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INFINITE NUMBER OF STABLE PERIODIC SOLUTIONS FOR AN EQUATION WITH NEGATIVE FEEDBACK

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Abstract. For all $\mu > 0$, a locally Lipschitz continuous map f with $xf(x) > 0$, $x \in \mathbb{R}\backslash\{0\}$, is constructed, such that the scalar equation $\dot{x}(t) = -\mu x(t) - f(x(t-1))$ with delayed negative feedback has an infinite number of periodic orbits. All periodic solutions defining these orbits oscillate slowly around 0 in the sense that they admit at most one sign change in each interval of length of 1. Moreover, if f is continuously differentiable, then the periodic orbits are hyperbolic and stable. In this example f is not bounded, but the Lipschitz constants for the restrictions of f to certain intervals are small. Based on this property, an infinite sequence of contracting return maps is given. Their fixed points are the initial segments of the periodic solutions.

Key words and phrases: delay differential equation, negative feedback, periodic solution, return map.

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1. Introduction

Set $\mu > 0$, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) = 0$ and $xf(x) > 0$ for all $x \in \mathbb{R} \setminus \{0\}$. A periodic solution $p : \mathbb{R} \to \mathbb{R}$ of the scalar delay differential equation

$$\dot{x}(t) = -\mu x(t) - f(x(t-1)) \tag{1.1}$$

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is called a slowly oscillating periodic (or SOP) solution if the successive zeros of p are spaced at distances larger than the delay 1.

In [8] Walther has given a class of Lipschitz continuous nonlinearities f for which Eq. (1.1) admits an SOP solution. A nonlinearity f in the function class considered is close to $a \cdot \operatorname{sgn}(x)$ outside a small neighborhood of 0; the Lipschitz constant for f is sufficiently small on $(-\infty, -\varepsilon) \cup (\varepsilon, \infty)$, $\varepsilon > 0$ small. Hence the associated return map is a contraction, and a periodic solution arises as the fixed point of the return map. In case f is C^1-smooth, the corresponding periodic orbit is hyperbolic and stable. In a subsequent paper [6], Ou and Wu have verified that the same result holds for a wider class of nonlinearities.

In case f in Eq. (1.1) is continuously differentiable with $f'(x) > 0$ for $x \in \mathbb{R}$, Cao [1] and Krisztin [3] have given sufficient conditions for the uniqueness of the SOP solution. In these works, $x \mapsto f(x)/x$ is strictly decreasing on $(0, \infty)$.

In this paper we follow the technique used by Walther in [8] to show that one may guarantee the existence of an arbitrary number of SOP solutions. For the nonlinearity f in the next theorem, $x \mapsto f(x)/x$ is not monotone.

Theorem 1.1. *Assume $\mu > 0$. There exists a locally Lipschitz continuous odd nonlinear map f satisfying $xf(x) > 0$ for all $x \in \mathbb{R} \setminus \{0\}$, for which Eq. (1.1) admits an infinite sequence of SOP solutions $(p^n)_{n=1}^{\infty}$ with $p^n(\mathbb{R}) \subsetneq p^{n+1}(\mathbb{R})$ for $n \geq 0$. If f is continuously differentiable, then the corresponding periodic orbits are stable and hyperbolic.*

We point out that a similar result appears in paper [5] of Nussbaum for the case $\mu = 0$. Although the construction of Nussbaum is different from the one presented here, $x \mapsto f(x)/x$ is likewise not monotone for the nonlinear map f given by him.

Suppose f in Theorem 1.1 is smooth with $f'(x) > 0$ for $x \in \mathbb{R}$. Based on [9], it can be confirmed that for the hyperbolic and stable SOP solutions p^n, p^{n+1} with ranges $p^n(\mathbb{R}) \subsetneq p^{n+1}(\mathbb{R})$, there exists an SOP solution p^* with range $p^n(\mathbb{R}) \subsetneq p^*(\mathbb{R}) \subsetneq p^{n+1}(\mathbb{R})$. Also, we have a Poincaré–Bendixson type result. For each globally defined bounded slowly oscillating solution (i.e., for each bounded solution defined on $\mathbb{R}$ with at most 1 sign change on each interval of length 1), the ω-limit set is either $\{0\}$ or a single periodic orbit defined by an SOP solution. Analogously for the α-limit set.

Moreover, the subset

$$\{x_0 : x : \mathbb{R} \to \mathbb{R} \text{ is a bounded, slowly oscillating solution of Eq. (1.1)}\} \cup \{0\}$$

of the phase space $C = C([-1,0], \mathbb{R})$ is homeomorphic to the 2-dimensional plane.

There are results similar to [8] for the positive feedback case, i.e., for equation $\dot{x}(t) = -\mu x(t) + f(x(t-1))$ with $\mu > 0$, $f \in C(\mathbb{R}, \mathbb{R})$ and $xf(x) > 0$ for $x \neq 0$, see e.g. Stoffer [7]. In [4] a feedback function f with $f(0) = 0$, $f'(x) > 0$, $x \in \mathbb{R}$, is given, for which there exist exactly two periodic orbits so that the corresponding periodic solutions oscillate slowly around zero in the sense that there are no 3 different zeros in any interval of length 1. The nonlinear map considered in [4] is close to the step function f^1 given by $f^1(x) = 0$ for $|x| \leq 1$, and $f^1(x) = K \cdot \operatorname{sgn}(x)$ for $|x| > 1$. Equations with such nonlinearities model neural networks of identical neurons that do not react upon small feedback; the feedback has to reach a certain threshold value to have a considerable effect [2]. Eq. (1.1) with nonlinearity f^1 is investigated in the next section.

The nonlinear map in Theorem 1.1 is close to the odd step function f^* with $f^*(x) = 0$ for all $x \in [0,1]$, and $f^*(x) = Kr^n$ for all $n \geq 0$ and $x \in (r^n, r^{n+1}]$. We conjecture that with similar nonlinearities, equation $\dot{x}(t) = -\mu x(t) + f(x(t-1))$ also admits an infinite number of periodic solutions oscillating slowly around zero in the sense that they have no 3 different zeros in any interval of length 1.

Some notations used in this paper are introduced.

The natural phase space for Eq. (1.1) is the space $C = C([-1,0], \mathbb{R})$ of continuous real functions defined on $[-1,0]$ equipped with the supremum norm $\|\varphi\| = \sup_{-1 \leq s \leq 0} |\varphi(s)|$.

If $I \subset \mathbb{R}$ is an interval, $u : I \to \mathbb{R}$ is continuous, then for $[t-1,t] \subset I$, segment $u_t \in C$ is defined by $u_t(s) = u(t+s)$, $-1 \leq s \leq 0$.

In the sequel we consider Eq. (1.1) with continuous or step function nonlinearities f. For any $\varphi \in C$, there is a unique solution $x^{\varphi,f} : [-1,\infty) \to \mathbb{R}$ with initial segment $x_0^{\varphi,f} = \varphi$ computed recursively using the variation-of-constants formula

$$x(t) = x(n) e^{-\mu(t-n)} + \int_n^t e^{-\mu(t-s)} f(x(s-1)) \, \mathrm{d}s \tag{1.2}$$

for all $n \geq 0$ and $t \in [n, n+1]$. Then $x^{\varphi,f}$ is absolutely continuous on $(0, \infty)$. If for some $(\alpha, \beta) \subset (0, \infty)$, the map $(\alpha, \beta) \ni t \mapsto f(x(t-1)) \in \mathbb{R}$ is continuous, then it is clear that $x^{\varphi,f}$ is continuously differentiable on (α, β), moreover, (1.1) holds for all $t \in (\alpha, \beta)$.

The solutions of Eq. (1.1) define the continuous semiflow

$$F = F_f : \mathbb{R}^+ \times C \ni (t, \varphi) \mapsto x_t^{\varphi,f} \in C. \tag{1.3}$$

For odd nonlinearities f, we have the following simple observation concluding from the variation-of-constants formula (1.2).

Remark 1.2. If $f : \mathbb{R} \to \mathbb{R}$ is odd, i.e. $f(-x) = -f(x)$ for all $x \in \mathbb{R}$, then for all $\varphi \in C$ and $t \geq -1$, $x^{-\varphi,f}(t) = -x^{\varphi,f}(t)$.

2. Periodic solutions for step functions

Fix $\mu > 0$ and

$$K > \mu \frac{e^{\mu} + \sqrt{2e^{2\mu} - 2e^{\mu} + 1}}{e^{\mu} - 1} \tag{2.1}$$

in this paper. As a starting point we look for periodic solutions of

$$\dot{x}(t) = -\mu x(t) - f^R(x(t-1)), \tag{2.2}$$

where $R > 0$ and

$$f^R(x) = \begin{cases} -KR & \text{if } x < -R, \\ 0 & \text{if } |x| \leq R, \\ KR & \text{if } x > R. \end{cases} \tag{2.3}$$

Remark 2.1. For each $R > 0$ and $x \in \mathbb{R}$, $f^R(x) = Rf^1(x/R)$. Hence all solutions of Eq. (2.2) are of the form $Rx(t)$, where $x(t)$ is a solution of

$$\dot{x}(t) = -\mu x(t) - f^1(x(t-1)). \tag{2.4}$$

In particular, all periodic solutions of Eq. (2.2) are of the form $Rx(t)$, where $x(t)$ is a periodic solution of Eq. (2.4). Thus the study of Eq. (2.2) is reduced to the investigation of Eq. (2.4).

Set $R = 1$ and $J_i = (f^1)^{-1}(i)$ for $i \in \{-K, 0, K\}$.

If $t_0 < t_1$ and $x : [t_0 - 1, t_1] \to \mathbb{R}$ is a solution of Eq. (2.4) such that for some $i \in \{-K, 0, K\}$, we have $x(t-1) \in J_{-i}$ for all $t \in (t_0, t_1)$, then Eq. (2.4) reduces to the ordinary differential equation

$$\dot{x}(t) = -\mu x(t) + i$$

on the interval (t_0, t_1), and thus

$$x(t) = \frac{i}{\mu} + \left(x(t_0) - \frac{i}{\mu}\right) e^{-\mu(t-t_0)} \qquad \text{for } t \in [t_0, t_1]. \tag{2.5}$$

In coherence with [4], we say that a function $x : [t_0, t_1] \to \mathbb{R}$ is of type (i/μ) on $[t_0, t_1]$ with $i \in \{-K, 0, K\}$ if (2.5) holds.

It is an easy calculation to show that if $\mu > 0$, and K satisfy (2.1), then $K > 2\mu$. As we shall see later, condition (2.1) comes from assumptions

$$K > 0 \quad \text{and} \quad \frac{K^2 - 2K\mu - \mu^2}{K^2 - \mu^2} > e^{-\mu}. \tag{2.6}$$

As for any $\mu > 0$ fixed, the second inequality is of second order in K, the solution formula gives (2.1) and (2.6) are equivalent.

Fix $\varphi \in C$ with $\varphi(s) > 1$ for $s \in [-1, 0)$ and $\varphi(0) = 1$. This choice implies that solution $x = x^{\varphi, f^1} : [-1, \infty) \mapsto \mathbb{R}$ is of type $(-K/\mu)$ on $[0, 1]$, that is

$$x(t) = -\frac{K}{\mu} + \left(1 + \frac{K}{\mu}\right) e^{-\mu t} \text{ for } t \in [0, 1]. \tag{2.7}$$

Clearly, x is strictly decreasing on $[0, 1]$. We claim that

$$x(1) = -\frac{K}{\mu} + \left(1 + \frac{K}{\mu}\right) e^{-\mu} \tag{2.8}$$

is smaller than -1, that is $e^{-\mu} < (K - \mu) / (K + \mu)$. Indeed, (2.6) (which condition is equivalent to the initial assumption (2.1)) gives

$$e^{-\mu} < \frac{K^2 - 2K\mu - \mu^2}{K^2 - \mu^2} < \frac{(K - \mu)^2}{K^2 - \mu^2} = \frac{K - \mu}{K + \mu}.$$

Therefore equation $x(t) = -1$ has a unique solution τ in $(0,1)$. It comes from (2.7) that

$$\tau = \frac{1}{\mu} \ln \frac{K+\mu}{K-\mu}. \tag{2.9}$$

Note that x maps $[0,\tau]$ onto $[-1,1]$. Hence x is of type (0) on $[1,\tau+1]$. Relations (2.5) and (2.8) yield

$$x(t) = x(1)\, e^{-\mu(t-1)} = -\frac{K}{\mu} e^{-\mu(t-1)} + \left(1 + \frac{K}{\mu}\right) e^{-\mu t} \text{ for } t \in [1, \tau+1]. \tag{2.10}$$

In particular,

$$x(\tau+1) = \frac{K-\mu}{\mu}\left(e^{-\mu} - \frac{K}{K+\mu}\right) \tag{2.11}$$

by (2.9).

Assumption (2.6) implies $x(\tau+1) < -1$. In addition, $x(1) < -1$ and (2.10) give that x is strictly increasing on $[1,\tau+1]$. So $x(t) < -1$ for $t \in [1,\tau+1]$. Also, $x(t) < -1$ for $t \in (\tau,1)$ because $x(\tau) = -1$, $\tau \in (0,1)$, and x strictly decreases on $[0,1]$.

In consequence, x is of type (K/μ) on $[\tau+1,\tau+2]$. Then (2.5), (2.9) and (2.11) imply

$$x(t) = \frac{K}{\mu} + \frac{1}{\mu}\left(K + \mu - \frac{2K^2 e^{\mu}}{K-\mu}\right) e^{-\mu t} \text{ for } t \in [\tau+1, \tau+2], \tag{2.12}$$

and

$$x(\tau+2) = \frac{1}{\mu}\left(K - \frac{2K^2}{K+\mu} e^{-\mu} + (K-\mu)\, e^{-2\mu}\right).$$

We claim $x(\tau+2) > -1$. This statement is equivalent to

$$(e^{\mu}-1)^2 K^2 + 2\mu e^{2\mu} K + \mu^2 \left(e^{2\mu} - 1\right) > 0.$$

So it suffices to show that

$$K > K_0(\mu) = \mu \frac{-e^{2\mu} + \sqrt{e^{4\mu} - (e^{\mu}-1)^2 (e^{2\mu}-1)}}{(e^{\mu}-1)^2}.$$

This condition is clearly fulfilled, as $K > 0$ and $K_0(\mu) < 0$ for all $\mu > 0$. Hence $x(\tau+2) > -1$.

Hypothesis (2.6) implies

$$K + \mu - \frac{2K^2 e^{\mu}}{K-\mu} < 0,$$

thus x is strictly increasing on $[\tau+1, \tau+2]$ by formula (2.12). This result and $x(\tau+1) < -1 < x(\tau+2)$ yield that there exists a unique $z \in (\tau+1, \tau+2)$ with $x(z) = -1$. From (2.12) we get

$$z = 1 + \frac{1}{\mu} \ln \left(\frac{2K^2}{K^2-\mu^2} - e^{-\mu} \right). \tag{2.13}$$

Clearly, $2 < \tau + 2$. We show that $z < 2$. Indeed, $z < 2$ is equivalent to

$$\mu \frac{\sqrt{e^{2\mu}+1}}{e^{\mu}-1} < K,$$

which is a direct consequence of (2.1). So the monotonicity of x on $[\tau+1, \tau+2]$ gives $x(2) > -1$.

It follows from the definition of z, from the estimate $x(t) < -1$ for $t \in (\tau, z)$ and from $z - \tau > 1$ that

$$x_z(s) < -1 \text{ for } s \in [-1,0), \text{ and } x_z(0) = -1.$$

Remark 1.2 and the previous argument give

$$x_{2z}(s) = x_z^{x_z, f^1}(s) > 1 \text{ for } s \in [-1,0), \text{ and } x_{2z}(0) = x_z^{x_z, f^1}(0) = 1.$$

Hence x can be extended to a periodic solution of Eq. (2.4) on $\mathbb{R}$. Let $x^1 : \mathbb{R} \to \mathbb{R}$ be a periodic function with minimal period $2z$, and with

$$x^1(t) = \begin{cases} x(t), & t \in [0, z], \\ -x(t-z), & t \in (z, 2z). \end{cases}$$

Then x^1 satisfies Eq. (2.4) for $t \in \mathbb{R}$.

Note that for all $\varphi \in C$ with $\varphi(s) > 1$ for $s \in [-1, 0)$ and $\varphi(0) = 1$, we have $x_t^{\varphi, f^1} = x_t^1$ for all $t \geq 1$.

By Remark 2.1, our reasoning gives the following result for Eq. (2.2).

Proposition 2.2. *Assume $R > 0$, $\mu > 0$, and K is chosen such that* (2.1) *holds. Let $\tau \in (0,1)$ and $z \in (\tau + 1, 2)$ be given by* (2.9) *and* (2.13), *respectively. Then Eq.* (2.2) *admits a periodic solution $x^R : \mathbb{R} \to \mathbb{R}$ with the following properties.*
(i) The minimal period of x^R is $2z$.
(ii) $x^R(0) = -x^R(\tau) = -x^R(z) = R$.
(iii) $x^R(t) > R$ on $[-1, 0)$, $x^R(t) \in (-R, R)$ on $(0, \tau)$, $x^R(t) < -R$ on (τ, z) and $x^R(t) > -R$ for all $t \in (z, 2]$.
(iv) x^R strictly decreases on $[0, 1]$, and it strictly increases on $[1, 2]$.
(v) $x^R(t) = Rx^1(t)$ for all $t \in \mathbb{R}$.
In consequence,
(vi) $\max_{t\in\mathbb{R}} \left|x^R(t)\right| = R \max_{t\in\mathbb{R}} \left|x^1(t)\right|$, *where*

$$\max_{t\in\mathbb{R}} \left|x^1(t)\right| = -x^1(1) = \frac{K}{\mu} - \frac{K+\mu}{\mu} e^{-\mu} \in \left(1, \frac{K}{\mu}\right).$$

Proposition 2.2 is applied in the next section with $R = r^n$, where $r > 1$ is fixed and $n \geq 0$. We are going to construct a feedback function f so that Eq. (1.1) has an SOP solution close to x^{r^n} in a sense to be clarified.

For technical purposes, we need the following notation. For $\xi \in (0,1)$, set $T_i(\xi) > 0$, $i \in \{1, 2, 3\}$, so that $T_1(\xi)$, $T_2(\xi)$, $T_3(\xi)$ is the time needed by a function of type $(-K/\mu)$ to decrease from 1 to $1 - \xi$, from $-1 + \xi$ to -1, and from -1 to $-1 - \xi$, respectively.

Using (2.5), one gets

$$T_1(\xi) = \frac{1}{\mu} \ln\left(1 + \frac{\mu\xi}{K + \mu(1 - \xi)}\right).$$

As $\ln(1 + x) < x$ for all $x > 0$, we obtain

$$T_1(\xi) < \frac{\xi}{K + \mu(1 - \xi)} < \frac{\xi}{K}. \tag{2.14}$$

Similarly,

$$T_2(\xi) < \frac{\xi}{K - \mu} \text{ and } T_3(\xi) < \frac{\xi}{K - 2\mu}. \tag{2.15}$$

As x^1 is of type $(-K/\mu)$ on $[0,1]$ (see (2.7)), and $x^R(t) = Rx^1(t)$ for all $R > 0$ and $t \in \mathbb{R}$, the definition of $T_i(\xi)$, $i \in \{1,2\}$, clearly gives

$$x^R(T_1(\xi)) = R(1-\xi) \text{ and } x^R(\tau - T_2(\xi)) = -R(1-\xi)$$

for $R > 0$, $\xi \in (0,1)$ and τ defined by (2.9). Analogously, $x^R(\tau + T_3(\xi)) = -R(1+\xi)$ for $R > 0$ and $\xi \in (0, \min\{1, |x^1(1)+1|\})$.

3. Slowly oscillating solutions for continuous nonlinearities

Now we turn attention to continuous nonlinearities. In addition to parameters $\mu > 0$ and K satisfying condition (2.1), fix a constant $M > K$.

For $r > 1$, $\varepsilon \in (0, r-1)$ and $\eta \in (0, M-K)$, let $N = N(r,\varepsilon,\eta)$ be the set of all continuous odd functions $f : \mathbb{R} \to \mathbb{R}$ with

$$|f(x)| < \eta \text{ for } x \in [0,1],$$

$$\left|\frac{f(x)}{r^n}\right| < M \text{ for all } x \in (r^n, r^n(1+\varepsilon)) \text{ and } n \geq 0$$

and with

$$\left|\frac{f(x)}{r^n} - K\right| < \eta \text{ for all } x \in \left[r^n(1+\varepsilon), r^{n+1}\right] \text{ and } n \geq 0.$$

Elements of N restricted to $[-r^n, r^n]$, $n \geq 1$, can be viewed as perturbations of $f^{r^{n-1}}$ introduced in the previous section.

Observe that

$$\max_{f \in N(r,\varepsilon,\eta),\, x \in [-r^n, r^n]} |f(x)| < Mr^{n-1} \text{ for all } n \geq 1. \tag{3.1}$$

For $f \in N(r,\varepsilon,\eta)$, we look for SOP solutions of Eq. (1.1) with initial functions in the nonempty closed convex sets $A_n = A_n(r,\varepsilon)$ defined as

$$A_n = \left\{\varphi \in C : \, r^n(1+\varepsilon) \leq \varphi(s) \leq r^{n+1} \text{ for } s \in [-1,0),\, \varphi(0) = r^n(1+\varepsilon)\right\}$$

for each $n \geq 0$.

Solutions of Eq. (1.1) with $f \in N(r,\varepsilon,\eta)$ and with initial segment in $A_n(r,\varepsilon)$ converge to x^{r^n} on $[0,2]$ as $r \to \infty$, $\varepsilon \to 0+$ and $\eta \to 0+$ in the following sense.

Proposition 3.1. *For each $\delta > 0$ there are $r_0 = r_0(\delta) > 1$, $\varepsilon_0 = \varepsilon_0(\delta) > 0$ and $\eta_0 = \eta_0(\delta) > 0$, such that for all $r > r_0$, $\varepsilon \in (0, \varepsilon_0)$, $\eta \in (0, \eta_0)$ and $n \geq 0$,*

$$\sup_{f \in N(r,\varepsilon,\eta),\, \varphi \in A_n(r,\varepsilon),\, t \in [0,2]} \left| x^{\varphi,f}(t) - x^{r^n}(t) \right| < \delta r^n.$$

Proof. Fix $\delta > 0$ arbitrarily. Set r, ε, η as in the definition of $N(r,\varepsilon,\eta)$, and choose r to be greater that $-x^1(1)$. In addition, assume that

$$\varepsilon + \eta < r + x^1(1), \text{ and } 2\varepsilon + \eta < \min\left\{1, \left|x^1(1) + 1\right|\right\}. \tag{3.2}$$

This is clearly possible. Fix any $n \geq 0$, $\varphi \in A_n(r,\varepsilon)$ and $f \in N(r,\varepsilon,\eta)$.

1. By Proposition 2.2 (iii), $x^{r^n}(t) > r^n$ for $t \in [-1, 0)$. Hence the definition of f^{r^n}, the definitions of the function classes $N(r,\varepsilon,\eta)$ and $A_n(r,\varepsilon)$ and the variation-of-constants formula give that

$$\begin{aligned}
\left| x^{\varphi,f}(t) - x^{r^n}(t) \right| &\leq \left| x^{\varphi,f}(0) - x^{r^n}(0) \right| e^{-\mu t} \\
&\quad + \left| \int_0^t e^{-\mu(t-s)} f(\varphi(s-1))\, \mathrm{d}s - \int_0^t e^{-\mu(t-s)} f^{r^n}\left(x^{r^n}(s-1)\right) \mathrm{d}s \right| \\
&\leq \varepsilon r^n e^{-\mu t} + \int_0^t e^{-\mu(t-s)} \left| f(\varphi(s-1)) - r^n K \right| \mathrm{d}s \qquad (3.3) \\
&< r^n(\varepsilon + \eta)
\end{aligned}$$

for $t \in [0,1]$.

2. Similarly, for $t \in [1,2]$ we have

$$\begin{aligned}
\left| x^{\varphi,f}(t) - x^{r^n}(t) \right| &\leq \left| x^{\varphi,f}(1) - x^{r^n}(1) \right| e^{-\mu(t-1)} \\
&\quad + \int_1^t e^{-\mu(t-s)} \left| f\left(x^{\varphi,f}(s-1)\right) - f^{r^n}\left(x^{r^n}(s-1)\right) \right| \mathrm{d}s \qquad (3.4) \\
&\leq \left\| x_1^{\varphi,f} - x_1^{r^n} \right\| + \int_0^1 \left| f\left(x^{\varphi,f}(s)\right) - f^{r^n}\left(x^{r^n}(s)\right) \right| \mathrm{d}s.
\end{aligned}$$

By the previous step, $\left\| x_1^{\varphi,f} - x_1^{r^n} \right\| < r^n(\varepsilon + \eta)$. Since $\left| x^{r^n}(t) \right| \leq r^n \left| x^1(1) \right|$ holds for all real t by Proposition 2.2 (vi) and since $\varepsilon + \eta < r + x^1(1)$ holds, it follows that

$$\left| x^{\varphi,f}(t) \right| < \left| x^{r^n}(t) \right| + r^n(\varepsilon + \eta) \leq r^n\left(-x^1(1) + \varepsilon + \eta\right) < r^{n+1} \quad \text{for } t \in [0,1]. \tag{3.5}$$

We give an upper estimate for the integral on the right hand side in (3.4).

2.a. First we consider interval $[0,\tau]$, where $\tau \in (0,1)$ is defined by (2.9). Recall from Proposition 2.2 (iii) that $x^{r^n}(t) \in [-r^n, r^n]$, thus $f^{r^n}\left(x^{r^n}(t)\right) = 0$ for $t \in [0,\tau]$.

Parameters ε, η are set so that $0 < \varepsilon + \eta < 1$, therefore $T_i(\varepsilon+\eta)$, $i \in \{1,2\}$, is defined, and $T_1(\varepsilon+\eta) < \tau - T_2(\varepsilon+\eta)$. By the monotonicity property of x^{r^n} on $[0,1]$ (see Proposition 2.2 (iv)) and the definitions of T_i, $i \in \{1,2\}$, we have

$$\left|x^{r^n}(t)\right| \leq r^n - r^n(\varepsilon+\eta) \quad \text{for } t \in [T_1(\varepsilon+\eta), \tau - T_2(\varepsilon+\eta)].$$

So with $T_1 = T_1(\varepsilon+\eta)$ and $T_2 = T_2(\varepsilon+\eta)$, the estimate given in the first step implies

$$\left|x^{\varphi,f}(t)\right| < \left|x^{r^n}(t)\right| + r^n(\varepsilon+\eta) \leq r^n \quad \text{for } t \in [T_1, \tau - T_2].$$

In case $n \geq 1$, property (3.1) yields

$$\left|f\left(x^{\varphi,f}(t)\right) - f^{r^n}\left(x^{r^n}(t)\right)\right| = \left|f\left(x^{\varphi,f}(t)\right)\right| < \frac{M}{r} r^n, \quad t \in [T_1, \tau - T_2].$$

For $n = 0$,

$$\left|f\left(x^{\varphi,f}(t)\right) - f^1\left(x^1(t)\right)\right| = \left|f\left(x^{\varphi,f}(t)\right)\right| < \eta r^0, \quad t \in [T_1, \tau - T_2],$$

by the definition of the function class $N(r,\varepsilon,\eta)$. As $0 < \tau - T_1 - T_2 < 1$, it follows that

$$\int_{T_1}^{\tau - T_2} \left|f\left(x^{\varphi,f}(s)\right) - f^{r^n}\left(x^{r^n}(s)\right)\right| \mathrm{d}s < \max\left\{\frac{M}{r}, \eta\right\} r^n \tag{3.6}$$

for each $n \geq 0$.

For $t \in [0, T_1) \cup (\tau - T_2, \tau]$, we have $\left|x^{\varphi,f}(t)\right| < r^{n+1}$ by (3.5). Hence (2.14), (2.15) and (3.1) imply

$$\begin{aligned}\left(\int_0^{T_1} + \int_{\tau-T_2}^{\tau}\right) \left|f\left(x^{\varphi,f}(s)\right) - f^{r^n}\left(x^{r^n}(s)\right)\right| \mathrm{d}s &= \left(\int_0^{T_1} + \int_{\tau-T_2}^{\tau}\right) \left|f\left(x^{\varphi,f}(s)\right)\right| \mathrm{d}s \\ &< M r^n (T_1 + T_2) < \frac{2M}{K-\mu}(\varepsilon+\eta) r^n.\end{aligned} \tag{3.7}$$

2.b. Estimates for the interval $(\tau, 1]$. For $t \in (\tau, 1]$, $x^{r^n}(t) < -r^n$, hence $f^{r^n}\left(x^{r^n}(t)\right) = -Kr^n$.

Parameters ε, η are fixed so that $0 < 2\varepsilon + \eta < \min\{1, |x^1(1) + 1|\}$ holds, thus $T_3(2\varepsilon + \eta)$ is defined and $\tau + T_3(2\varepsilon + \eta) < 1$. The fact that x^{r^n} strictly decreases on $[0, 1]$ and the definition of T_3 give that

$$x^{r^n}(t) \leq -r^n - r^n(2\varepsilon + \eta) \quad \text{for } t \in [\tau + T_3(2\varepsilon + \eta), 1].$$

Hence

$$x^{\varphi,f}(t) < x^{r^n}(t) + r^n(\varepsilon + \eta) \leq -r^n(1 + \varepsilon) \quad \text{for } t \in [\tau + T_3, 1],$$

where $T_3 = T_3(2\varepsilon + \eta)$. Also, $x^{\varphi,f}(t) > -r^{n+1}$ for t in this interval. It follows from the definition of $N(r, \varepsilon, \eta)$ that

$$\left|f\left(x^{\varphi,f}(t)\right) - f^{r^n}\left(x^{r^n}(t)\right)\right| = \left|f\left(x^{\varphi,f}(t)\right) - (-Kr^n)\right| < r^n\eta$$

for $t \in [\tau + T_3, 1]$, and

$$\int_{\tau+T_3}^{1} \left|f\left(x^{\varphi,f}(s)\right) - f^{r^n}\left(x^{r^n}(s)\right)\right| \mathrm{d}s < (1 - \tau - T_3)\, r^n\eta < r^n\eta. \tag{3.8}$$

It remains to consider the interval $(\tau, \tau + T_3)$. From (2.15), (3.1) and (3.5) we obtain that

$$\begin{aligned}\int_{\tau}^{\tau+T_3} \left|f\left(x^{\varphi,f}(s)\right) - f^{r^n}\left(x^{r^n}(s)\right)\right| \mathrm{d}s &\leq \int_{\tau}^{\tau+T_3} \left(\left|f\left(x^{\varphi,f}(s)\right)\right| + \left|f^{r^n}\left(x^{r^n}(s)\right)\right|\right) \mathrm{d}s \\ &< T_3(M + K)\, r^n < \frac{M + K}{K - 2\mu}(2\varepsilon + \eta)\, r^n. \end{aligned} \tag{3.9}$$

Set $r_0, \varepsilon_0, \eta_0$ as in the definition of $N(r, \varepsilon, \eta)$ with $r_0 > -x^1(1)$ and $M/r_0 < \delta/2$. If necessary, decrease $\varepsilon_0 > 0$ and $\eta_0 > 0$ so that (3.2) holds for $r_0, \varepsilon_0, \eta_0$, and

$$(\varepsilon_0 + \eta_0) + \eta_0 + \frac{2M}{K - \mu}(\varepsilon_0 + \eta_0) + \eta_0 + \frac{M + K}{K - 2\mu}(2\varepsilon_0 + \eta_0) < \frac{\delta}{2}.$$

Then summing up the estimates (3.3), (3.4) and (3.6)-(3.9), we conclude that

$$\left|x^{\varphi,f}(t) - x^{r^n}(t)\right| < \delta r^n \text{ on } [0, 2]$$

for all $r > r_0$, $\varepsilon \in (0, \varepsilon_0)$, $\eta \in (0, \eta_0)$, $n \geq 0$, $\varphi \in A_n(r, \varepsilon)$ and $f \in N(r, \varepsilon, \eta)$. □

Fix any $w \in (\tau, z-1)$. Then $w+1 \in (\tau+1, z)$, and $x^{r^n}(t) < -r^n$ on $[w, w+1]$ for all $n \geq 0$ by Proposition 2.2 (iii).

In the subsequent result, we apply Proposition 3.1 and confirm that with an appropriate choice of parameters r, ε and η, we have $x^{\varphi,f}(t) < -r^n(1+\varepsilon)$ on $[w, w+1]$ for all $f \in N(r, \varepsilon, \eta)$, $\varphi \in A_n(r, \varepsilon)$ and $n \geq 0$. The same proposition and $x^{r^n}(2) > -r^n$ guarantee $x^{\varphi,f}(2) > -r^n$. Hence there exists $q \in (w+1, 2)$ with $x_q^{\varphi,f} \in -A_n(r, \varepsilon)$.

Before reading the proof, recall that $x^{r^n}(t) = r^n x^1(t)$, $t \in \mathbb{R}$, and

$$\frac{K}{\mu} > \left|x^1(1)\right| \geq x^1(2) > -1 > x^1(1).$$

Proposition 3.2. *There exist $r_1 > 1$, $\varepsilon_1 > 0$ and $\eta_1 > 0$ so that for each $r > r_1$, $\varepsilon \in (0, \varepsilon_1)$, $\eta \in (0, \eta_1)$, $n \geq 0$, $f \in N(r, \varepsilon, \eta)$ and $\varphi \in A_n(r, \varepsilon)$, the solution $x^{\varphi,f}$: $[-1, \infty) \to \mathbb{R}$ of Eq.* (1.1) *has the following properties.*
(i) $-r^{n+1} < x^{\varphi,f}(t) < r^{n+1}$ for $t \in [0, 2]$.
(ii) $x^{\varphi,f}(t) < -r^n(1+\varepsilon)$ for $t \in [w, w+1]$, and $x^{\varphi,f}(2) > -r^n$.
(iii) $\dot{x}^{\varphi,f}(t) < 0$ for $t \in (0, 1)$, and $\dot{x}^{\varphi,f}(t) > 0$ for $t \in (w+1, 2]$.
(iv) If $q = q(\varphi, f) \in (1+w, 2)$ is set so that $x^{\varphi,f}(q) = -r^n(1+\varepsilon)$, then q is unique, and $x_q^{\varphi,f} \in -A_n(r, \varepsilon)$.
(v) If in addition $\psi \in A_n(r, \varepsilon)$, then for the semiflow (1.3) *the equality $F(1+w, \psi) = F(1+w, \varphi)$ implies $q(\psi, f) = q(\varphi, f)$.*

Proof. Assume

$$0 < \delta < \min\left\{\frac{1}{2}\left(\frac{K}{\mu} + x^1(1)\right), -\frac{1}{2}\left(\max_{t \in [w, w+1]} x^1(t) + 1\right), 1 + x^1(2)\right\}.$$

Note that all expressions on the right hand side are positive.

Choose $r_1 = \max\{K/\mu, r_0(\delta)\}$,

$$\varepsilon_1 = \min\left\{\varepsilon_0(\delta), -\frac{1}{2}\left(\max_{t \in [w, w+1]} x^1(t) + 1\right)\right\}, \; \eta_1 = \min\left\{\eta_0(\delta), \frac{1}{2}\left(K + \mu x^1(1)\right)\right\},$$

where $r_0(\delta)$, $\varepsilon_0(\delta)$ and $\eta_0(\delta)$ are given by Proposition 3.1. Consider $r > r_1$, $\varepsilon \in (0, \varepsilon_1)$, $\eta \in (0, \eta_1)$, $n \geq 0$, $f \in N(r, \varepsilon, \eta)$ and $\varphi \in A_n(r, \varepsilon)$.

(i) For $t \in [0, 2]$, it follows from Proposition 2.2 (vi) and Proposition 3.1, that

$$\left|x^{\varphi,f}(t)\right| < x^{r^n}(t) + r^n \delta \leq r^n\left(\left|x^1(1)\right| + \delta\right).$$

As we chose δ to be smaller than $K/\mu + x^1(1) \leq r + x^1(1)$, we deduce that $\left|x^{\varphi,f}(t)\right| < r^{n+1}$.

(ii) For $t \in [w, w+1]$ we get

$$x^{\varphi,f}(t) < x^{r^n}(t) + r^n \delta \leq r^n \left(\max_{t \in [w,w+1]} x^1(t) + \delta \right) < -r^n (1+\varepsilon)$$

because $\delta + \varepsilon < -\max_{t \in [w,w+1]} x^1(t) - 1$. For $t = 2$ we obtain that

$$x^{\varphi,f}(2) > x^{r^n}(2) - r^n \delta \geq r^n \left(x^1(2) - \delta\right) > -r^n,$$

as $\delta < 1 + x^1(2)$.

(iii) For $t \in (0,1)$,

$$\begin{aligned}
\dot{x}^{\varphi,f}(t) &= -\mu x^{\varphi,f}(t) - f(\varphi(t-1)) \\
&< -\mu \left(x^{r^n}(t) - r^n \delta\right) - r^n (K - \eta) \\
&\leq r^n \left(-\mu x^1(1) + \mu \delta - K + \eta\right) < 0,
\end{aligned}$$

as the parameters are set so that

$$\delta + \frac{\eta}{\mu} < \frac{K}{\mu} + x^1(1).$$

For $t \in (w+1, 2]$, we have $t - 1 \in (w, 1]$. Thus $-r^{n+1} < x^{\varphi,f}(t-1) < -r^n(1+\varepsilon)$ by assertions (i) and (ii) of this proposition, and

$$\begin{aligned}
\dot{x}^{\varphi,f}(t) &= -\mu x^{\varphi,f}(t) - f\left(x^{\varphi,f}(t-1)\right) \\
&> -\mu \left(x^{r^n}(t) + r^n \delta\right) + r^n (K - \eta) \\
&\geq r^n \left(-\mu x^1(2) - \mu \delta + K - \eta\right) > 0,
\end{aligned}$$

since

$$\delta + \frac{\eta}{\mu} < \frac{K}{\mu} + x^1(1) < \frac{K}{\mu} - x^1(2).$$

Assertion (iv) now follows immediately.

(v) If $\psi \in A_n(r, \varepsilon)$ and $F(1+w, \psi) = F(1+w, \varphi)$, then $x^{\psi,f}(t) = x^{\varphi,f}(t)$ for $t \geq 1 + w$. As $q(\psi, f) > 1 + w$ and $q(\varphi, f) > 1 + w$, $q(\psi, f) = q(\varphi, f)$ follows. $\square$

4. Lipschitz continuous return maps

Recall that $\mu > 0$, and (2.1) holds in this paper. In addition, from now on we assume that $K > \mu e^{\mu}$. $M > K$ is fixed as before.

Set $r > r_1$, $\varepsilon \in (0, \varepsilon_1)$ and $\eta \in (0, \eta_1)$ in this section, where r_1, ε_1 and η_1 are specified by Proposition 3.2. Following Walther [8] and based on the results of Proposition 3.2, we introduce the Lipschitz continuous return map

$$R_f^n : A_n(r,\varepsilon) \ni \varphi \mapsto -F(q(\varphi, f), \varphi) \in A_n(r,\varepsilon)$$

for each $f \in N(r,\varepsilon,\eta)$ and $n \geq 0$. As it is discussed in [8], the fixed point of R_f^n, $n \geq 0$, is the initial segment of a periodic solution p^n of Eq. (1.1) with minimal period $2q$ and special symmetry $p^n(t) = -p^n(t+q)$, $t \in \mathbb{R}$. As p^n has at most 1 zero on $[0, q]$ and $q > 1$, the special symmetry property implies that p^n is an SOP solution.

In order to verify the Lipschitz continuity of R_f^n, we define the map

$$s_f^n : F(1+w, A_n(r,\varepsilon)) \ni \psi \mapsto q(\varphi, f) - 1 - w \in (0, 1-w), \text{ where } \psi = F(1+w, \varphi),$$

for each $n \geq 0$ and $f \in N(r,\varepsilon,\eta)$. Also, set

$$F_1^n : A_n(r,\varepsilon) \ni \varphi \mapsto F(1,\varphi) \in C,$$

$$F_w^n : F(1, A_n(r,\varepsilon)) \ni \varphi \mapsto F(w,\varphi) \in C,$$

$$S_f^n : F(1+w, A_n(r,\varepsilon)) \ni \varphi \mapsto -F\left(s_f^n(\varphi), \varphi\right) \in A_n(r,\varepsilon)$$

for all $f \in N(r,\varepsilon,\eta)$ and $n \geq 0$. Proposition 3.2 implies that s_f^n and S_f^n are well-defined. Then R_f^n is the composite of F_1^n, followed by F_w^n, then by S_f^n.

We give Lipschitz constants for the maps above. As next result we state Proposition 3.1 of [8] without proof.

Proposition 4.1. *Set $r > r_1$, $\varepsilon \in (0,\varepsilon_1)$ and $\eta \in (0,\eta_1)$. Assume $n \geq 0$, and $f \in N(r,\varepsilon,\eta)$ is locally Lipschitz continuous. If $L^n = L^n(f)$ and $L_*^n = L_*^n(f)$ are Lipschitz constants for the restrictions $f|_{[-r^{n+1}, r^{n+1}]}$ and $f|_{[r^n(1+\varepsilon), r^{n+1}]}$, respectively, then L_*^n is a Lipschitz constant for F_1^n, and $1 + wL^n$ is a Lipschitz constant for F_w^n.*

The following result is analogous to Proposition 3.2 in [8], and the proof needs only slight modifications.

Proposition 4.2. *Let $r > r_1$, $\varepsilon \in (0, \varepsilon_1)$, $\eta \in (0, \eta_1)$ and $n \geq 0$. Assume in addition that*

$$K - \eta > (1 + \varepsilon)\, \mu e^{\mu}.$$

If $f|_{[r^n(1+\varepsilon), r^{n+1}]}$ is Lipschitz continuous with Lipschitz constant $L_*^n = L_*^n(f)$, then s_f^n is Lipschitz continuous with Lipschitz constant

$$L\left(s_f^n\right) = \frac{1 + e^{\mu} L_*^n}{r^n \left[K - \eta - \mu e^{\mu}(1 + \varepsilon)\right]},$$

and S_f^n is Lipschitz continuous with Lipschitz constant

$$\frac{1 + e^{\mu} L_*^n}{\left[K - \eta - \mu e^{\mu}(1 + \varepsilon)\right]}(\mu r + M) + 1 + L_*^n.$$

Proof. Choose $\varphi, \bar{\varphi} \in F(1 + w, A_n(r, \varepsilon))$. With $s = s_f^n(\varphi) \in (0, 1 - w) \subset (0, 1)$ and $\bar{s} = s_f^n(\bar{\varphi}) \in (0, 1 - w) \subset (0, 1)$, we have

$$-(1 + \varepsilon)\, r^n = \varphi(0)\, e^{-\mu s} - \int_0^s e^{-\mu(s - \xi)} f(\varphi(\xi - 1))\, \mathrm{d}\xi,$$

and

$$-(1 + \varepsilon)\, r^n = \bar{\varphi}(0)\, e^{-\mu \bar{s}} - \int_0^{\bar{s}} e^{-\mu(\bar{s} - \xi)} f(\bar{\varphi}(\xi - 1))\, \mathrm{d}\xi.$$

Hence

$$\begin{aligned}
(1 + \varepsilon)\, r^n \left|e^{\mu s} - e^{\mu \bar{s}}\right| &\geq \left|\int_0^s e^{\mu \xi} f(\varphi(\xi - 1))\, \mathrm{d}\xi - \int_0^{\bar{s}} e^{\mu \xi} f(\varphi(\xi - 1))\, \mathrm{d}\xi\right| \\
&- \left|\varphi(0) - \bar{\varphi}(0)\right| \\
&- \left|\int_0^{\bar{s}} e^{\mu \xi} \left\{f(\varphi(\xi - 1)) - f(\bar{\varphi}(\xi - 1))\right\} \mathrm{d}\xi\right| \\
&\geq \left|\int_{\bar{s}}^s e^{\mu \xi} f(\varphi(\xi - 1))\, \mathrm{d}\xi\right| \\
&- \left\|\varphi - \bar{\varphi}\right\| \\
&- \left|\int_0^{\bar{s}} e^{\mu \xi} \left\{f(\varphi(\xi - 1)) - f(\bar{\varphi}(\xi - 1))\right\} \mathrm{d}\xi\right|.
\end{aligned}$$

Since $-r^{n+1} < \varphi(t) < -r^n(1 + \varepsilon)$ and $-r^{n+1} < \bar{\varphi}(t) < -r^n(1 + \varepsilon)$ for each $t \in [-1, 0]$, we conclude that

$$(1 + \varepsilon)\, r^n \left|e^{\mu s} - e^{\mu \bar{s}}\right| \geq |s - \bar{s}|\, r^n (K - \eta) - \left\|\varphi - \bar{\varphi}\right\| - e^{\mu} L_*^n \left\|\varphi - \bar{\varphi}\right\|.$$

On the other hand, $|e^{\mu s} - e^{\mu \bar{s}}| \leq \mu e^{\mu} |s - \bar{s}|$. Thus

$$|s - \bar{s}| \leq \frac{1 + e^{\mu} L_*^n}{r^n \left[K - \eta - \mu e^{\mu} (1 + \varepsilon)\right]} \|\varphi - \bar{\varphi}\| ,$$

and the proof of the first assertion is complete.

If $\varphi = F(1 + w, \psi)$ with $\psi \in A_n(r, \varepsilon)$, then for $t \in [-1, 0]$,

$$\begin{aligned} F(\bar{s}, \varphi)(t) - F(s, \varphi)(t) &= x^{\psi}_{1+w+\bar{s}}(t) - x^{\psi}_{1+w+s}(t) \\ &= \int_{1+w+s}^{1+w+\bar{s}} \dot{x}^{\psi}(\xi) \, \mathrm{d}\xi \\ &= \int_{1+w+s}^{1+w+\bar{s}} \left\{-\mu x^{\psi}(\xi) - f\left(x^{\psi}(\xi - 1)\right)\right\} \mathrm{d}\xi. \end{aligned}$$

So Proposition 3.2 (i) and (3.1) imply

$$|F(\bar{s}, \varphi)(t) - F(s, \varphi)(t)| \leq |s - \bar{s}| (\mu r + M) r^n \leq L\left(s_f^n\right) (\mu r + M) r^n \|\varphi - \bar{\varphi}\|$$

for $t \in [-1, 0]$. Also, it is easy to see using $\bar{s} \in (0, 1)$, $-r^{n+1} < \varphi(t), \bar{\varphi}(t) < -r^n (1 + \epsilon)$, $t \in [-1, 0]$, the oddness of f and the variation-of-constants formula, that

$$\|F(\bar{s}, \varphi) - F(\bar{s}, \bar{\varphi})\| \leq (1 + L_*^n) \|\varphi - \bar{\varphi}\| .$$

Hence

$$\begin{aligned} \|S(\varphi) - S(\bar{\varphi})\| &\leq \|F(s, \varphi) - F(\bar{s}, \varphi)\| + \|F(\bar{s}, \varphi) - F(\bar{s}, \bar{\varphi})\| \\ &\leq \left\{\frac{1 + e^{\mu} L_*^n}{K - \eta - \mu e^{\mu} (1 + \varepsilon)} (\mu r + M) + 1 + L_*^n\right\} \|\varphi - \bar{\varphi}\| , \end{aligned}$$

and the proof is complete. □

It follows that under the assumptions of the last two propositions, R_f^n is Lipschitz continuous, and

$$L\left(R_f^n\right) = L_*^n (1 + w L^n) \left(\frac{1 + e^{\mu} L_*^n}{K - \eta - \mu e^{\mu} (1 + \varepsilon)} (\mu r + M) + 1 + L_*^n\right)$$

is a Lipschitz constant for R_f^n. Clearly, if $L\left(R_f^n\right) < 1$, then R_f^n is a strict contraction with a unique fixed point in $A_n(r, \varepsilon)$, and Eq. (1.1) has an SOP solution with initial function in $A_n(r, \varepsilon)$.

Proof of Theorem 1.1. Choose $r > r_1$, $\varepsilon \in (0, \varepsilon_1)$ and $\eta \in (0, \eta_1)$ with

$$K - \eta > (1 + \varepsilon)\, \mu e^{\mu}.$$

We give a nonlinearity $f \in N(r, \varepsilon, \eta)$ so that R_f^n is a contraction for each $n \geq 0$. The function f is defined recursively on $[-r^n, r^n]$ for $n \geq 1$.

First step. Let $f : [-1-\varepsilon, 1+\varepsilon] \to \mathbb{R}$ be a Lipschitz continuous odd function with $|f(x)| < \eta$ for $x \in [0,1]$, $|f(x)| < M$ for all $x \in (1, 1+\varepsilon)$ and $f(1+\varepsilon) \in (K-\eta, K+\eta)$. Let L_{**}^0 be a Lipschitz constant for $f|_{[-1-\varepsilon, 1+\varepsilon]}$. Extend the definition of f to domain $[-r, r]$ so that f remains odd, $|f(x) - K| < \eta$ for $x \in [1+\varepsilon, r]$, and $f|_{[1+\varepsilon, r]}$ is Lipschitz continuous with Lipschitz constant L_*^0 satisfying

$$L_*^0 \left(1 + w \max\left\{L_*^0, L_{**}^0\right\}\right) \left(\frac{1 + e^{\mu} L_*^0}{K - \eta - \mu e^{\mu}(1+\varepsilon)} (\mu r + M) + 1 + L_*^0\right) < 1.$$

This is possible by choosing L_*^0 sufficiently small. Then $L^0 = \max\{L_*^0, L_{**}^0\}$ is a Lipschitz constant for $f|_{[-r,r]}$, and R_f^0 is a strict contraction.

Recursive step. If f is defined for $[-r^n, r^n]$ with some $n \geq 1$, extend the definition of f to the domain $[-r^{n+}, r^{n+1}]$ so that f remains odd, Lipschitz continuous,

$$\left|\frac{f(x)}{r^n}\right| < M \text{ for all } x \in (r^n, r^n(1+\varepsilon)),$$

$$\left|\frac{f(x)}{r^n} - K\right| < \eta \text{ for all } x \in \left[r^n(1+\varepsilon), r^{n+1}\right],$$

and if L_{**}^n is a Lipschitz constant for $f|_{(r^n, r^n(1+\varepsilon))}$, then $f|_{[r^n(1+\varepsilon), r^{n+1}]}$ has a Lipschitz constant L_*^n with

$$L_*^n \left(1 + w \max_{0 \leq k \leq n} \left\{L_*^k, L_{**}^k\right\}\right) \left(\frac{1 + e^{\mu} L_*^n}{K - \eta - \mu e^{\mu}(1+\varepsilon)} (\mu r + M) + 1 + L_*^n\right) < 1.$$

Then $L^n = \max_{0 \leq k \leq n} \left\{L_*^k, L_{**}^k\right\}$ is a Lipschitz constant for $f|_{[-r^{n+1}, r^{n+1}]}$, and R_f^n is a strict contraction.

Thereby we obtain a locally Lipschitz continuous odd function f for which R_f^n is a strict contraction for all $n \geq 0$. For such f, Eq. (1.1) has an infinite sequence of SOP solutions with initial segments in $A_n(r, \varepsilon)$, $n \geq 0$. It is clear that one may set f in this construction so that $xf(x) > 0$ holds for all $x \in \mathbb{R} \setminus \{0\}$.

It follows from Section 4 in [8], that if f is continuously differentiable, then the corresponding periodic orbits are stable and hyperbolic. □

5. A possible modification

As before, set $K > 0$ satisfying condition (2.1) and choose $M > K$. For $r > 1$, $\varepsilon \in (0, r-1)$ and $\eta \in (0, M-K)$, let $\widetilde{N}(r,\varepsilon,\eta)$ be the set of all continuous odd functions $f : \mathbb{R} \to \mathbb{R}$ with

$$\left|\frac{f(x)}{r^n}\right| < M \text{ for all } x \in (r^n, r^n(1+\varepsilon)) \text{ and } n \in \mathbb{Z}$$

and with

$$\left|\frac{f(x)}{r^n} - K\right| < \eta \text{ for all } x \in \left[r^n(1+\varepsilon), r^{n+1}\right] \text{ and } n \in \mathbb{Z}.$$

Then minor modifications of our results in Section 3 and in Section 4 yield the subsequent theorem.

Theorem 5.1. *Assume $\mu > 0$. There exists a locally Lipschitz continuous odd nonlinear map $f \in \widetilde{N}(r,\varepsilon,\eta)$ satisfying $xf(x) > 0$ for all $x \in \mathbb{R} \setminus \{0\}$, for which Eq. (1.1) admits a two-sided infinite sequence of SOP solutions $(p^n)_{-\infty}^{\infty}$ with*

$$\lim_{n\to-\infty} \max_{x\in\mathbb{R}} |p^n(x)| = 0, \quad \lim_{n\to\infty} \max_{x\in\mathbb{R}} |p^n(x)| = \infty,$$

and with $p^n(\mathbb{R}) \subsetneq p^{n+1}(\mathbb{R})$ for $n \in \mathbb{Z}$.

It is easy to see that the elements of $\widetilde{N}(r,\varepsilon,\eta)$ are not differentiable at $x = 0$. Hence the hyperbolicity and stability of the periodic orbits given by the theorem does not follow directly from paper [8] of Walther. Still we conjecture that these periodic orbits are hyperbolic and stable.

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