problem (4.1) has at least three distinct nonnegative solutions $u_1,\ \ u_2,\ \ u_3 \in [x_1,\ y_2]$. Moreover, for $t \in \mathbb{R}^+$

$$x_1(t) \leq u_1(t) < y_1(t),\ \ x_2(t) < u_2(t) \leq y_2(t),\ \ x_2(t) \nleq u_3(t) \nleq y_1(t).$$

□

Reference

[1] R.P. Agarwal, D. O'Regan, *Infinite interval problems modeling phenomena which arise in the theory of plasma and electrical potential theory*, Stud. Appl. Math., 111 (2003) 339-358.

[2] B. Ahmad, J.J. Nieto *Existence results for nonlinear boundary value problems of fractional integro-differential equations with integral boundary conditions*, Bound. Value Probl., 2009, Art. ID 708576, 1-11.

[3] H. Amann, *Fixed point equation and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev., 18 (1976) 620-709.

[4] A. Arara, M. Benchohra, N. Hamidi, J.J. Nieto, *Fractional order differential equations on an unbounded domain*, Nonlinear Anal., 72 (2010) 580-586.

[5] M. Benchohra, S. Hamania, S.K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Anal., 71 (2009) 2391-2396.

[6] Y. Guo, C. Yu, J. Wang, *Existence of three positive solutions for m-point boundary value problems on infinite intervals*, Nonlinear Anal., 71 (2009) 717-722.

[7] M. Feng, X. Zhang, W. Ge, *New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions*, Bound. Value Probl., 2011, Art. ID 720702, 1-20.

[8] S. Hamani, M. Benchohra, J.R. Graef, *Existence results for boundary-value problems with nonlinear fractional differential inclusions and integral conditions*, Electron. J. Differential Equations, 2010, No. 20, 1-16.

[9] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier B.V., Amsterdam, 2006.

[10] L. Liu, X. Hao, Y. Wu, *Unbounded solutions of second-order multipoint boundary value problem on the half-line*, Bound. Value Probl., 2010, Art. ID 236560, 1-15.

[11] L. Liu, Z. Wang, Y. Wu, *Multiple positive solutions of the singular boundary value problems for second-order differential equations on the the half-line*, Nonlinear Anal., 71 (2009) 2564-2575.

[12] X. Liu, M. Jia, *Multiple solutions for fractional differential equations with nonlinear boundary conditions*, Comput. Math. Appl., 59 (2010) 2880-2886.

[13] X. Liu, M. Jia, E.R. Kaufmann, *Boundary value problems for second-order functional differential equations on infinite intervals*, Nonlinear Stud., 16 (2009) 171-186.

[14] X. Liu, M. Jia, B. Wu, *Existence and uniqueness of solution for fractional differential equations with integral boundary value conditions*, Electron. J. Qual. Theory Differ. Equ. No. 69 (2009) 1-10.

[15] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.

[16] J.R.L. Webb, *A unified approach to nonlocal boundary value problems*, Proc. Dynam. Systems Appl., 5 (2008) 510-515.

[17] J.R.L. Webb, G. Infante, *Positive solutions of nonlocal boundary value problems involving integral conditions*, NoDEA Nonlinear Differential Equations Appl., 15 (2008) 45-67.

[18] J.R.L. Webb, G. Infante, *Positive solutions of nonlocal boundary value problems: a unified approach*, J. London Math. Soc., 74 (2006) 673-693.

[19] J. Xu, Z. Yang, *Positive solutions for singular Sturm-Liouville boundary value problems on the half line*, Electron. J. Differential Equations, 2010, No. 171, 1-8.

[20] S. Xi, M. Jia, H. Ji, *Second-order boundary value problems of differential equations with integral boundary condition on the the half-line*, Electron. J. Qual. Theory Differ. Equ., 2009, No. 31, 1-13.

[21] S. Xi, M. Jia, H. Ji, *Multiple positive solutions for boundary value problems of second-order differential equations system on the the half-line*, Electron. J. Qual. Theory Differ. Equ., 2010, No. 17, 1-15.

[22] X. Xu, J. Sun, *Solutions for an operator equation under the conditions of pairs of paralleled lower and upper solutions*, Nonlinear Anal., 69 (2008) 2251-2266.

[23] A. Yang, H. Wang, *Positive solutions for higher-order nonlinear fractional differential equation with integral boundary condition*, Electron. J. Qual. Theory Differ. Equ., 2011, No. 1, 1-15.

[24] Z. Yang, *Positive solutions of a second-order integral boundary value problem*, J. Math. Anal. Appl., 321 (2006) 751-765.

[25] S. Zhang, *Existence results of positive solutions to fractional differential equation with integral boundary conditions*, Math. Bohem., 135 (2010), No. 3, 299-317.

[26] X. Zhang, *Existence of positive solutions for boundary value problems of second order nonlinear differential equations on the half line*, Electron. J. Differential Equations, 2009, No. 141, 1-10.

[27] X. Zhao, W. Ge, *Unbounded solutions for a fractional boundary value problems on the infinite interval*, Acta Appl. Math., 109 (2010) 495-505.

[28] Y. Zhou, F. Jiao, J. Li, *Existence and uniqueness for fractional neutral differential equations with infinite delay*, Nonlinear Anal., 71 (2009) 3249-3256.

(Received February 20, 2011)