Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **23**, 1-17; **http://www.math.u-szeged.hu/ejqtde/**

On the Zeros of Solutions of Any Order of Derivative of Second Order Linear Differential Equations Taking Small Functions*

Jin Tu[a] Hong-Yan Xu[b] Cui-Yan Zhang[a]

[a]College of Mathematics and Information Science, Jiangxi Normal University, Nanchang 330022, China
[b] Department of Informatics and Engineering, Jingdezhen Ceramic Institute, Jingdezhen 333403, China
E-mail: tujin2008@sina.com; xhyhhh@126.com; zhcy0618@163.com

Abstract

In this paper, we investigate the hyper-exponent of convergence of zeros of $f^{(j)}(z) - \varphi(z)(j \in N)$, where f is a solution of second or $k(\geq 2)$ order linear differential equation, $\varphi(z) \not\equiv 0$ is an entire function satisfying $\sigma(\varphi) < \sigma(f)$ or $\sigma_2(\varphi) < \sigma_2(f)$. We obtain some precise results which improve the previous results in [3, 5] and revise the previous results in [11, 13]. More importantly, these results also provide us a method to investigate the hyper-exponent of convergence of zeros of $f^{(j)}(z) - \varphi(z)(j \in N)$.

Key words: linear differential equations, hyper-order, hyper-exponent of convergence of zeros
AMS Subject Classification(2000): 30D35, 34M05

1. Introduction and results

In this paper, we shall assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna's theory of meromorphic functions (see [7, 10]). In addition, we use $\sigma(f)$ to denote the order of meromorphic function $f(z)$ and $\tau(f)$ to denote the type of an entire function $f(z)$ with $\sigma(f) = \sigma$, which is defined to be (see [7])

$$\tau(f) = \overline{\lim_{r\to\infty}} \frac{\log M(r,f)}{r^{\sigma}}.$$

We use $\sigma_2(f)$ to denote the hyper-order of entire function $f(z)$, which is defined to be (see [14])

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log_3 M(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_2 T(r,f)}{\log r},$$

*Corresponding author is Jin Tu. E-mail: tujin2008@sina.com
This work was supported by the Natural Science Foundations of Jiang-Xi Province in China (Grant No. 2009GQS0013, Grant No. 2010GQS0119), and by the Foundation of Education Department of Jiang-xi Province in China (Grant No.GJJ08161, Grant No. GJJ09463, Grant No. GJJ10223).

where $\log_i r = \log_{i-1}(\log r)(i \in N)$. It is easy to see that $\sigma(f) = \infty$ if $\sigma_2(f) > 0$. Assume that $\varphi(z)$ is an entire function with $\sigma(\varphi) < \sigma(f)$ or $\sigma_2(\varphi) < \sigma_2(f)$, the hyper-exponent of convergence of zeros of $f(z) - \varphi(z)$ is defined to be

$$\lambda_2(f-\varphi) = \overline{\lim_{r\to\infty}} \frac{\log_2 N(r, \frac{1}{f-\varphi})}{\log r},$$

especially if $\varphi(z) = z$, the hyper-exponent of convergence of fixed points of $f(z)$ is defined to be (see [2])

$$\lambda_2(f-z) = \overline{\lim_{r\to\infty}} \frac{\log_2 N(r, \frac{1}{f-z})}{\log r}.$$

The hyper-exponent of convergence of distinct zeros of $f(z) - \varphi(z)$ and the hyper-exponent of convergence of distinct fixed points of $f(z)$ is respectively defined to be

$$\overline{\lambda}_2(f-\varphi) = \overline{\lim_{r\to\infty}} \frac{\log_2 \overline{N}(r, \frac{1}{f-\varphi})}{\log r}, \quad \overline{\lambda}_2(f-z) = \overline{\lim_{r\to\infty}} \frac{\log_2 \overline{N}(r, \frac{1}{f-z})}{\log r}.$$

We denote the linear measure of a set $E \subset [1, \infty)$ by $mE = \int_E dt$ and the logarithmic measure of E by $m_l E = \int_E \frac{dt}{t}$.

By the definition of the hyper-order, we can estimate the hyper-order and the hyper-exponent of convergence of zeros of the solutions of linear differential equation more precisely. For second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.1}$$

where $A(z), B(z) \not\equiv 0$ are entire functions, it is well known that every nonconstant solution f of (1.1) has infinite order if $\sigma(A) < \sigma(B)$ or $A(z)$ is a polynomial and $B(z)$ is transcendental.

In 1996, K. H. Kwon investigated the hyper-order of the solutions of (1.1) and obtained the following result.

Theorem A.[9] If $A(z)$ and $B(z)$ are entire functions such that $\sigma(A) < \sigma(B)$ or $\sigma(B) < \sigma(A) < \frac{1}{2}$, then every entire function $f \not\equiv 0$ of (1.1) satisfies $\sigma_2(f) \geq \max\{\sigma(A), \sigma(B)\}$.

Up to now, we have known that every nonconstant solution of (1.1) satisfies $\sigma_2(f) = \sigma(B)$ if $\sigma(A) < \sigma(B)$ or $\sigma_2(f) = \sigma(A)$ if $\sigma(B) < \sigma(A) < \frac{1}{2}$ (see [2]).

In 2000, Chen Z. X. firstly investigated the fixed points of solutions of (1.1) and obtained the following results.

Theorem B.[3] If $P(z)$ is a polynomial with degree $n \geq 1$, then every non-trivial solution of

$$f'' + P(z)f = 0 \tag{1.2}$$

has infinitely many fixed points and satisfies $\overline{\lambda}(f-z) = \lambda(f-z) = \sigma(f) = \frac{n+2}{2}$.

Theorem C.[3] If $A(z)$ is a transcendental entire function with $\sigma(A) = \sigma < +\infty$, then every non-trivial solution of

$$f'' + A(z)f = 0 \tag{1.3}$$

has infinitely many fixed points and satisfies $\overline{\lambda}_2(f-z)=\lambda_2(f-z)=\sigma_2(f)=\sigma$.

Up to now, many authors investigated the fixed points of hyper-exponents of convergence of zeros of the solutions of (1.2) and (1.3)(see [11,13]). In 2006, Chen Z. X. investigated the solutions of second order linear differential equation and obtained the following results.

Theorem D.[5] Let $A_j(z)\not\equiv 0 (j=1,2)$ be entire functions with $\sigma(A_j)<1$, suppose that a,b are complex numbers and satisfy $ab\neq 0$ and $\arg a\neq \arg b$ or $a=cb \quad (0<c<1)$. If $\varphi(z)\not\equiv 0$ is an entire function of finite order, then every non-trivial solution f of

$$f''+A_1(z)e^{az}f'+A_2(z)e^{bz}f=0 \tag{1.4}$$

satisfies $\overline{\lambda}(f-\varphi)=\overline{\lambda}(f'-\varphi)=\overline{\lambda}(f''-\varphi)=\infty$.

Theorem E.[5] If $A_1(z)\not\equiv 0, \varphi(z)\not\equiv 0, Q(z)$ are entire functions with $\sigma(A_1)<1$ and $1<\sigma(Q)<\infty$, then every non-trivial solution f of

$$f''+A_1(z)e^{az}f'+Q(z)f=0 \tag{1.5}$$

satisfies $\overline{\lambda}(f-\varphi)=\overline{\lambda}(f'-\varphi)=\overline{\lambda}(f''-\varphi)=\infty$, where $a\neq 0$ is a complex number.

Theorem F.[5] Let $A_1(z)\not\equiv 0, \varphi(z)\not\equiv 0, Q(z)$ be entire functions with $\sigma(A_1)<1$, $\sigma(Q)>1$ and $\sigma(\varphi)<1$, and if $a_j(z)(j=0,1,2)$ are polynomials which are not all equal to zero, then every solution $f\not\equiv 0$ of (1.5) satisfies $\overline{\lambda}(g-\varphi)=\infty$, where $g(z)=a_2f''+a_1f'+a_0f$.

In the same year, Liu M. S. and Zhang X. M. investigated the fixed points when the coefficients of the equations are meromorphic functions and obtained the following results.

Theorem G.[11] Suppose that $k\geq 2$ and $A(z)$ be a transcendental meromorphic function satisfying $\delta(\infty,A)=\underline{\lim}_{r\to\infty}\frac{m(r,A)}{T(r,A)}=\delta>0$, $\sigma(A)=\sigma<+\infty$. Then every meromorphic solution $f\not\equiv 0$ of the equation

$$f^{(k)}+A(z)f=0 \tag{1.6}$$

satisfies that f and $f',f'',\cdots,f^{(k)}$ all have infinitely many fixed points and $\overline{\lambda}(f^{(j)}-z)=\sigma (j=0,1,\cdots,k)$.

Theorem H.[11] Suppose that $P(z)=\frac{P_1(z)}{P_2(z)}\not\equiv 0$ be a rational function with $n=$di(P), where di$(P)=$deg$P_1(z)-$deg$P_2(z)$, and k be an integer with $k\geq 2$. Then:

(1) If $n\neq -k$, then every meromorphic solution $f\not\equiv 0$ of the following equation

$$f^{(k)}+P(z)f=0 \tag{1.7}$$

satisfies that f and $f',f'',\cdots,f^{(k)}$ all have infinitely many fixed points and $\overline{\lambda}(f^{(j)}-z)=\max\left\{\frac{n+k}{k},0\right\} (j=0,1,\cdots,k)$.

(2) If $n=-k$, then every transcendental meromorphic solution f of (1.7) satisfies that f and $f',f'',\cdots,f^{(k-2)}$ all have infinitely many fixed points and $\overline{\lambda}(f^{(k-2)}-z)=0$.

In this paper, we investigate the hyper-exponent of convergence of zeros of $f^{(j)}(z)-\varphi(z)(j \in N)$, where f is a solution of (1.1) and $\varphi(z) \not\equiv 0$ is an entire function satisfying $\sigma(\varphi)<\sigma(f)$ or $\sigma_2(\varphi)<\sigma_2(f)$, and obtain the following results.

Theorem 1.1. Let $A(z)$ and $B(z)$ be entire functions with finite order. If $\sigma(A)<\sigma(B)<\infty$ or $0<\sigma(A)=\sigma(B)<\infty$ and $\tau(A)<\tau(B)$, then for every solution $f \not\equiv 0$ of (1.1) and for any entire function $\varphi(z) \not\equiv 0$ satisfying $\sigma_2(\varphi)<\sigma(B)$, we have
(i) $\overline{\lambda}_2(f-\varphi)=\overline{\lambda}_2(f'-\varphi)=\overline{\lambda}_2(f''-\varphi)=\overline{\lambda}_2(f'''-\varphi)=\sigma_2(f)=\sigma(B)$;
(ii) $\overline{\lambda}_2(f^{(j)}-\varphi)=\sigma_2(f)=\sigma(B)(j>3, j \in N)$.

Theorem 1.2. Let $A(z)$ be a polynomial, $B(z)$ be a transcendental entire function, then for every solution $f \not\equiv 0$ of (1.1) and for any entire function $\varphi(z)$ of finite order, we have
(i) $\overline{\lambda}(f-\varphi)=\lambda(f-\varphi)=\sigma(f)=\infty$;
(ii) $\overline{\lambda}(f^{(j)}-\varphi)=\lambda(f^{(j)}-\varphi)=\sigma(f^{(j)}-\varphi)=\infty(j \geq 1, j \in N)$.

Theorem 1.3. Let $A(z)$ and $B(z)$ be meromorphic functions satisfying $\sigma(A)<\sigma(B)$ and $\delta(\infty, B)>0$. Then for every meromorphic solution $f \not\equiv 0$ of (1.1) and for any meromorphic function $\varphi(z) \not\equiv 0$ satisfying $\sigma_2(\varphi)<\sigma(B)$, we have $\overline{\lambda}_2(f^{(j)}-\varphi)=\lambda_2(f^{(j)}-\varphi) \geq \sigma(B)(j=0,1,2,\cdots)$.

Theorem 1.4. Let $P(z)$ be a polynomial with degree $n \geq 1$ and $k \geq 2$ is an integer. Then for every solution $f \not\equiv 0$ of (1.7) and for any entire function $\varphi(z) \not\equiv 0$ with $\sigma(\varphi)<\frac{n+k}{k}$, we have
(i) $\overline{\lambda}(f-\varphi)=\overline{\lambda}(f'-\varphi)=\overline{\lambda}(f''-\varphi)=\overline{\lambda}(f'''-\varphi)=\sigma(f)=\frac{n+k}{k}$;
(ii) $\overline{\lambda}(f^{(j)}-\varphi)=\sigma(f)=\frac{n+k}{k}(j>3, j \in N)$.

Corollary 1.1. Under the hypotheses of Theorem 1.1, if $\varphi(z)=z$, for every solution $f \not\equiv 0$ of (1.1), we have
(i) $\overline{\lambda}_2(f-z)=\overline{\lambda}_2(f'-z)=\overline{\lambda}_2(f''-z)=\overline{\lambda}_2(f'''-z)=\sigma_2(f)=\sigma(B)$;
(ii) $\overline{\lambda}_2(f^{(j)}-z)=\sigma_2(f)=\sigma(B)(j>3, j \in N)$.

Corollary 1.2. Under the hypotheses of Theorem 1.2, if $\varphi(z)=z$, for every solution $f \not\equiv 0$ of (1.1), we have
(i) $\overline{\lambda}(f-z)=\lambda(f-z)=\sigma(f)=\infty$;
(ii) $\overline{\lambda}(f^{(j)}-z)=\lambda(f^{(j)}-z)=\sigma(f^{(j)}-z)=\infty(j \geq 1, j \in N)$.

Corollary 1.3. Under the hypotheses of Theorem 1.3, if $\varphi(z)=z$, for every meromorphic solution $f \not\equiv 0$ of (1.1), we have $\overline{\lambda}_2(f^{(j)}-z)=\lambda_2(f^{(j)}-z) \geq \sigma(B)(j=0,1,2,\cdots)$.

Corollary 1.4. Let $P(z)=\frac{P_1(z)}{P_2(z)}$ be a rational function with $\mathrm{di}(P(z))=n \geq 1$. Then for every meromorphic solution $f \not\equiv 0$ of (1.7) and for any meromorphic function $\varphi(z) \not\equiv 0$ with $\sigma(\varphi)<\frac{n+k}{k}$, we have $\overline{\lambda}(f^{(j)}-\varphi)=\lambda(f^{(j)}-\varphi)=\sigma(f)=\frac{n+k}{k}(j=0,1,2,\cdots)$.

Corollary 1.5. Under the hypotheses of Theorem 1.1, let $L(f)=a_k f^{(k)}+a_{k-1} f^{(k-1)}+\cdots+a_0 f$, where a_j are entire functions which are not all equal to zero and satisfy $\sigma(a_j)<\sigma(B)(i=0,1,\cdots,k)$, then for every solution $f \not\equiv 0$ of (1.1), we have $\sigma_2(L(f))=\sigma_2(f)=\sigma(B)$.

Remark 1.1. Theorem 1.1 is an improvement of Theorem C. In Theorem D, if $ab \neq 0$ and $a = cb(0 < c < 1)$, it is easy to see that $\sigma(A_1 e^{az}) = \sigma(A_1 e^{bz}) = 1$ and $\tau(A_1 e^{az}) = a < \tau(A_1 e^{bz}) = b$. By Theorem 1.1, for every solution $f \not\equiv 0$ of (1.1) and for any entire function $\varphi(z) \not\equiv 0$ with $\sigma_2(\varphi) < 1$, we have $\overline{\lambda}_2(f - \varphi) = \overline{\lambda}_2(f' - \varphi) = \overline{\lambda}_2(f'' - \varphi) = 1$. Therefore, Theorem 1.1 is also a partial extension of Theorem D. Theorem 1.3 and Theorem 1.4 are the improvements of Theorem G and Theorem H respectively.

Remark 1.2. Theorem 1.1 and Theorem 1.2 also provide us a method to investigate the hyper-exponent convergence of zeros of $f^{(j)} - \varphi (j = 0, 1, 2, \cdots)$. If we can find an equation (1.1) with coefficients $A(z), B(z)$ satisfying the hypotheses of Theorem 1.1 or Theorem 1.2 such that f is an entire function of (1.1), then we have $\overline{\lambda}_2(f^{(j)} - \varphi) = \sigma(B)$ or $\overline{\lambda}(f^{(j)} - \varphi) = \infty (j = 0, 1, 2, \cdots)$. For example, set $f(z) = e^{a(z)}$, $a(z)$ is a transcendental entire function, then $f(z)$ is a solution of $f'' - (a'' + a'^2)f = 0$, then by Theorem 1.1 and by Lemma 2.8 (ii), for any entire function $\varphi(z) \not\equiv 0$ with $\sigma_2(\varphi) < \sigma(a)$ if $\sigma(a) > 0$ or $\sigma(\varphi) < \infty$ if $\sigma(a) = 0$, we have $\overline{\lambda}_2(f^{(j)} - \varphi) = \sigma(a)(j = 0, 1, 2, \cdots)$.

2. Lemmas for the proofs of theorems

Lemma 2.1.[6] Let $G(r) : (0, +\infty) \to R$, $H(r) : (0, +\infty) \to R$ be monotone increasing functions such that $G(r) \leq H(r)$ outside of an exceptional set E_0 of finite linear measure, then for any given $\alpha > 1$, there exists a $r > r_0$ such that $G(r) \leq H(\alpha r)$ for all $r > r_0$.

Lemma 2.2.[2] Let $f(z)$ be an entire function with $\sigma_2(f) = \sigma$, and $\nu_f(r)$ denote the central index of $f(z)$. Then

$$\overline{\lim_{r\to\infty}} \frac{\log_2 \nu_f(r)}{\log r} = \sigma.$$

Lemma 2.3. Let $f(z)$ be a transcendental entire function with $\sigma(f) = \sigma \geq 0$, then there exists a set $E_1 \subset [1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_1$, we have

$$\lim_{r\to\infty} \frac{\log T(r, f)}{\log r} = \sigma, \quad r \in E_1.$$

Proof. By $\sigma(f) = \sigma$, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $(1 + \frac{1}{n})r_n < r_{n+1}$ and

$$\lim_{r\to\infty} \frac{\log T(r_n, f)}{\log r_n} = \sigma(f),$$

there exists a n_1 such that for all $n \geq n_1$ and for any $r \in [r_n, (1 + \frac{1}{n})r_n]$, we have

$$\frac{\log T(r_n, f)}{\log(1 + \frac{1}{n})r_n} \leq \frac{\log T(r, f)}{\log r} \leq \frac{\log T((1 + \frac{1}{n})r_n, f)}{\log r_n}.$$

Set $E_1 = \bigcup_{n=n_1}^{\infty} [r_n, (1+\frac{1}{n})r_n]$, we have

$$\lim_{r\to\infty} \frac{\log T(r,f)}{\log r} = \lim_{r\to\infty} \frac{\log T(r_n,f)}{\log r_n} = \sigma(f), \quad r \in E_1$$

and $m_l E_1 = \sum_{n=n_1}^{\infty} \int_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \sum_{n=n_1}^{\infty} \log(1+\frac{1}{n}) = \infty$, thus we complete the proof of this lemma.

By the same reasoning in Lemma 2.3, we have the following result.

Lemma 2.4. Let $f(z)$ be a transcendental entire function with $\sigma_2(f) = \sigma \geq 0$, then there exists a set $E_2 \subset [1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_2$, we have

$$\lim_{r\to\infty} \frac{\log_2 T(r,f)}{\log r} = \sigma, \quad r \in E_2.$$

Lemma 2.5. Let $A_0, A_1, \cdots A_{k-1}, F \not\equiv 0$ be meromorphic functions, if f is a meromorphic solution of the equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = F, \tag{2.1}$$

then we have the following statements:

(i) if $\max\{\sigma(F), \sigma(A_j); j = 0, 1, \cdots, k-1\} < \sigma(f) = \sigma \leq \infty$, then $\sigma(f) = \lambda(f) = \overline{\lambda}(f)$;

(ii) if $\max\{\sigma_2(F), \sigma_2(A_j); j = 0, 1, \cdots, k-1\} < \sigma_2(f) = \sigma$, then $\sigma_2(f) = \lambda_2(f) = \overline{\lambda}_2(f)$.

Proof. Since the proof of (i) and (ii) is the same, then we only prove (ii) here. By (2.1), we have

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_0\right). \tag{2.2}$$

By (2.2), we get

$$N(r, \frac{1}{f}) \leq k\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{F}) + \sum_{j=0}^{k-1} N(r, A_j). \tag{2.3}$$

By the theorem on logarithmic derivative and (2.2), we have that

$$m(r, \frac{1}{f}) \leq m(r, \frac{1}{F}) + \sum_{j=0}^{k-1} m(r, A_j) + O\{\log(rT(r,f))\}, \quad r \notin E_3 \tag{2.4}$$

holds for $|z| = r$ outside a set $E_3 \subset (0, \infty)$ of finite linear measure. By (2.3) and (2.4), we have

$$T(r,f) = T(r, \frac{1}{f}) + o(1) \leq k\overline{N}(r, \frac{1}{f}) + T(r, \frac{1}{F}) + \sum_{j=0}^{k-1} T(r, A_j) + O\{\log(rT(r,f))\}$$

$$= k\overline{N}(r, \frac{1}{f}) + T(r, F) + \sum_{j=0}^{k-1} T(r, A_j) + O\{\log(rT(r,f))\}, \quad r \notin E_3. \tag{2.5}$$

By (2.5), we have $\sigma_2(f) \leq \max\{\overline{\lambda}_2(f), \sigma_2(A_j), \sigma_2(F)\}$. Since $\max\{\sigma_2(F), \sigma_2(A_j); j = 0, 1, \cdots, k - 1\} < \sigma_2(f)$, we get $\sigma_2(f) \leq \overline{\lambda}_2(f)$. Therefore $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f)$.

Lemma 2.6.[3] Let $f(z)$ be an entire function of order $\sigma(f) = \alpha < +\infty$. Then for any given $\varepsilon > 0$, there is a set $E_4 \subset [1, \infty)$ that has finite linear measure and finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_4$, we have

$$\exp\{-r^{\alpha+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\alpha+\varepsilon}\}.$$

Lemma 2.7. Let $f(z)$ be an entire function with $\sigma_2(f) = \alpha > 0$, and let $L(f) = a_2 f'' + a_1 f' + a_0 f$, where a_0, a_1, a_2 are entire functions which are not all equal to zero and satisfy $\max\{\sigma(a_j), j = 0, 1, 2\} = b < \alpha$, then $\sigma_2(L(f)) = \sigma_2(f) = \alpha$.

Proof. $L(f)$ can be written as

$$L(f) = f(a_2 \frac{f''}{f} + a_1 \frac{f'}{f} + a_0). \tag{2.6}$$

By Wiman-Valiron Lemma (see [7,10]), for all z satisfying $|z| = r$ and $|f(z)| = M(r, f)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)), \quad j \in N, \quad r \notin E_5, \tag{2.7}$$

where E_5 is a set of finite logarithmic measure. From the (1.4.5) in [8,pp.26] , for any given $\varepsilon > 0$, we have that

$$\nu_f(r) < [\log \mu_f(r)]^{1+\varepsilon} \tag{2.8}$$

holds outside a set E_6 with finite logarithmic measure, where $\mu_f(r)$ is the maximum term of f. By Cauchy 's inequality, we have $\mu_f(r) \leq M(r, f)$. Substituting it into (2.8), we have

$$\nu_f(r) < [\log M(r, f)]^{1+\varepsilon}, \quad r \notin E_6. \tag{2.9}$$

By Lemma 2.4, there exists a set E_2 having infinite logarithmic measure such that for all $|z| = r \in E_2$ and for all sufficiently large r, we have

$$\sigma_2(f) = \lim_{r\to\infty} \frac{\log_2 \nu_f(r)}{\log r} = \alpha, \quad r \in E_2. \tag{2.10}$$

By (2.10) and Lemma 2.6, for any given $0 < \varepsilon < \alpha - b$, we have

$$\exp\{-r^{b+\varepsilon}\} < |a_j(z)| < \exp\{r^{b+\varepsilon}\} < \exp\{r^{\alpha-\varepsilon}\} < \nu_f(r) < \exp\{r^{\alpha+\varepsilon}\},$$

$$j = 0, 1, 2, \cdots \quad r \in E_2 - \bigcup_{i=4}^{6} E_i. \tag{2.11}$$

Substituting (2.11) into (2.9), we have

$$\exp_2\{r^{\alpha-2\varepsilon}\} < M(r,f), \quad r \in E_2 - \bigcup_{i=4}^{6} E_i, \tag{2.12}$$

where $\exp_2\{r\} = \exp\{\exp\{r\}\}$. By (2.6), we have

$$|L(f)| = |f| \left| a_2 \frac{f''}{f} + a_1 \frac{f'}{f} + a_0 \right| \geq |f| \left[\left| a_2 \frac{f''}{f} + a_1 \frac{f'}{f} \right| - |a_0| \right]. \tag{2.13}$$

Substituting (2.7), (2.11), (2.12) into (2.13), for all z satisfying $|f(z)| = M(r,f)$ and $|z| = r \in E_2 - \bigcup_{i=4}^{6} E_i$, we have

$$\begin{aligned} |L(f)| &\geq |f| \left[\left| \frac{\nu_f(r)}{z} \left(a_2 \frac{\nu_f(r)}{z} + a_1 \right) \right| - |a_0| \right| \\ &\geq |f| \left[\left| \frac{\nu_f(r)}{z} \right| \left| \left| a_2 \frac{\nu_f(r)}{z} \right| - |a_1| \right| - |a_0| \right] \\ &\geq \exp_2\{r^{\alpha-2\varepsilon}\} \left[\exp\{r^{\alpha-\varepsilon}\} - \exp\{r^{b+\varepsilon}\} \right]. \end{aligned} \tag{2.14}$$

By (2.14), we have $\sigma_2(L(f)) \geq \sigma_2(f)$.

On the other hand, it is easy to get $\sigma_2(L(f)) \leq \sigma_2(f)$. Hence $\sigma_2(L(f)) = \sigma_2(f)$.

By the similar proof in Lemma 2.7, we can easily get the following result.

Lemma 2.8. (i) Let $f(z)$ be an entire function with $\sigma_2(f) = \alpha > 0$, and let $L(f) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_0 f$, where $a_0, a_1, \cdots, a_k$ are entire functions which are not all equal zero and satisfy $b = \max\{a_j(z), j = 0, 1, \cdots, k\} < \alpha$, then $\sigma_2(L(f)) = \sigma_2(f) = \alpha$.

(ii) Let $f(z)$ be an entire function with $\sigma(f) = \alpha \leq \infty$, and let $L(f) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_0 f^2$, where $a_0, a_1, \cdots, a_k$ are entire functions which are not all equal zero and satisfy $b = \max\{a_j(z), j = 0, 1, \cdots, k\} < \alpha$, then $\sigma(L(f)) = \sigma(f) = \alpha$.

Remark 2.1. The assumption $\sigma(a_j) < \sigma_2(f)(j = 0, 1, 2)$ in Lemma 2.7 is necessary. For example, $f(z) = e^{e^z}$ satisfies $\sigma_2(f) = 1$ and $f'' - f' - e^{2z} f = 0$, where $a_2 = 1, a_1 = 1, a_0 = -e^{2z}$, and a_0 satisfies $\sigma(a_0) = \sigma_2(f) = 1$, however, we have $\sigma_2(L(f)) = 0 < 1$.

Lemma 2.9.[11] Let $f(z)$ be an entire function with $\sigma(f) = \sigma, \tau(f) = \tau, 0 < \sigma < \infty, 0 < \tau < \infty$, then for any given $\beta < \tau$, there exists a set $E_7 \subset [1, +\infty)$ that has infinite logarithmic measure such that for all $r \in E_7$, we have

$$\log M(r,f) > \beta r^\sigma.$$

Remark 2.2. Lemma 2.9 also holds if $\tau(f) = \infty$.

Lemma 2.10.[12] Let $A(z), B(z)$ be entire functions satisfying $0 < \sigma(A) = \sigma(B) < \infty, \tau(A) < \tau(B)$, then every solution $f \not\equiv 0$ of (1.1) satisfies $\sigma_2(f) = \sigma(B)$.

Lemma 2.11.[6] Let $f(z)$ be a transcendental meromorphic function and $\alpha > 1$ be a given constant, for any given $\varepsilon > 0$, there exists a set $E_8 \subset [1, \infty)$ that has finite logarithmic measure and a constant $B > 0$ that depends only on α and $(m, n)(m, n \in \{0, \cdots, k\}$ with $m < n)$ such that for all z satisfying $|z| = r \notin [0, 1] \cup E_8$, we have

$$\left| \frac{f^{(n)}(z)}{f^{(m)}(z)} \right| \leq B \left(\frac{T(\alpha r, f)}{r} (\log^\alpha r) \log T(\alpha r, f) \right)^{n-m}.$$

Lemma 2.12.[6] Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$, $\Gamma = (k_1, j_1), \cdots, (k_m, j_m)\}$ be a finite set of distinct pairs of integers which satisfy $k_i > j_i \geq 0$ for $i = 1, \cdots, m$. And let $\varepsilon > 0$ be a given constant, then there exists a set $E_9 \subset (1, \infty)$ that has finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_9$ and $(k, j) \in \Gamma$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma - 1 + \varepsilon)}.$$

Lemma 2.13. Let $U(z), V(z)$ be meromorphic functions of finite order. If $\overline{\lim\limits_{r\to\infty}} \frac{\log m(r,U)}{\log r} = \beta_1$, and there exists a set E_{10} with infinite logarithmic measure such that $\lim\limits_{r\to\infty} \frac{\log m(r,V)}{\log r} = \beta_2 > \beta_1$ holds for all $r \in E_{10}$, then every meromorphic solution of

$$f'' + Uf' + Vf = 0 \tag{2.15}$$

satisfies $\sigma_2(f) \geq \beta_2$.

Proof. Assume that $f(z)$ is a meromorphic solution of (2.15), by (2.15), we have

$$m(r, V) \leq m(r, \frac{f''}{f}) + m(r, \frac{f'}{f}) + m(r, U). \tag{2.16}$$

By the theorem on logarithmic derivative and (2.16), we have

$$m(r, V) \leq O\{\log rT(r, f)\} + m(r, U), \quad r \notin E_3, \tag{2.17}$$

where $E_3 \subset [1, +\infty)$ is a set having finite logarithmic measure. By the hypotheses of Lemma 2.13, there exists a set E_{10} having infinite logarithmic measure such that for all $|z| = r \in E_{10} - E_3$, we have

$$r^{\beta_2 - \varepsilon} \leq O\{\log rT(r, f)\} + 4r^{\beta_1 + \varepsilon}, \tag{2.18}$$

where $0 < 2\varepsilon < \beta_2 - \beta_1$. By (2.18), we have $\sigma_2(f) \geq \beta_2$.

Lemma 2.14. Let $U(z), V(z)$ be meromorphic functions of finite order. If there exist positive constants $\sigma, \beta_3, \beta_4 (0 < \beta_3 < \beta_4)$ and a set E_{11} with infinite logarithmic measure such that

$$|U(z)| \le \exp\{\beta_3 r^\sigma\}, \quad |V(z)| \ge \exp\{\beta_4 r^\sigma\}$$

hold for all $|z| = r \in E_{11}$, then every meromorphic solution of (2.15) satisfies $\sigma_2(f) \ge \sigma$.

Proof. Assume that $f(z)$ is a meromorphic solution of (2.15), by (2.15), we have

$$|V(z)| \le \left|\frac{f''}{f}\right| + |U(z)| \left|\frac{f'}{f}\right|. \tag{2.19}$$

By Lemma 2.11, there exists a set E_8 having finite logarithmic measure such that for all $|z| = r \notin E_8$, we have

$$\left|\frac{f''}{f}\right| \le B[T(2r,f)]^2, \quad \left|\frac{f'}{f}\right| \le B[T(2r,f)]. \tag{2.20}$$

where $B > 0$ is a constant. By (2.19)-(2.20) and the hypotheses in Lemma 2.14, for all $|z| = r \in E_{11} - E_8$, we have

$$\exp\{\beta_4 r^\sigma\} \le 2B[T(2r,f)]^2 \exp\{\beta_3 r^\sigma\}. \tag{2.21}$$

Since $0 < \beta_3 < \beta_4$, by (2.21), we have $\sigma_2(f) \ge \sigma$.

Lemma 2.15. Let $A(z), B(z)$ be meromorphic functions with $\sigma(A) < \sigma(B)$ and $\delta(\infty, B) = \varliminf_{r\to\infty} \frac{m(r,B)}{T(r,B)} > 0$. Then every meromorphic solution f of (1.1) satisfies $\sigma_2(f) \ge \sigma(B)$.

Proof. Let f be a meromorphic solution of (1.1), by (1.1), we have

$$\begin{aligned} m(r,B) &\le m(r,\frac{f''}{f}) + m(r,\frac{f'}{f}) + m(r,A) \\ &\le O\{\log rT(r,f)\} + T(r,A), \quad r \notin E_3, \end{aligned} \tag{2.22}$$

where $E_3 \subset [1, +\infty)$ is a set having finite linear measure. By Lemma 2.3, there exists a set E_1 having infinite logarithmic measure such that for all $|z| = r \in E_1$, we have

$$\lim_{r\to\infty} \frac{\log T(r,B)}{\log r} = \sigma(B), \quad r \in E_1. \tag{2.23}$$

Since $\delta(\infty, B) > 0$, then for any given $\varepsilon (0 < 2\varepsilon < \sigma(B) - \sigma(A))$ and for all $r \in E_1$, by (2.23), we have

$$m(r,B) \ge r^{\sigma(B)-\varepsilon}. \tag{2.24}$$

From (2.22) and (2.24), we have

$$r^{\sigma(B)-\varepsilon} \le O\{\log rT(r,f)\} + r^{\sigma(A)+\varepsilon}, \quad r \in E_1 - E_3, \tag{2.25}$$

by (2.25), we have $\sigma_2(f) \ge \sigma(B)$.

Remark 2.3. We have to note that we can only obtain $\sigma_2(f) \geq \sigma(B)$ if $A(z), B(z)$ in (1.1) are transcendental meromorphic functions and satisfy the hypotheses of Lemma 2.15. Since if f is a meromorphic solution of (1.1), we can only obtain $\overline{\lambda}(\frac{1}{f}) \leq \max\{\overline{\lambda}(\frac{1}{A}), \lambda(\frac{1}{B})\}$ instead of $\lambda(\frac{1}{f}) \leq \max\{\lambda(\frac{1}{A}), \lambda(\frac{1}{B})\}$, we can not use the Wiman-Valiron Lemma on meromorphic function to obtain $\sigma_2(f) \leq \sigma(B)$. Thus, the conclusion of Lemma 2.2 in [11, pp, 194] that every meromorphic solution $f \not\equiv 0$ of (1.6) satisfies $\sigma_2(f) = \sigma(A)$ remains open.

Lemma 2.16.[1] Let $P(z)$ be a rational function with di(P)=$n \geq 1$, then every meromorphic solution $f \not\equiv 0$ of (1.7) satisfies $\sigma(f) = \frac{n+k}{k}$.

Remark 2.4. Especially if $P(z)$ is a polynomial, then every solution $f \not\equiv 0$ of (1.7) satisfies $\sigma(f) = \frac{n+k}{k}$.

By the similar proof in Lemma 2.16, we can easily obtain the following result.

Lemma 2.17. Let $A_j(z)(j = 0, 1, \cdots, k-1)$ be rational functions satisfying $\mathrm{di}(A_0) = n_0 \geq 1$ and $\mathrm{di}(A_j) = n_j \leq 0 (j = 1, 2, \cdots, k-1)$, then every meromorphic solution f of

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0 \tag{2.26}$$

satisfies $\sigma(f) = \frac{n_0+k}{k}$.

3. Proof of Theorem 1.1

Now we divide the proof of Theorem 1.1 into two cases. Case (i): $\sigma(A) < \sigma(B) < \infty$; Case (ii): $0 < \sigma(A) = \sigma(B) < \infty$, and $\tau(A) < \tau(B)$.

Case (i): (1)Now we prove that $\overline{\lambda}_2(f - \varphi) = \sigma_2(f)$. Assume that $f \not\equiv 0$ is a solution of (1.1), then $\sigma_2(f) = \sigma(B)$ (see [2]). Let $g = f - \varphi$, since $\sigma_2(\varphi) < \sigma(B)$, then $\sigma_2(g) = \sigma_2(f) = \sigma(B)$, $\overline{\lambda}_2(g) = \overline{\lambda}_2(f - \varphi)$. Substituting $f = g + \varphi, f' = g' + \varphi', f'' = g'' + \varphi''$ into (1.1), we have

$$g'' + Ag' + Bg = -(\varphi'' + A\varphi' + B\varphi). \tag{3.1}$$

If $\varphi'' + A\varphi' + B\varphi \equiv 0$, we have $\sigma_2(\varphi) = \sigma(B)$ (see [2]), this is a contradiction. By Lemma 2.5 (ii) and $\varphi'' + A\varphi' + B\varphi \not\equiv 0$, we have $\overline{\lambda}_2(g) = \lambda_2(g) = \sigma_2(g) = \sigma(B)$, therefore $\overline{\lambda}_2(f - \varphi) = \lambda_2(f - \varphi) = \sigma_2(f) = \sigma(B)$.

(2) Now we prove that $\overline{\lambda}_2(f' - \varphi) = \sigma_2(f)$. Let $g_1 = f' - \varphi$, then $\sigma_2(g_1) = \sigma_2(f) = \sigma(B)$ and

$$f' = g_1 + \varphi, f'' = g_1' + \varphi', f''' = g_1'' + \varphi''. \tag{3.2}$$

By (1.1), we get

$$f = -\frac{1}{B}(f'' + Af'). \tag{3.3}$$

The derivation of (1.1) is

$$f''' + Af'' + (A' + B)f' + B'f = 0. \tag{3.4}$$

Substituting (3.2),(3.3) into (3.4), we obtain

$$g_1'' + \left(A - \frac{B'}{B}\right) g_1' + \left(A' + B - \frac{AB'}{B}\right) g_1 = -\left(\varphi'' + \left(A - \frac{B'}{B}\right) \varphi' + \left(A' + B - \frac{AB'}{B}\right) \varphi\right). \tag{3.5}$$

Set $V_1 = A' + B - \frac{AB'}{B}, U_1 = A - \frac{B'}{B}$, then U_1, V_1 are meromorphic functions of finite order. By the theorem on the logarithmic derivative and Lemma 2.3, it is easy to see that there exists a set E_1 having infinite logarithmic measure such that $\lim\limits_{r\to\infty} \frac{\log m(r,V_1)}{\log r} = \sigma(B) > \overline{\lim\limits_{r\to\infty}} \frac{\log m(r,U_1)}{\log r} = \sigma(A)$ holds for all $r \in E_1$. Let $F_1 = \varphi'' + U_1\varphi' + V_1\varphi$, we affirm that $F_1 \not\equiv 0$. If $F_1 \equiv 0$, by Lemma 2.13, we have $\sigma_2(\varphi) \geq \sigma(B)$, this is a contradiction with $\sigma_2(\varphi) < \sigma(B)$, therefore $F_1 \not\equiv 0$. By Lemma 2.5 (ii), we get $\overline{\lambda}_2(f' - \varphi) = \lambda_2(f' - \varphi) = \sigma_2(f)$.

(3) Now we prove that $\overline{\lambda}_2(f'' - \varphi) = \sigma_2(f)$. Let $g_2 = f'' - \varphi$, then $\sigma_2(g_2) = \sigma_2(f) = \sigma(B)$ and

$$f'' = g_2 + \varphi, f''' = g_2' + \varphi', f^{(4)} = g_2'' + \varphi''. \tag{3.6}$$

Substituting (3.3) and $V_1 = A' + B - \frac{AB'}{B}, U_1 = A - \frac{B'}{B}$ into (3.4), we have

$$f''' + U_1 f'' + V_1 f' = 0. \tag{3.7}$$

The derivation of (3.7) is

$$f^{(4)} + \left(U_1 - \frac{V_1'}{V_1}\right) f''' + \left(U_1' + V_1 - \frac{V_1'U_1}{V_1}\right) f'' = 0. \tag{3.8}$$

Set $U_2 = U_1 - \frac{V_1'}{V_1}, V_2 = U_1' + V_1 - \frac{V_1'U_1}{V_1}$, then U_2, V_2 are meromorphic functions of finite order. It is easy to see that there exists a set E_1 having infinite logarithmic measure such that $\lim\limits_{r\to\infty} \frac{\log m(r,V_2)}{\log r} = \sigma(B) > \overline{\lim\limits_{r\to\infty}} \frac{\log m(r,U_2)}{\log r} = \sigma(A)$ holds for all $r \in E_1$. Then by (3.8), we get

$$f^{(4)} + U_2 f''' + V_2 f'' = 0. \tag{3.9}$$

Substituting (3.6) into (3.9), we have

$$g_2'' + U_2 g_2' + V_2 g_2 = -\left(\varphi'' + U_2\varphi' + V_2\varphi\right). \tag{3.10}$$

Let $F_2 = \varphi'' + U_2\varphi' + V_2\varphi$, if $F_2 \equiv 0$, by Lemma 2.13, we have $\sigma_2(\varphi) \geq \sigma(B)$, this is a contradiction with $\sigma_2(\varphi) < \sigma(B)$, therefore $F_2 \not\equiv 0$. By Lemma 2.5 (ii), we get $\overline{\lambda}_2(f'' - \varphi) = \lambda_2(f'' - \varphi) = \sigma_2(f)$.

(4) Now we prove that $\overline{\lambda}_2(f''' - \varphi) = \sigma_2(f)$. Let $g_3 = f''' - \varphi$, then $\sigma_2(g_3) = \sigma_2(f) = \sigma(B)$ and

$$g_3' = f^{(4)} - \varphi', g_3'' = f^{(5)} - \varphi''. \tag{3.11}$$

The derivation of (3.9) is

$$f^{(5)} + U_2 f^{(4)} + (U_2' + V_2) f''' + V_2' f'' = 0. \tag{3.12}$$

By (3.9), we have

$$f'' = -\frac{1}{V_2}\left(f^{(4)} + U_2 f'''\right). \tag{3.13}$$

Substituting (3.13) into (3.12), we have

$$f^{(5)} + \left(U_2 - \frac{V_2'}{V_2}\right) f^{(4)} + \left(U_2' + V_2 - \frac{V_2'U_2}{V_2}\right) f''' = 0. \tag{3.14}$$

Set $U_3 = U_2 - \frac{V_2'}{V_2}, V_3 = U_2' + V_2 - \frac{V_2'U_2}{V_2}$, then U_3, V_3 are meromorphic functions of finite order, and it is easy to see that there exists a set E_1 having infinite logarithmic measure such that $\lim_{r\to\infty} \frac{\log m(r,V_3)}{\log r} = \sigma(B) > \overline{\lim_{r\to\infty}} \frac{\log m(r,U_3)}{\log r} = \sigma(A)$ holds for all $r \in E_1$. By (3.14), we have

$$f^{(5)} + U_3 f^{(4)} + V_3 f''' = 0. \tag{3.15}$$

Substituting (3.11) into (3.15), we have

$$g_3'' + U_3 g_3' + V_3 g_3 = -\left(\varphi'' + U_3\varphi' + V_3\varphi\right). \tag{3.16}$$

Let $F_3 = \varphi'' + U_3\varphi' + V_3\varphi$. By Lemma 2.13, we have $F_3(z) \not\equiv 0$. And by Lemma 2.5 (ii), we get $\overline{\lambda}_2(f''' - \varphi) = \lambda_2(f''' - \varphi) = \sigma_2(f)$.

(5) Now we prove that $\overline{\lambda}_2(f^{(j)} - \varphi) = \sigma_2(f)(j > 3, j \in N)$. Let $f^{(j)} = g_j + \varphi$, $f^{(j+1)} = g_j' + \varphi', f^{(j+2)} = g_j'' + \varphi''(j > 3, j \in N)$, then $\sigma_2(g_j) = \sigma_2(f^{(j)}) = \sigma(B)(j > 3, j \in N)$. By successive derivation on (3.15) and set $U_j = U_{j-1} - \frac{V_{j-1}'}{V_{j-1}}, V_j = U_{j-1}' + V_{j-1} - \frac{V_{j-1}'U_{j-1}}{V_{j-1}}$, we have

$$g_j'' + U_j g_j' + V_j g_j = -\left(\varphi'' + U_j\varphi' + V_j\varphi\right), \tag{3.17}$$

where U_j, V_j are meromorphic functions of finite order, and it is easy to see that there exists a set E_1 having infinite logarithmic measure such that $\lim_{r\to\infty} \frac{\log m(r,V_j)}{\log r} = \sigma(B) > \overline{\lim_{r\to\infty}} \frac{\log m(r,U_j)}{\log r} = \sigma(A)$ holds for all $r \in E_1 (j > 3, j \in N)$. Let $F_j(z) = \varphi'' + U_j\varphi' + V_j\varphi$. By Lemma 2.13, we have $F_j \not\equiv 0$. Then by Lemma 2.5 (ii), we get $\overline{\lambda}_2(f^{(j)} - \varphi) = \lambda_2(f^{(j)} - \varphi) = \sigma_2(f) = \sigma(B)(j > 3, j \in N)$.

Case (ii): (1)Now we prove that $\overline{\lambda}_2(f - \varphi) = \sigma_2(f)$. Assume that $f \not\equiv 0$ is a solution of (1.1), by Lemma 2.10, we know that $\sigma_2(f) = \sigma(B) > 0$. Let $g = f - \varphi$, $\varphi \not\equiv 0$ is an entire function with $\sigma_2(\varphi) < \sigma(B)$, then we have $\sigma_2(g) = \sigma_2(f) = \sigma(B)$, $\overline{\lambda}_2(g) = \overline{\lambda}_2(f - \varphi)$. Substituting $f = g + \varphi, f' = g' + \varphi', f'' = g'' + \varphi''$ into (1.1), we have

$$g'' + Ag' + Bg = -(\varphi'' + A\varphi' + B\varphi). \tag{3.18}$$

We affirm that $\varphi'' + A\varphi' + B\varphi \not\equiv 0$. If $\varphi'' + A\varphi' + B\varphi \equiv 0$, by Lemma 2.10, we have $\sigma_2(\varphi) = \sigma(B)$, this is a contradiction with $\sigma_2(\varphi) < \sigma(B)$. By Lemma 2.5 (ii), we have $\overline{\lambda}_2(g) = \lambda_2(g) = \sigma_2(g) = \sigma(B)$, therefore $\overline{\lambda}_2(f - \varphi) = \lambda_2(f - \varphi) = \sigma_2(f) = \sigma(B)$.

(2) Now we prove that $\overline{\lambda}_2(f' - \varphi) = \sigma_2(f)$. Let $g_1 = f' - \varphi$, then $\sigma_2(g_1) = \sigma_2(f) = \sigma(B)$ and

$$f' = g_1 + \varphi, f'' = g_1' + \varphi', f''' = g_1'' + \varphi''. \tag{3.19}$$

From (3.3)-(3.5) in case (i), we set $F_1 = \varphi'' + U_1\varphi' + V_1\varphi$, where $U_1 = A - \frac{B'}{B}, V_1 = A' + B - \frac{AB'}{B}$ are moromorphic functions of finite order. By Lemma 2.9 and Lemma 2.12, it is easy to obtain that for all $|z| = r \in E_7 - E_9$ and for any given $\varepsilon(0 < 2\varepsilon < \tau(B) - \tau(A))$, we have

$$|U_1(z)| \le \exp\{(\tau(A) + \varepsilon)r^{\sigma(b)}\}, \quad |V_1(z)| \ge \exp\{(\tau(B) - \varepsilon)r^{\sigma(B)}\}. \tag{3.20}$$

where E_7 is a set having infinite logarithmic measure, E_9 is a set having finite logarithmic measure. If $F_1 \equiv 0$, by Lemma 2.14, we have $\sigma_2(\varphi) \ge \sigma(B)$, this is a contradiction with $\sigma_2(\varphi) < \sigma(B)$.

Therefore $F_1 \not\equiv 0$. By (3.5) and Lemma 2.5 (ii) , we have $\overline{\lambda}_2(f'-\varphi) = \lambda_2(f'-\varphi) = \sigma_2(f) = \sigma(B)$. The following cases $\overline{\lambda}_2(f^{(j)} - \varphi) = \lambda_2(f^{(j)} - \varphi) = \sigma_2(f) = \sigma(B)(j \geq 2, j \in N)$ can be obtained by the above similar proof.

4. Proofs of Theorems 1.2-1.3

Using the similar proof in Theorem 1.1 and Lemma 2.5 (i), we can obtain Theorem 1.2. Using the similar proof in Theorem 1.1 and Lemma 2.15, we can easily obtain Theorem 1.3.

5. Proof of Theorem 1.4

(1) Now we prove that $\overline{\lambda}(f - \varphi) = \lambda(f - \varphi) = \sigma(f) = \frac{n+k}{k}$. Assume that $f \not\equiv 0$ is a solution of (1.7), by Lemma 2.16, we have $\sigma(f) = \frac{n+k}{k}$. Let $g = f - \varphi$, since $\sigma(\varphi) < \frac{n+k}{k}$, then $\sigma(g) = \sigma(f) = \frac{n+k}{k}$, $\overline{\lambda}(g) = \overline{\lambda}(f - \varphi)$. Substituting $f = g + \varphi, f^{(k)} = g^{(k)} + \varphi^{(k)}$ into (1.7), we have

$$g^{(k)} + Pg = -(\varphi^{(k)} + P\varphi). \tag{5.1}$$

If $\varphi^{(k)} + P\varphi \equiv 0$, by Lemma 2.16, we have $\sigma(\varphi) = \frac{n+k}{k}$, this is a contradiction with $\sigma(\varphi) < \frac{n+k}{k}$. By Lemma 2.5 (i) and $\varphi^{(k)} + P\varphi \not\equiv 0$, we have $\overline{\lambda}(g) = \lambda(g) = \sigma(g) = \frac{n+k}{k}$, therefore $\overline{\lambda}(f - \varphi) = \lambda(f - \varphi) = \sigma(f) = \frac{n+k}{k}$.

(2) Now we prove that $\overline{\lambda}(f' - \varphi) = \lambda(f' - \varphi) = \sigma(f) = \frac{n+k}{k}$. Let $g_1 = f' - \varphi$, by Lemma 2.16 and $\sigma(\varphi) < \frac{n+k}{k}$, we have $\sigma(g_1) = \sigma(f') = \sigma(f) = \frac{n+k}{k}$ and

$$f' = g_1 + \varphi, f^{(k+1)} = g_1^{(k)} + \varphi^{(k)}, f^{(k)} = g_1^{(k-1)} + \varphi^{(k-1)}. \tag{5.2}$$

By (1.7), we get

$$f = -\frac{f^{(k)}}{P}. \tag{5.3}$$

The derivation of (1.7) is

$$f^{(k+1)} + P'f + Pf' = 0. \tag{5.4}$$

Substituting (5.2), (5.3) into (5.4), we obtain

$$g_1^{(k)} - \frac{P'}{P} g_1^{(k-1)} + Pg_1 = -\left(\varphi^{(k)} - \frac{P'}{P}\varphi^{(k-1)} + P\varphi\right). \tag{5.5}$$

Let $F_1 = -\left(\varphi^{(k)} - \frac{P'}{P}\varphi^{(k-1)} + P\varphi\right)$. We affirm that $F_1 \not\equiv 0$. If $F_1 \equiv 0$, by Lemma 2.17, we have $\sigma(\varphi) = \frac{n+k}{k}$, this is a contradiction, therefore $F_1 \not\equiv 0$. By Lemma 2.5 (i), we get $\overline{\lambda}(f' - \varphi) = \lambda(f' - \varphi) = \sigma(f) = \frac{n+k}{k}$.

(3) Now we prove that $\overline{\lambda}(f'' - \varphi) = \lambda(f'' - \varphi) = \sigma(f) = \frac{n+k}{k}$. Let $g_2 = f'' - \varphi$, by Lemma 2.16 and $\sigma(\varphi) < \frac{n+k}{k}$, we have $\sigma(g_2) = \sigma(f) = \frac{n+k}{k}$ and

$$f'' = g_2 + \varphi, f^{(k+2)} = g^{(k)} + \varphi^{(k)}, f^{(k+1)} = g^{(k-1)} + \varphi^{(k-1)}. \tag{5.6}$$

Substituting (5.3) into (5.4), we have

$$f^{(k+1)} - \frac{P'}{P} f^{(k)} + P f' = 0. \tag{5.7}$$

The derivation of (5.7) is

$$f^{(k+2)} - \frac{2P'}{P} f^{(k+1)} + \left[-\frac{P''}{P} + 2\left(\frac{P'}{P}\right)^2 \right] f^{(k)} + P f'' = 0. \tag{5.8}$$

Substituting (5.6) into (5.8), we have

$$g_2^{(k)} - \frac{2P'}{P} g_2^{(k-1)} + \left[-\frac{P''}{P} + 2\left(\frac{P'}{P}\right)^2 \right] g_2^{(k-2)} + P g_2$$

$$= -\left\{ \varphi^{(k)} - \frac{2P'}{P} \varphi^{(k-1)} + \left[-\frac{P''}{P} + 2\left(\frac{P'}{P}\right)^2 \right] \varphi^{(k-2)} + P\varphi \right\}. \tag{5.9}$$

If $F_2(z) = \varphi^{(k)} - \frac{2P'}{P}\varphi^{(k-1)} + \left[-\frac{P''}{P} + 2\left(\frac{P'}{P}\right)^2 \right] \varphi^{(k-2)} + P\varphi \equiv 0$, by Lemma 2.17, we have $\sigma(\varphi) = \frac{n+k}{k}$, this is a contradiction. Therefore $F_2 \not\equiv 0$. By Lemma 2.5 (i), we get $\overline{\lambda}(f'' - \varphi) = \lambda(f'' - \varphi) = \sigma(f) = \frac{n+k}{k}$.

(4) Now we prove that $\overline{\lambda}(f''' - \varphi) = \lambda(f''' - \varphi) = \sigma(f) = \frac{n+k}{k}$. Let $g_3 = f''' - \varphi$, by Lemma 2.16 and $\sigma(\varphi) < \frac{n+k}{k}$, we have $\sigma(g_3) = \sigma(f) = \frac{n+k}{k}$ and

$$f''' = g_3 + \varphi, f^{(k+3)} = g_3^{(k)} + \varphi^{(k)}, f^{(k+2)} = g_3^{(k-1)} + \varphi^{(k-1)}. \tag{5.10}$$

The derivation of (5.8) is

$$f^{(k)} + \left[\left(-\frac{P'}{P}\right) + 2\left(\frac{P'}{P}\right)^2 \right] f^{(k+1)} + P' f'' + P f''' = 0. \tag{5.11}$$

By (5.8), we have

$$f'' = -\frac{1}{P} \left\{ f^{(k+2)} - \frac{2P'}{P} f^{(k+1)} + \left[-\frac{P''}{P} + 2\left(\frac{P'}{P}\right)^2 \right] f^{(k)} \right\}. \tag{5.12}$$

Substituting (5.12) into (5.11), we have

$$f^{(k+3)} - \frac{3P'}{P} f^{(k+2)} + \left[-\frac{3P''}{P} + 6\left(\frac{P'}{P}\right)^2 \right] f^{(k+1)} + \left[-\frac{P'''}{P} + \frac{6P'P''}{P^2} - 6\left(\frac{P'}{P}\right)^3 \right] f^{(k)} + P f''' = 0. \tag{5.13}$$

Substituting (5.10) into (5.13), we have

$$g_3^{(k)} - \frac{3P'}{P} g_3^{(k-1)} + \left[-\frac{3P''}{P} + 6\left(\frac{P'}{P}\right)^2 \right] g_3^{(k-2)} + \left[-\frac{P'''}{P} + \frac{6P'P''}{P^2} - 6\left(\frac{P'}{P}\right)^3 \right] g_3^{(k-3)} + Pg_3$$

$$= -\left\{ \varphi^{(k)} - \frac{3P'}{P}\varphi^{(k-1)} + \left[-\frac{3P''}{P} + 6\left(\frac{P'}{P}\right)^2 \right] \varphi^{(k-2)} + \left[-\frac{P'''}{P} + \frac{6P'P''}{P^2} - 6\left(\frac{P'}{P}\right)^3 \right] \varphi^{(k-3)} + P\varphi \right\}. \tag{5.14}$$

Let $F_3(z) = \varphi^{(k)} - \frac{3P'}{P}\varphi^{(k-1)} + \left[-\frac{3P''}{P} + 6\left(\frac{P'}{P}\right)^2 \right] \varphi^{(k-2)} + \left[-\frac{P'''}{P} + \frac{6P'P''}{P^2} - 6\left(\frac{P'}{P}\right)^3 \right] \varphi^{(k-3)} + P\varphi$. If $F_3(z) \equiv 0$, by Lemma 2.17, we have $\sigma(\varphi) = \frac{n+k}{k}$, this is a contradiction. Therefore $F_3(z) \not\equiv 0$. By Lemma 2.5 (i), we get $\overline{\lambda}(f''' - \varphi) = \lambda(f''' - \varphi) = \sigma(f) = \frac{n+k}{k}$.

(5) Now we prove that $\overline{\lambda}(f^{(j)} - \varphi) = \lambda(f^{(j)} - \varphi) = \sigma(f) = \frac{n+k}{k} (j > 3, j \in N)$. Let $f^{(j)} = g_j + \varphi$, $f^{(k+j)} = g_j^{(k)} + \varphi^{(k)}$, $f^{(k+j-1)} = g_j^{(k-1)} + \varphi^{(k-1)} (j > 3, j \in N)$. By derivation on (5.13), we can also get the following equation which have similar form with (5.14),

$$g_j^{(k)} + \frac{MP'}{P} g_j^{(k-1)} + \left[-\frac{MP''}{P} + 2M\left(\frac{P'}{P}\right)^2 \right] g_j^{(k-2)} + \cdots + Pg_j$$

$$= -\left\{ \varphi^{(k)} - \frac{MP'}{P}\varphi^{(k-1)} + \left[-\frac{MP''}{P} + 2M\left(\frac{P'}{P}\right)^2 \right] \varphi^{(k-2)} + \cdots + P\varphi \right\}. \tag{5.15}$$

Let $F_j = \varphi^{(k)} - \frac{MP'}{P}\varphi^{(k-1)} + \left[-\frac{MP''}{P} + 2M\left(\frac{P'}{P}\right)^2 \right] \varphi^{(k-2)} + \cdots + P\varphi$. If $F_j \equiv 0$, by Lemma 2.17, we have $\sigma(\varphi) = \frac{n+k}{k}$, this is a contradiction. Therefore $F_j \not\equiv 0$. By Lemma 2.5 (i), we get $\overline{\lambda}(f^{(j)} - \varphi) = \lambda(f^{(j)} - \varphi) = \sigma(f) = \frac{n+k}{k} (j > 3, j \in N)$.

6. Proof of Corollary 1.5

By the similar proof in Theorem 1.1 and Lemma 2.8 (i), we can easily obtain the Corollary 1.5.

Acknowledgements
The authors thank for the referee's useful suggestions which improved the paper.

References

[1] Chen Z. X., Complex oscillation of solutions of a class of differential equation of meromorphic coefficients, Chinese Ann. Math., 14(A5), 1993, 584-594 (Chinese).

[2] Chen Z. X., C. C. Yang, Quantitative estimations on the zeros and growths of entire solutions of linear differential equations. Complex Variables and Elliptic Equations, 42, 2000, 119-133.

[3] Chen Z. X., The fixed points and hyper order of solutions of second order complex differential equations. Acta Math. Scientia., 20(A3), 2000, 425-432 (Chinese).

[4] Chen Z. X., On the hyper order of solutions of some second order linear differential equations, Acta Math. Sinica, 18(B1), 2002, 79-88 (Chinese).

[5] Chen Z. X. and Shon K. H., The relation between solutions of a class of second order differential equation with functions of small growth. Chinese. Ann. Math., 27(A4), 2006, 431-442 (Chinese).

[6] Gundersen G., Estimates for the logarithmic derivative of a meromorphic funtion, plus similar estimates. J. London Math. Soc., 305(2), 1998, 88-104.

[7] Hayman W., Meromorphic Functions. Oxford: Clarendon Press., 1964.

[8] He Y. Z. and Xiao X. Z., Algebroid functions and ordinary differential equations. Beijing, Science Press., 1988 (Chinese).

[9] Kwon K. H., On the growth of entire functions satisfying second order linear differential equation. Bull. Korean. Math. Soc., 3, 1996, 487-496.

[10] Laine I., Nevanlinna theory and complex differential equations. Walter de Gruyter, Berlin, New York, 1993.

[11] Liu M. S. and Zhang X. M., Fixed points of meromorphic solutions of higher order linear differential equations. Ann.Acad Sci. Fenn. Ser. A. I. Math., 31, 2006, 191-211.

[12] Tu. J and Yi C. F., On the growth of solutions of a class of higher order linear differential equations with coefficients having the same order. J. Math. Anal. Appl., 340, 2008, 487-497.

[13] Wang J. and Lü W. R., The fixed points and hyper-order of solutions of second order linear differential equations with meromorphic functions. Acta. Math. Appl. Sinica., 27, 2004, 72-80 (Chinese).

[14] Yi H. X. and Yang C. C., Uniqueness theory of meromorphic functions. Beijing: Science Press 1995, New York: Kluwer Academic Publishers 2003.

(Received October 11, 2010)