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Time Periodic Solutions for a Viscous Diffusion Equation with Nonlinear Periodic Sources*

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Abstract

In this paper, we prove the existence of nontrivial nonnegative classical time periodic solutions to the viscous diffusion equation with strongly nonlinear periodic sources. Moreover, we also discuss the asymptotic behavior of solutions as the viscous coefficient k tends to zero.

Keywords: Viscous diffusion equation; Periodicity; Existence; Asymptotic behavior

1 Introduction

This paper deals with the following viscous diffusion equation in one spatial dimension

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} = D^2 u + m(x,t)u^q, \qquad (x,t) \in Q \equiv (0,1) \times \mathbb{R}^+ \tag{1.1}$$

under the homogeneous boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad t \geq 0 \tag{1.2}$$

and the time periodic condition

$$u(x,t+\omega) = u(x,t), \qquad (x,t) \in Q, \tag{1.3}$$

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where $q > 1$, $D = \partial/\partial x$, $k > 0$ denotes the viscous coefficient, $m(x,t) \in C^1(\overline{Q})$ is a positive function satisfies $m(x, t + \omega) = m(x,t)$ for any $(x,t) \in Q$, ω is a positive constant. The purpose of this paper is to investigate the solvability of the time periodic problem (1.1)–(1.3) and the asymptotic behavior of solutions as the viscous coefficient k tends to zero.

Equations of the form (1.1) can also be called pseudo-parabolic equations [1, 2], or Sobolev type equations [3, 4]. They model many mathematical and physical phenomena, such as the seepage of homogeneous fluids through a fissured rock [5, 6], or the heat conduction involving a thermodynamic temperature $\theta = u - k\Delta u$ and a conductive temperature u [7, 8], or the populations with the tendency to form crowds [9, 10]. Furthermore, according to experimental results, some researchers have recently proposed modifications to Cahn's model which incorporate out-of-equilibrium viscoelastic relaxation effects, and thus obtained this type of equations (see [11]). This paper deals with such equations with strong nonlinear time periodic sources, i.e. $q > 1$. From the early 19th century so far, diffusion equations have been widely investigated, among them periodic problems have been paid much attention. The researches on second order periodic diffusion equations are extensive, and many profound results have been obtained ([12, 13, 14, 15]). When $k = 0$, i.e. there isn't any viscosity, the equation (1.1) in multi-spatial dimension becomes

$$\frac{\partial u}{\partial t} = \Delta u + m(x,t)u^q,$$

which has been studied in [16, 17]. The authors proved the existence of nontrivial nonnegative time periodic solutions when $q \in \left(1, \frac{N}{N-2}\right)$, where N is the spatial dimension. When $k > 0$, i.e. pseudo-parabolic equations, Matahashi and Tsutsumi established the existence theorems of time periodic solutions for the linear case

$$\frac{\partial u}{\partial t} - \frac{\partial D^2 u}{\partial t} - D^2 u = f(x,t),$$

and the semilinear case

$$\frac{\partial u}{\partial t} - \frac{\partial D^2 u}{\partial t} - D^2 u + |u|^p u - f(x,t) = 0$$

with $0 < p < 2$, in [18](1978) and [19](1979), respectively. There are also some other early works that related to the periodic problems of the following well-known BBM equation which also has the viscous term

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} - D^2 u + Du + uDu = 0$$

with periodicity conditions with respect to space but not time variable, see for example [20], [21]-[24]. As far as we know, there are a few investigations devoted to time periodic problems of this kind of viscous diffusion equations. Furthermore, notice that such equations can be used to describe models which are sensitive to time periodic factors (for example seasons), such as aggregating populations ([9, 10]), etc, and there are some numerical results and the stability of solutions ([20, 25, 26]) which indicate that time periodic solutions should exist, so it is reasonable to study the periodic problems of the equation (1.1).

This paper is organized as follows. In Section 2, we apply topological degree method to prove the existence of nontrivial nonnegative strong time periodic solutions to the problem (1.1)–(1.3). We further prove that the strong solution is classical. Consequently, in Section 3, we discuss the asymptotic behavior of solutions as the viscous coefficient k tends to zero.

2 Existence of Periodic Solutions

This section is devoted to the existence of time periodic solutions of the problem (1.1)–(1.3). Due to the time periodicity of the solutions under consideration, we only need to consider the problem on $Q_\omega = (0,1)\times(0,\omega)$. In fact, the existence results we obtained are finally for the classical solutions, but due to the proof procedure, we first need to define strong solutions of the problem (1.1)–(1.3).

Definition 2.1 *Let $E = C_\omega(\overline{Q}_\omega)$ be the set of all functions which are continuous in $[0,1]\times\mathbb{R}$ and ω-periodic with respect to t. A function u is said to be a strong solution of the problem* (1.1)–(1.3), *if $u \in \overset{\circ}{W}{}_2^{2,1}(Q_\omega)\cap C_\omega(\overline{Q}_\omega)$ with Du_t and D^2u_t in $L^2(Q_\omega)$, and satisfies*

$$\iint_{Q_\omega}\frac{\partial u}{\partial t}\varphi dxdt - k\iint_{Q_\omega}\frac{\partial D^2u}{\partial t}\varphi dxdt = \iint_{Q_\omega} D^2u\varphi dxdt + \iint_{Q_\omega} m(x,t)u^q\varphi dxdt,$$

for any $\varphi\in C(\overline{Q}_\omega)$, with $\varphi(x,0)=\varphi(x,\omega)$ and $\varphi(0,t)=\varphi(1,t)=0$ for $t\in[0,\omega]$.

Our main result is as follows.

Theorem 2.1 *The problem* (1.1)–(1.3) *admits at least one nontrivial nonnegative classical time periodic solution u in $C^{2+\alpha,1+\alpha/2}(\overline{Q}_\omega)$ with its derivative $\frac{\partial u}{\partial t}$ in $C^{2+\alpha,\alpha/2}(\overline{Q}_\omega)$, where $\alpha\in(0,1)$.*

In order to prove this theorem, we employ the topological degree method to get the existence of nontrivial strong time periodic solutions. Finally, by lifting the regularity of the strong solution, we get the classical solution. Actually, the topological degree method enables us to study the problem by considering a simpler equation with parameter

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2u}{\partial t} = D^2u + \tau f(x,t), \qquad (x,t)\in Q_\omega, \tag{2.1}$$

where $\tau\in[0,1]$ and $f\in E$. Define

$$F : E\times[0,1]\longrightarrow E, \qquad (f,\tau)\longmapsto u.$$

In the following, we prove that the map F is completely continuous. Furthermore, it is easy to see that if we set $f=\Phi(u)=m(x,t)|u|^q$, then the map $F(\Phi(u),\tau)$ is also completely continuous.

Lemma 2.1 *For any* $\tau \in [0,1]$, $f \in E$, *the equation* (2.1) *subject to the conditions* (1.2) *and* (1.3) *has a unique strong solution* $u \in C^{\alpha,\alpha/2}(\overline{Q}_\omega)$, *where* $\alpha \in (0,1)$.

Proof. The existence and uniqueness results can be found in [18]. Next, we discuss the regularity of the solutions. Multiplying (2.1) by u and integrating the result with respect to x over $(0,1)$, by using Young's inequality and Poincaré's inequality, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_0^1 \left(u^2 + k|Du|^2\right) dx + \int_0^1 |Du|^2 dx = \tau \int_0^1 f u dx \leq \frac{1}{2}\int_0^1 |Du|^2 dx + C, \tag{2.2}$$

here and below, C is a constant independent of u and τ. Then we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_0^1 \left(u^2 + k|Du|^2\right) dx \leq C, \qquad \forall t \in (0,\omega). \tag{2.3}$$

Integrating (2.2) over $(0,\omega)$ and noticing the periodicity of u, we get

$$\iint_{Q_\omega} |Du|^2 dxdt \leq C,$$

which with Poincaré's inequality imply

$$\iint_{Q_\omega} \left(u^2 + k|Du|^2\right) dxdt \leq C. \tag{2.4}$$

Set

$$F(t) = \int_0^1 (u^2(x,t) + k|Du(x,t)|^2) dx, \quad \forall t \in [0,\omega].$$

From (2.4), by the mean value theorem, we see that there exists a point $\hat{t} \in (0,\omega)$ such that

$$F(\hat{t}) = \frac{1}{\omega}\int_0^\omega F(t)dt \leq C.$$

For any $t \in (\hat{t},\omega]$, integrating (2.3) from $\hat{t}$ to t gives

$$F(t) \leq C + F(\hat{t}) \leq C, \qquad \forall t \in [\hat{t},\omega].$$

Noticing the periodicity of $F(t)$, we arrive

$$F(0) = F(\omega) \leq C.$$

Hence, integrating (2.3) over $(0,t)$, we obtain

$$F(t) \leq C, \qquad \forall t \in [0,\omega].$$

Recalling the definition of $F(t)$ and $k > 0$, we have

$$\int_0^1 |Du(x,t)|^2 dx \leq C, \quad \forall t \in [0,\omega]. \tag{2.5}$$

Noticing that $u(0,t)=0$, there holds

$$|u(x,t)| = \left|\int_0^x Du(y,t)dy\right| \le \left(\int_0^1 |Du(x,t)|^2 dx\right)^{1/2} \le C, \qquad \forall (x,t) \in Q_\omega. \tag{2.6}$$

Multiplying (2.1) with D^2u and integrating the result with respect to x over $(0,1)$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(|Du|^2 + k|D^2u|^2\right) dx + \int_0^1 |D^2u|^2 dx = -\tau\int_0^1 fD^2u dx \le \frac{1}{2}\int_0^1 |D^2u|^2 dx + C,$$

Similar to the above discussion, we can obtain

$$\iint_{Q_\omega} |D^2u|^2 dxdt \le C, \tag{2.7}$$

$$\int_0^1 |D^2u(x,t)|^2 dx \le C, \qquad \forall t \in [0,\omega]. \tag{2.8}$$

From the inequality (2.5) and (2.8), we can conclude that

$$|Du(x,t)| \le C, \qquad \forall (x,t) \in Q_\omega. \tag{2.9}$$

Multiplying (2.1) by $\dfrac{\partial u}{\partial t}$ and integrating over Q_ω, noticing the periodicity of u, we have

$$\iint_{Q_\omega} \left|\frac{\partial u}{\partial t}\right|^2 dxdt + k\iint_{Q_\omega} \left|\frac{\partial Du}{\partial t}\right|^2 dxdt = \tau\iint_{Q_\omega} f\frac{\partial u}{\partial t} dxdt \le \frac{1}{2}\iint_{Q_\omega} \left|\frac{\partial u}{\partial t}\right|^2 dxdt + C,$$

from which we have

$$\iint_{Q_\omega} \left|\frac{\partial u}{\partial t}\right|^2 dxdt \le C \tag{2.10}$$

$$\iint_{Q_\omega} \left|\frac{\partial Du}{\partial t}\right|^2 dxdt \le C. \tag{2.11}$$

We rewrite the equation (2.1) into the following form

$$\frac{\partial D^2u}{\partial t} = \frac{1}{k}\frac{\partial u}{\partial t} - \frac{1}{k}D^2u - \frac{\tau}{k}f(x,t).$$

By using (2.7), (2.10) and recalling $k>0$, we get

$$\iint_{Q_\omega} \left|\frac{\partial D^2u}{\partial t}\right|^2 dxdt \le C. \tag{2.12}$$

From (2.9), we have

$$|u(x_1,t) - u(x_2,t)| \le C|x_1 - x_2|, \quad \forall t \in [0,\omega], \quad x_1, x_2 \in [0,1]. \tag{2.13}$$

For any $(x,t_1),(x,t_2)\in\overline{Q}_\omega$, we consider the case of $0\le x\le 1/2$. Denote $\Delta t=t_2-t_1>0$ satisfying $(\Delta t)^\beta\le 1/4, \beta\in(0,1)$. For any $y\in(x,x+(\Delta t)^\beta)$, integrating the equation (1.1) over $(y,y+(\Delta t)^\beta)\times(t_1,t_2)$ yields

$$\begin{aligned}
&\int_y^{y+(\Delta t)^\beta}(u(z,t_2)-u(z,t_1))dz\\
=&k\int_y^{y+(\Delta t)^\beta}\int_{t_1}^{t_2}\frac{\partial D^2u}{\partial t}(z,t)dtdz+\int_y^{y+(\Delta t)^\beta}\int_{t_1}^{t_2}D^2u(z,t)dtdz\\
&+\tau\int_y^{y+(\Delta t)^\beta}\int_{t_1}^{t_2}f(z,t)dtdz.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(\Delta t)^\beta\int_0^1\left[u(y+\theta(\Delta t)^\beta,t_2)-u(y+\theta(\Delta t)^\beta,t_1)\right]d\theta\\
\le&(\Delta t)^{\frac{1+\beta}{2}}\left[k\left(\iint_{Q_\omega}\left|\frac{\partial D^2u}{\partial t}\right|^2dxdt\right)^{1/2}+\left(\iint_{Q_\omega}|D^2u|^2dxdt\right)^{1/2}\right]+\|f\|_{L^\infty(Q_\omega)}(\Delta t)^{1+\beta}.
\end{aligned}$$

Integrating the above equality with respect to y over $(x,x+(\Delta t)^\beta)$, from (2.7),(2.12), and by using the mean value theorem, we get

$$|u(x^*,t_2)-u(x^*,t_1)|\le C|t_2-t_1|^{\frac{1-\beta}{2}},$$

where $x^*=y^*+\theta^*(\Delta t)^\beta$, $y^*\in(x,x+(\Delta t)^\beta)$, $\theta^*\in(0,1)$. Recalling $\beta\in(0,1)$, we have $(1-\beta)/2\in(0,1/2)$. Combining the above inequality with (2.13), we have

$$\begin{aligned}
|u(x,t_1)-u(x,t_2)|&\le|u(x,t_1)-u(x^*,t_1)|+|u(x^*,t_1)-u(x^*,t_2)|+|u(x^*,t_2)-u(x,t_2)|\\
&\le C|t_1-t_2|^{\min\left\{\beta,\frac{1-\beta}{2}\right\}}.
\end{aligned}$$

Hence,

$$|u(x_1,t_1)-u(x_2,t_2)|\le C(|x_1-x_2|+|t_1-t_2|^{\alpha/2}) \tag{2.14}$$

for all $(x_i,t_i)\in\overline{Q}_\omega(i=1,2)$, $\alpha\in(0,1)$. Thus we have $u\in C^{\alpha,\alpha/2}(\overline{Q}_\omega)$. The proof is complete. □

Lemma 2.2 *The map $F:E\times[0,1]\to E$ is completely continuous.*

Proof. By Lemma 2.1, the periodicity of u in t, and the Arzelá-Ascoli theorem, we can see that F maps any bounded set of $E\times[0,1]$ into a compact set of E.

Suppose that $\{f_n\}_{n=1}^\infty\subset C_\omega(\overline{Q}_\omega)$, $\{\tau_n\}_{n=1}^\infty\subset[0,1]$, $f\in C_\omega(\overline{Q}_\omega)$, $\tau\in[0,1]$, and

$$\lim_{n\to\infty}|f_n-f|_0=0,\qquad \lim_{n\to\infty}\tau_n=\tau.$$

Denote $u_n = F(f_n, \tau_n)$, $u = F(f, \tau)$. Similar to the proof of Lemma 2.1, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_0^1 \left(|u_n - u|^2 + k|Du_n - Du|^2\right)dx + \int_0^1 |Du_n - Du|^2 dx \tag{2.15}$$
$$= \int_0^1 (\tau_n f_n - \tau f)(u_n - u)dx \le \frac{1}{2}\int_0^1 |Du_n - Du|^2 dx + \frac{1}{2}\int_0^1 |\tau_n f_n - \tau f|^2 dx,$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_0^1 \left(|u_n - u|^2 + k|Du_n - Du|^2\right)dx \le C\int_0^1 (\tau_n^2|f_n - f|^2 + |f|^2|\tau_n - \tau|^2)dx \to 0$$

and

$$\iint_{Q_\omega} |Du_n - Du|^2 dxdt \le C\iint_{Q_\omega} (\tau_n^2|f_n - f|^2 + |f|^2|\tau_n - \tau|^2)dxdt \to 0$$

as $n \to \infty$. Using the method to prove (2.6), we have

$$\lim_{n\to\infty} |u_n - u|_0 = 0.$$

The proof is complete. □

Before using the topological degree method, we should remark that if we set $f = \Phi(u) = m(x,t)|u|^q$, then the nontrivial strong time periodic solution we obtained are just the nontrivial nonnegative classical solution.

Proposition 2.1 *If $u \in C^{\alpha,\alpha/2}(\overline{Q}_\omega)$ is the nontrivial strong time periodic solution of*

$$\frac{\partial u}{\partial t} - kD^2u = D^2u + m(x,t)|u|^q, \tag{2.16}$$

subject to (1.2), (1.3), *then u is just the nontrivial nonnegative classical time periodic solution.*

Proof. As is well known, $(I - kD^2)^{-1}$ is bounded from $C^{\alpha,\alpha/2}(\overline{Q}_\omega)$ to $C^{2+\alpha,\alpha/2}(\overline{Q}_\omega)$, then the strong solution u in $C^{\alpha,\alpha/2}(\overline{Q}_\omega)$ satisfies

$$\frac{\partial u}{\partial t} = -\frac{1}{k}u + (I - kD^2)^{-1}(\frac{1}{k}u + m(x,t)|u|^q). \tag{2.17}$$

Thus we have

$$\frac{\partial u}{\partial t} + \frac{1}{k}u \in C^{2+1/2,\alpha}(\overline{Q}_\omega). \tag{2.18}$$

Multiplying $e^{t/k}$ on both sides of (2.17), we get

$$\frac{\partial}{\partial t}(e^{t/k}u) = e^{t/k}(I - kD^2)^{-1}(\frac{1}{k}u + m(x,t)|u|^q).$$

For any $t \in [0,\omega]$, integrating the above equation in $[t, t+\omega]$ and using the periodicity of u yield

$$u(x,t) = (e^{(t+\omega)/k} - e^{t/k})^{-1}\int_t^{t+\omega} e^{s/k}(I - kD^2)^{-1}\left(\frac{1}{k}u(x,s) + m(x,s)|u|^q(x,s)\right)\mathrm{d}s,$$

which with (2.18) imply that

$$u \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_\omega), \qquad \frac{\partial u}{\partial t} \in C^{2+\alpha,\alpha/2}(\overline{Q}_\omega).$$

Hence u is the classical solution and we conclude that $u \geq 0$. Suppose to the contrary, there exists a pair of points $(x_0, t_0) \in (0,1) \times (0,\omega)$ such that

$$u(x_0, t_0) < 0.$$

Since u is continuous, then there exists an interval $(x_0 - \delta_1, x_0 + \delta_2)$ such that $u(x, t_0) < 0$ in $(x_0 - \delta_1, x_0 + \delta_2)$ and $u(x_0 - \delta_1, t_0) = u(x_0 + \delta_2, t_0) = 0$. Multiplying (2.16) by φ which is the principle eigenfunction of $-D^2$ in $(x_0 - \delta_1, x_0 + \delta_2)$ with homogeneous Dirichlet boundary condition, and integrating on $(x_0 - \delta_1, x_0 + \delta_2)$, we can get

$$(1 + k\lambda_r) \int_{x_0-\delta_1}^{x_0+\delta_2} u_t \varphi dx + \lambda_r \int_{x_0-\delta_1}^{x_0+\delta_2} u\varphi dx = \int_{x_0-\delta_1}^{x_0+\delta_2} m(x,t)|u|^q \varphi dx, \tag{2.19}$$

where λ_r is the first eigenvalue. Integrating the above inequality from 0 to ω and using the periodicity of u, we have

$$\lambda_r \int_0^\omega \int_{x_0-\delta_1}^{x_0+\delta_2} u\varphi dxdt > 0.$$

By the mean value theorem, there exists a point $t^* \in (0, \omega)$ such that

$$\int_{x_0-\delta_1}^{x_0+\delta_2} u(x, t^*)\varphi dx > 0.$$

Actually (2.19) is equivalent to

$$\int_{x_0-\delta_1}^{x_0+\delta_2} \frac{\partial e^{t\lambda_r/(1+k\lambda_r)} u}{\partial t} \varphi dx = \frac{1}{1 + k\lambda_r} \int_{x_0-\delta_1}^{x_0+\delta_2} e^{t\lambda_r/(1+k\lambda_r)} m(x,t)|u|^q \varphi dx. \tag{2.20}$$

Integrating the above inequality from t^* to ω implies that

$$\int_{x_0-\delta_1}^{x_0+\delta_2} e^{\omega\lambda_r/(1+k\lambda_r)} u(x,\omega)\varphi dx > 0.$$

Recalling the periodicity of u, we see that

$$\int_{x_0-\delta_1}^{x_0+\delta_2} u(x, 0)\varphi dx > 0.$$

Then integrating (2.20) over $(0, t)$ implies that

$$\int_{x_0-\delta_1}^{x_0+\delta_2} e^{t\lambda_r/(1+k\lambda_r)} u(x,t)\varphi dx > 0, \qquad t \in (0, \omega)$$

which is contradict with $u(x, t_0) < 0$ in $(x_0 - \delta_1, x_0 + \delta_2)$. The proof is complete. □

In the following, we are going to establish the existence of nontrivial strong time periodic solutions by calculating the topological degree. For this purpose, we denote the ball in $C(\overline{Q}_\omega)$ with center zero and radius R by $B_R(0)$. Firstly, we calculate $\deg(I - F(\Phi(\cdot), 1),\ B_r(0),\ 0)$ for r appropriately small.

Proposition 2.2 *There exists a constant $r > 0$ such that*

$$\deg(I - F(\Phi(\cdot), 1),\ B_r(0),\ 0) = 1.$$

Proof. Owing to the complete continuity of the map $F(\Phi(u), \sigma)$, where $\sigma \in [0, 1]$ is a parameter, the homotopy invariance of degree implies

$$\deg(I - F(\Phi(\cdot), 1),\ B_r(0),\ 0) = \deg(I,\ B_r(0),\ 0) = 1,$$

provided that

$$F(\Phi(u), \sigma) \neq u, \qquad \forall\, \sigma \in [0, 1],\ u \in \partial B_r(0). \tag{2.21}$$

Therefore, we need only to prove that there exists a constant $r > 0$ such that (2.21) holds. In fact, it suffices to take

$$r < \left(\frac{\pi^2}{\overline{m}}\right)^{\frac{1}{q-1}},$$

where $\overline{m}$ is the upper bound of $m(x, t)$. Suppose $u \in \partial B_r(0)$, namely $\|u\|_{C(\overline{Q}_\omega)} = r$, satisfies $F(\Phi(u), \sigma) = u$ for any $\sigma \in [0, 1]$. Multiplying the equation

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} = D^2 u + \sigma m(x, t)|u|^q$$

with u and integrating over Q_ω, by the time periodicity of u, and noticing that the first eigenvalue of the Laplacian equation with homogeneous Dirichlet boundary value conditions in $(0, 1)$ is π^2, we have

$$\begin{aligned} 0 =& -\iint_{Q_\omega} |Du|^2 dxdt + \sigma \iint_{Q_\omega} m(x, t)|u|^q u dxdt \\ \leq& -\pi^2 \iint_{Q_\omega} u^2 dxdt + \overline{m} r^{q-1} \iint_{Q_\omega} u^2 dxdt \\ =& \left(\overline{m} r^{q-1} - \pi^2\right) \iint_{Q_\omega} u^2 dxdt < 0, \end{aligned}$$

which is a contradiction. The proof is complete. □

Next, we calculate $\deg(I - F(\Phi(\cdot), 1),\ B_R(0),\ 0)$ for appropriately large R. In order to do this, we need the following maximum norm estimate.

Lemma 2.3 *If u is a time periodic solution of the equation*

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} = D^2 u + m(x, t)|u|^q + (1 - \tau)(\pi^2|u| + 1) \tag{2.22}$$

subject to the conditions (1.2) *and* (1.3), *then*

$$\|u\|_{L^\infty(Q_\omega)} \leq M_1, \qquad \forall \tau \in [0, 1],$$

where M_1 is a positive constant independent of u, k and τ.

For proving this lemma, we need the following result, the proof of which is similar to [29].

Lemma 2.4 *If $q > 1$, $a(x)$ is appropriately smooth and satisfies $0 < \underline{a} \le a(x) \le \overline{a}$, where $\underline{a}$ and $\overline{a}$ are positive constants, then the problem*

$$\begin{cases} \int_{\mathbb{R}} DvD\varphi dx = \int_{\mathbb{R}} a(x)v^q \varphi dx, & \forall \varphi \in C_0^1(\mathbb{R}),\ \varphi \ge 0, \\ v > 0, & \forall x \in \mathbb{R} \end{cases} \tag{2.23}$$

has no solution $v \in C^1(\mathbb{R})$.

Proof. For any $\psi \in C_0^1(\mathbb{R})$, $\psi \ge 0$, taking $\varphi = v^{-\lambda}\psi$ with $0 < \lambda < \frac{q-1}{2}$, we have

$$\int_{\mathbb{R}} v^{-\lambda} DvD\psi dx - \lambda \int_{\mathbb{R}} v^{-\lambda-1}|Dv|^2 \psi dx = \int_{\mathbb{R}} a(x)v^{q-\lambda}\psi dx.$$

It follows that

$$\begin{aligned} &\lambda \int_{\mathbb{R}} v^{-\lambda-1}|Dv|^2\psi dx + \int_{\mathbb{R}} a(x)v^{q-\lambda}\psi dx \le \int_{\mathbb{R}} v^{-\lambda}|Dv||D\psi| dx \\ \le& \frac{\lambda}{2}\int_{\mathbb{R}} v^{-\lambda-1}|Dv|^2\psi dx + C\int_{\mathbb{R}} v^{-\lambda+1}\frac{|D\psi|^2}{\psi} dx \\ \le& \frac{\lambda}{2}\int_{\mathbb{R}} v^{-\lambda-1}|Dv|^2\psi dx + \frac{1}{2}\int_{\mathbb{R}} a(x)v^{q-\lambda}\psi dx + C\int_{\mathbb{R}} \frac{|D\psi|^{\frac{2(q-\lambda)}{q-1}}}{\psi^{\frac{2(q-\lambda)}{q-1}-1}} dx, \end{aligned}$$

which implies that

$$\lambda \int_{\mathbb{R}} v^{-\lambda-1}|Dv|^2\psi dx + \int_{\mathbb{R}} a(x)v^{q-\lambda}\psi dx \le C\int_{\mathbb{R}} \frac{|D\psi|^{\frac{2(q-\lambda)}{q-1}}}{\psi^{\frac{2(q-\lambda)}{q-1}-1}} dx, \tag{2.24}$$

where C is constant independent of v. Furthermore, replacing φ by ψ in (2.23), we see that

$$\begin{aligned} &\int_{\mathbb{R}} a(x)v^q\psi dx = \int_{\mathbb{R}} DvD\psi dx \\ \le& \left(\int_{\mathbb{R}} v^{-\lambda-1}|Dv|^2\psi dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} v^{\lambda+1}\frac{|D\psi|^2}{\psi} dx\right)^{\frac{1}{2}} \\ \le& \left(\int_{\mathbb{R}} v^{-\lambda-1}|Dv|^2\psi dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} a(x)v^{q-\lambda}\psi dx\right)^{\frac{\lambda+1}{2(q-\lambda)}} \left(\int_{\mathbb{R}} \frac{|D\psi|^{\frac{2(q-\lambda)}{q-2\lambda-1}}}{\psi^{\frac{2(q-\lambda)}{q-2\lambda-1}-1}} dx\right)^{\frac{q-2\lambda-1}{2(q-\lambda)}}. \end{aligned}$$

Combining the above inequality with (2.24), we obtain

$$\int_{\mathbb{R}} a(x)v^q\psi dx \le C\left(\int_{\mathbb{R}} \frac{|D\psi|^{\frac{2(q-\lambda)}{q-1}}}{\psi^{\frac{2(q-\lambda)}{q-1}-1}} dx\right)^{\frac{q+1}{2(q-\lambda)}} \left(\int_{\mathbb{R}} \frac{|D\psi|^{\frac{2(q-\lambda)}{q-2\lambda-1}}}{\psi^{\frac{2(q-\lambda)}{q-2\lambda-1}-1}} dx\right)^{\frac{q-2\lambda-1}{2(q-\lambda)}}. \tag{2.25}$$

For $A > 0$, define

$$\xi(x) = \xi_0\left(\frac{|x|}{A}\right),$$

where $\xi_0 \in C^1(\mathbb{R}^+)$ with $0 \le \xi_0 \le 1$ satisfying

$$\xi_0(\nu) = \begin{cases} 1, & 0 \le \nu \le 1, \\ 0, & \nu \ge 2. \end{cases}$$

Taking $\psi = \xi^\kappa$ with κ appropriately large, through a simple calculation, we get

$$\int_{\mathbb{R}} \frac{|D\psi|^{2\rho}}{\psi^{2\rho-1}} dx \le CA^{1-2\rho}, \qquad \forall \rho > 0.$$

Recalling (2.25), we obtain

$$\int_{-A}^{A} v^q dx \le \frac{1}{\underline{a}} \int_{\mathbb{R}} a(x) v^q \psi dx \le CA^{-\frac{q+1}{q-1}},$$

where C is independent of v and A. Letting $A \to \infty$ and noticing $q > 1$, we arrive

$$\int_{\mathbb{R}} v^q dx = 0.$$

Then we have $v \equiv 0$ for any $x \in \mathbb{R}$, which is a contradiction. The proof is complete. □

Remark 2.1 *The result of Lemma* 2.4 *is also correct for* $v \in H^1_{\text{loc}}(\mathbb{R})$.

Proof of Lemma 2.3 Suppose that the periodic solution u is not uniformly bounded. Then, there exist unbounded real number collection $\{\rho_n\}_{n=1}^\infty$, a sequence $\{\tau_n\}_{n=1}^\infty (\tau_n \in [0,1])$ and the periodic solution sequence $\{u_n\}_{n=1}^\infty$ of the problem (2.22), (1.2) and (1.3), such that

$$\rho_n = \max_{\overline{Q}_\omega} u_n(x,t) = u_n(x_n, t_n) \longrightarrow \infty, \qquad n \to \infty.$$

Since $x_n \in (0,1)$, there exists a subsequence of $\{x_n\}_{n=1}^\infty$, denoted by itself for simplicity, and $x_0 \in (0,1)$ such that $x_n \to x_0$ as $n \to \infty$. For any fixed n, define

$$v_{nj}(y,s) = \rho_n^{-1} u_n(x_0 + \rho_n^{-\frac{q-1}{2}} y,\ t_j + js),$$
$$\tilde{m}_{nj}(y,s) = m(x_0 + \rho_n^{-\frac{q-1}{2}} y,\ t_j + js),$$

where $t_j \in \{t_n\}_{n=1}^\infty$ and

$$(y,s) \in Q_{nj} = \Omega_n \times \left(-\frac{t_j}{j}, \frac{\omega - t_j}{j}\right), \quad \Omega_n = \{y;\ y = \rho_n^{\frac{q-1}{2}}(x - x_0),\ x \in (0,1)\}.$$

Obviously, $\|v_{nj}\|_{L^\infty(Q_{nj})} = 1$ and v_{nj} satisfies

$$\rho_n^{1-q}\frac{\partial v_{nj}}{\partial s} - k\frac{\partial D^2 v_{nj}}{\partial s} = jD^2 v_{nj} + j\tilde{m}_{nj}(y,s)|v_{nj}|^q + j(1-\tau_n)(\pi^2\rho_n^{1-q}|v_{nj}| + \rho_n^{-q}),$$

here and below, we denote $D = \dfrac{\partial}{\partial y}$. Similar to the proof of Lemma 2.1 and Proposition 2.1, we can deduce that $v_{nj} \geq 0$. Thus, for simplicity, in what follows, we would throw off the symbol of absolute value of v_{nj}. Therefore, for any $\phi(y,s) \in C^1(Q_{nj})$ satisfying $\phi|_{\partial\Omega_n} = 0$, we have

$$\begin{aligned}
&\rho_n^{1-q}\iint_{Q_{nj}} \frac{\partial v_{nj}}{\partial s}\phi dyds + k\iint_{Q_{nj}} \frac{\partial D v_{nj}}{\partial s} D\phi dyds + j\iint_{Q_{nj}} Dv_{nj} D\phi dyds \\
=&j\iint_{Q_{nj}} \tilde{m}_{nj}(y,s) v_{nj}^q \phi dyds + j(1-\tau_n)\iint_{Q_{nj}} (\pi^2\rho_n^{1-q} v_{nj} + \rho_n^{-q})\phi dyds.
\end{aligned}$$

Taking $\phi = v_{nj}$, by virtue of the periodicity of v_{nj}, we get

$$\begin{aligned}
j\iint_{Q_{nj}} |Dv_{nj}|^2 dyds =& j\iint_{Q_{nj}} \tilde{m}_{nj}(y,s) v_{nj}^{q+1} dyds \\
&+ j(1-\tau_n)\iint_{Q_{nj}} (\pi^2\rho_n^{1-q} v_{nj}^2 + \rho_n^{-q} v_{nj}) dyds \\
\leq& C\omega|\Omega_n| + C\omega(\rho_n^{1-q} + \rho_n^{-q})|\Omega_n|.
\end{aligned}$$

Hence, by means of the integral mean value theorem, there exists a point $s_j \in \left(-\frac{t_j}{j}, \frac{\omega - t_j}{j}\right)$ such that

$$\int_{\Omega_n} |Dv_{nj}(y,s_j)|^2 dy \leq C|\Omega_n| + C(\rho_n^{1-q} + \rho_n^{-q})|\Omega_n|.$$

Noticing that for any $s > s_j$, by taking $\phi = \chi_{(s_j,s)}\dfrac{\partial v_{nj}}{\partial s}$, we have

$$\begin{aligned}
\int_{\Omega_n} |Dv_{nj}(y,s)|^2 dy \leq& \int_{\Omega_n} |Dv_{nj}(y,s_j)|^2 dy + \frac{2}{q+1}\int_{\Omega_n} \tilde{m}_{nj}(y,s) v_{nj}^{q+1}(y,s) dy \\
&- \frac{2}{q+1}\int_{\Omega_n} \tilde{m}_{nj}(y,s_j) v_{nj}^{q+1}(y,s_j) dy \\
&- \frac{2j}{q+1}\int_{s_j}^{s}\int_{\Omega_n} \frac{\partial \tilde{m}_{nj}}{\partial t}(y,s) v_{nj}^{q+1} dyds \\
&+ (1-\tau_n)\pi^2\rho_n^{1-q}\int_{\Omega_n} (v_{nj}^2(y,s) - v_{nj}^2(y,s_j)) dy \\
&+ 2(1-\tau_n)\rho_n^{-q}\int_{\Omega_n} (v_{nj}(y,s) - v_{nj}(y,s_j)) dy \\
\leq& C|\Omega_n| + C(\rho_n^{1-q} + \rho_n^{-q})|\Omega_n|,
\end{aligned}$$

where C is a constant independent of j, n and $|\Omega_n|$. By the periodicity of v_{nj}, we get

$$\int_{\Omega_n} |Dv_{nj}(y, \ -t_j/j)|^2 dy \le C|\Omega_n| + C(\rho_n^{1-q} + \rho_n^{-q})|\Omega_n|.$$

Similar to the above argument, we obtain

$$\int_{\Omega_n} |Dv_{nj}(y,s)|^2 dy \le C|\Omega_n| + C(\rho_n^{1-q} + \rho_n^{-q})|\Omega_n| \tag{2.26}$$

for any $s \in \left[-\frac{t_j}{j}, \frac{\omega - t_j}{j}\right]$. On the other hand, noticing that for any $\varphi \in C_0^1(\Omega_n)$, there holds

$$\begin{aligned} j \iint_{Q_{nj}} Dv_{nj} D\varphi dyds =& j \iint_{Q_{nj}} \tilde{m}_{nj}(y,s) v_{nj}^q \varphi dyds \\ &+ j(1-\tau_n) \iint_{Q_{nj}} (\pi^2 \rho_n^{1-q} v_{nj} + \rho_n^{-q}) \varphi dyds. \end{aligned}$$

Fixing $j_0 > 0$, for any $j = lj_0$, where l is a positive integer, we get

$$\begin{aligned} j_0 \iint_{Q_{nj_0}} Dv_{nj} D\varphi dyds =& j_0 \iint_{Q_{nj_0}} \tilde{m}_{nj}(y,s) v_{nj}^q \varphi dyds \\ &+ j_0(1-\tau_n) \iint_{Q_{nj_0}} (\pi^2 \rho_n^{1-q} v_{nj} + \rho_n^{-q}) \varphi dyds. \end{aligned}$$

Recalling (2.26), there exists a function $v_n \in H^1(Q_{nj_0})$ satisfying $\|v_n\|_{L^\infty(Q_{nj_0})} = 1$, such that

$$Dv_{nj} \rightharpoonup Dv_n \quad \text{in } L^2(Q_{nj_0}), \qquad v_{nj} \to v_n \quad \text{in } L^\gamma(Q_{nj_0}) \quad \text{for } \gamma > 0,$$

as $j \to \infty$. Meanwhile, since $\tilde{m}_{nj}$ is continuous on $\overline{Q}_{nj_0}$, then there exists a function $\tilde{m}_n$, such that

$$\tilde{m}_{nj} \to \tilde{m}_n \quad \text{as } j \to \infty.$$

Hence, taking $l \to \infty$, we have

$$\begin{aligned} j_0 \iint_{Q_{nj_0}} Dv_n D\varphi dyds =& j_0 \iint_{Q_{nj_0}} \tilde{m}_n(y,s) v_n^q \varphi dyds \\ &+ j_0(1-\tau_n) \iint_{Q_{nj_0}} (\pi^2 \rho_n^{1-q} v_n + \rho_n^{-q}) \varphi dyds. \end{aligned}$$

Then, by virtue of the arbitrariness of j_0, taking $j_0 \to \infty$, we arrive

$$\int_{\Omega_n} Dv_n D\varphi dy = \int_{\Omega_n} \tilde{m}_n(y,0) v_n^q \varphi dy + (1-\tau_n) \int_{\Omega_n} (\pi^2 \rho_n^{1-q} v_n + \rho_n^{-q}) \varphi dy.$$

Choose $\varphi = v_n \eta^2$, where

$$\eta(y) = \begin{cases} 1, & y \in (-R, R), \\ 0, & y \overline{\in} (-2R, 2R), \end{cases}$$

and $0 \leq \eta \leq 1$ is smooth enough, $|\eta'(y)| \leq \frac{C}{R}$. Then, for n large enough, we have $(-2R, 2R) \subset \Omega_n$ and

$$\begin{aligned}\int_{-2R}^{2R} |Dv_n|^2\eta^2 dy =& - \int_{-2R}^{2R} 2v_n Dv_n \eta\eta' dy + \int_{-2R}^{2R} \tilde{m}_n(y,0)v_n^{q+1}\eta^2 dy \\ &+ (1-\tau_n)\int_{-2R}^{2R} (\pi^2\rho_n^{1-q}v_n^2 + \rho_n^{-q}v_n)\eta^2 dy \\ \leq& \frac{1}{2}\int_{-2R}^{2R} |Dv_n|^2\eta^2 dy + CR + \frac{C}{R} + C(\rho_n^{1-q} + \rho_n^{-q})R,\end{aligned}$$

which implies that

$$\int_{-R}^{R} |Dv_n|^2 dy \leq CR + \frac{C}{R} + C(\rho_n^{1-q} + \rho_n^{-q})R,$$

where C is a constant independent of n and R. Therefore, there exists a function $\hat{v} \in H^1_{\mathrm{loc}}(\mathbb{R})$ (pass to a subsequence if necessary) such that

$$Dv_n \rightharpoonup D\hat{v} \ \text{ in } L^2(-R,R), \qquad v_n \to \hat{v} \ \text{ in } L^\gamma(-R,R) \ \text{ for } \gamma > 0,$$

as $n \to \infty$. Since $\tilde{m}_n(y,0)$ is continuous on $[-R,R]$, then there exists a function $\tilde{m}(y,0)$ such that $\tilde{m}_n(y,0) \to \tilde{m}(y,0)$ as $n \to \infty$. Thus, we have

$$\begin{cases} \displaystyle\int_{-R}^{R} D\hat{v}D\varphi dy = \int_{-R}^{R} \tilde{m}(y,0)\hat{v}^q\varphi dy, & \forall\varphi \in C_0^1(-R,R), \\ \|\hat{v}\|_{L^\infty(-R,R)} = 1, \qquad \text{and} \qquad \hat{v} \geq 0, & \forall y \in (-R,R). \end{cases}$$

Moreover, since $\hat{v} \not\equiv 0$, by the strong maximum principle we have $\hat{v} > 0$ for any $x \in (-R,R)$. Taking R larger and larger and repeating the argument for the subsequence $\hat{v}_k$ obtained at the previous step, we get a Cantor diagonal subsequence, for simplicity, we still denote it by $\{\hat{v}_k\}_{k=1}^\infty$, which converges in $H^1_{\mathrm{loc}}(\mathbb{R})$ to a function $v \in H^1_{\mathrm{loc}}(\mathbb{R})$ as $k \to \infty$ and

$$\begin{cases} \displaystyle\int_{\mathbb{R}} DvD\varphi dy = \int_{\mathbb{R}} \tilde{m}(y,0)v^q\varphi dy, & \forall\varphi \in C_0^1(\mathbb{R}), \\ \|v\|_{L^\infty(\mathbb{R})} = 1, \qquad \text{and} \qquad v > 0, & \forall y \in \mathbb{R}. \end{cases}$$

Thus, thanks to Lemma 2.4, we see that for $q > 1$, the above problem has no solution, which is a contradiction. The proof is complete. □

Proposition 2.3 *There exists a constant $R > r$ such that*

$$\deg(I - F(\Phi(\cdot),1),\ B_R(0),\ 0) = 0.$$

Proof. Set $\Psi(u) = \pi^2|u| + 1$, $H(u,\tau) = F(\Phi(u) + (1-\tau)\Psi(u), 1)$. Then the map $H(u,\tau)$ is completely continuous. Hence, the homotopy invariance of degree implies

$$\deg(I - F(\Phi(\cdot),1),\ B_R(0),\ 0) = \deg(I - H(\cdot,0),\ B_R(0),\ 0),$$

provided that

$$H(u,\tau) \neq u, \qquad \forall\, \tau \in [0,1], \quad u \in \partial B_R(0).$$

In fact, Lemma 2.3 implies that the above inequality holds for $R > \max\{M_1,\ r\}$.

On the other hand, when $\tau = 0$, the equation (2.22) becomes

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} = D^2 u + m(x,t)|u|^q + \pi^2|u| + 1. \tag{2.27}$$

Similar to the proof of Lemma 2.1 and Proposition 2.1, we can deduce that $u > 0$. Multiplying the above equation by $\sin \pi x$ and integrating over Q_ω, by the periodicity of u, we have

$$\begin{aligned}
0 &= \iint_{Q_\omega} D^2 u \sin \pi x dxdt + \iint_{Q_\omega} m(x,t)|u|^q \sin \pi x dxdt \\
&\quad + \iint_{Q_\omega} \pi^2 u \sin \pi x dxdt + \iint_{Q_\omega} \sin \pi x dxdt \\
&= \iint_{Q_\omega} m(x,t)|u|^q \sin \pi x dxdt + \iint_{Q_\omega} \sin \pi x dxdt \\
&= \iint_{Q_\omega} m(x,t)|u|^q \sin \pi x dxdt + \frac{2\omega}{\pi} > 0,
\end{aligned}$$

which is a contradiction. Therefore, the equation (2.27) with the Dirichlet boundary value conditions (1.2) doesn't admit nonnegative periodic solutions. Hence,

$$\deg(I - H(\cdot, 0),\ B_R(0),\ 0) = 0.$$

Consequently,

$$\deg(I - F(\Phi(\cdot), 1),\ B_R(0),\ 0) = 0.$$

The proof is complete. □

Finally, we prove our main result of this section.

Proof of Theorem 2.1 From Proposition 2.2 and Proposition 2.3, we see that there exist constants R and r satisfying $R > r > 0$ such that

$$\deg(I - F(\Phi(\cdot), 1),\ B_R(0)/B_r(0),\ 0) = -1,$$

which implies that the problem (1.1)–(1.3) admits at least one nontrivial strong time periodic solution $u \in E$ such that $r \le \|u\|_{C_\omega(\overline{Q}_\omega)} \le R$.

Basing on the discussion in Lemma 2.1 and Proposition 2.1, u is just the nontrivial nonnegative classical time periodic solution. The proof of Theorem 2.1 is complete. □

3 Asymptotic Behavior

In this section, we are interested in the asymptotic behavior of solutions as the viscous coefficient k tends to zero. Here, we denote by C a constant independent of u and k.

Theorem 3.1 *If u_k is a nontrivial nonnegative classical time periodic solution of the problem (1.1)–(1.3), then $u_k(x,t)$ is uniformly convergent in Q_ω as $k \to 0$, and the limit function $u(x,t)$ is a nontrivial nonnegative classical periodic solution of the following problem*

$$\frac{\partial u}{\partial t} = D^2 u + m(x,t)u^q, \qquad (x,t) \in Q_\omega, \tag{3.1}$$

$$u(0,t) = u(1,t) = 0, \qquad t \in [0,\omega], \tag{3.2}$$

$$u(x,\omega) = u(x,0), \qquad x \in [0,1]. \tag{3.3}$$

Proof. Similar to the proof of Lemma 2.3, we can prove that the time periodic solution u_k satisfies

$$\|u_k\|_{L^\infty(Q_\omega)} \le M_1,$$

where M_1 is independent of k. Multiplying (1.1) by $D^2 u_k$ and integrating the result with respect to x over $(0,1)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 (|Du_k|^2 + k|D^2u_k|^2)dx + \int_0^1 |D^2u_k|^2 dx \\ = &-\int_0^1 m(x,t)u_k^q D^2u_k dx \le \frac{1}{2}\int_0^1 |D^2u_k|^2 dx + C, \end{aligned} \tag{3.4}$$

from which we get

$$\frac{d}{dt}\int_0^1 (|Du_k|^2 + k|D^2u_k|^2)dx \le C, \quad \forall t \in (0,\omega). \tag{3.5}$$

Moreover, integrating (3.4) over $(0,\omega)$ yields

$$\iint_{Q_\omega} |D^2u_k|^2 dxdt \le C. \tag{3.6}$$

From $\|u_k\|_{L^\infty(Q_\omega)} \le M_1$, (3.6) and Young's inequality, we get

$$\iint_{Q_\omega} |Du_k|^2 dxdt = -\iint_{Q_\omega} u_k D^2u_k dxdt \le \frac{1}{2}\iint_{Q_\omega} u_k^2 dxdt + \frac{1}{2}\iint_{Q_\omega} |D^2u_k|^2 dxdt \le C.$$

Then, we have

$$\iint_{Q_\omega} (|Du_k|^2 + k|D^2u_k|^2)dxdt \le C + Ck. \tag{3.7}$$

From (3.5) and (3.7), by the similar proof in Lemma 2.1, we can deduce that

$$\int_0^1 |Du_k(x,t)|^2 dx \le C + Ck, \quad \forall t \in [0,\omega]. \tag{3.8}$$

Taking $\frac{\partial u_k}{\partial t}$ as a test function, we can derive

$$\iint_{Q_\omega} \left|\frac{\partial u_k}{\partial t}\right|^2 dxdt \le C. \tag{3.9}$$

By means of the equation (1.1), we can further obtain

$$k \iint_{Q_\omega} \left| \frac{\partial D^2 u_k}{\partial t} \right|^2 dxdt \le C. \tag{3.10}$$

Similar to the proof in Lemma 2.1, we can prove that

$$|u_k(x_1,t_1) - u_k(x_2,t_2)| \le (C + Ck)(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/2})$$

for all $(x_i,t_i) \in \overline{Q}_\omega (i = 1,2)$, $\alpha \in (0,1)$. Therefore, there exists a function $u \in H^{2,1}(Q_\omega) \cap C^{\alpha,\alpha/2}(\overline{Q}_\omega)$ such that

$$\begin{aligned} &u_k \to u && \text{uniformly in } Q_\omega, \\ &\frac{\partial u_k}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}, \quad D^2 u_k \rightharpoonup D^2 u && \text{weakly in } L^2(Q_\omega), \end{aligned} \tag{3.11}$$

as $k \to 0$. Recalling the equation (1.1), we see that for any $\varphi \in C^2(\overline{Q}_\omega)$ satisfying $\varphi(x,\omega) = \varphi(x,0)$ and $\varphi(0,t) = \varphi(1,t) = 0$ for $t \in [0,\omega]$, we have

$$\iint_{Q_\omega} \frac{\partial u_k}{\partial t} \varphi dxdt - k \iint_{Q_\omega} \frac{\partial u_k}{\partial t} D^2 \varphi dxdt = \iint_{Q_\omega} D^2 u_k \varphi dxdt + \iint_{Q_\omega} m(x,t) u_k^q \varphi dxdt.$$

Taking $k \to 0$, by (3.11), we arrive

$$\iint_{Q_\omega} \frac{\partial u}{\partial t} \varphi dxdt = \iint_{Q_\omega} D^2 u \varphi dxdt + \iint_{Q_\omega} m(x,t) u^q \varphi dxdt,$$

which implies that u satisfies the equation (3.1) in the sense of distribution. It is obvious that u satisfies the conditions (3.2) and (3.3). Therefore, from the classical theory of the parabolic equation, $u(x,t)$ is a nontrivial nonnegative classical time periodic solution of the problem (3.1)–(3.3). The proof of this theorem is complete. □

References

[1] R. E. Showalter and T. W. Ting, Pseudoparabolic partial differential equations, *SIAM J. Math. Anal.*, **1**(1970), 1–26.

[2] T. W. Ting, Parabolic and pseudo-parabolic partial differential equations, *J. Math. Soc. Japan*, **21**(1969), 440–453.

[3] E. I. Kaikina, P. I. Naumkin and I. A. Shishmarev, The Cauchy problem for an equation of Sobolev type with power non-linearity, *Izv. Math.*, **1**(69)(2005), 59–111.

[4] S. L. Sobolev, Some new problems in mathematical physics, *Izv. Akad. Nauk SSSR Ser. Math.*, **18**(1954), 3–50.

[5] G. I. Barenblatt, V .M. Entov, V. M. Ryzhik, Theory of fluid flow through natural rocks, Kluwer, Dordrecht, 1990.

[6] G. I. Barenblatt, Iv. P. Zheltov and I. N. Kochina, Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks, *J. Appl. Math. Mech.*, **24**(1960), 1286–1303.

[7] P. J. Chen and M. E. Gurtin, On a theory of heat conduction involving two temperatures, *Z. Angew. Math. Phys.*, **19**(1968), 614–627.

[8] T. W. Ting, A cooling process according to two-temperature theory of heat conduction, *J. Math. Anal. Appl.*, **45**(1974), 23–31.

[9] V. Padrón, Sobolev regularization of a nonlinear ill-posed parabolic problem as a model for aggregating populations, *Commun. in Partial Differential Equations*, **23**(1998), 457–486.

[10] V. Padrón, Effect of aggregation on population recovery modeled by a forward-backward pseudoparabolic equation, *Trans. Amer. Math. Soc.*, **356**(7)(2004), 2739–2756.

[11] A. Novick-Cohen and R. L. Pego, Stable patterns in a viscous diffusion equation, *Trans. Amer. Math. Soc.*, **324**(1991), 331–351.

[12] D. W. Bange, Periodic solutions of a quasilinear parabolic differential equation, *J. Diff. Eqs.*, **17**(1975), 61–72.

[13] H. Amann, Periodic solutions of semilinear parabolic equations, Nonlinear Analysis, Academic Press, New York, 1978.

[14] P. Hess, M. A. Pozio and A. Tesei, Time periodic solutions for a class of degenerate parabolic problems, *Houston J. Math.*, **21**(2)(1995), 367–394.

[15] M. Badii, Periodic solutions for a class of degenerate evolution problem, *Nonlinear Analysis TMA*, **44**(2001), 499–508.

[16] M. Esteban, On periodic solutions of superlinear parabolic problems, *Trans. Amer. Math. Soc.*, **293**(1986), 171–189.

[17] M. Esteban, A remark on the existence of positive periodic solutions of superlinear parabolic problems, *Proc. Amer. Math. Soc.*, **102**(1988), 131–136.

[18] T. Matahashi, M. Tsutsumi, On a periodic problem for pseudo-parabolic equations of Sobolev-Galpern type, *Math. Japonica,* **22**(1978), 535–553.

[19] T. Matahashi, M. Tsutsumi, Periodic solutions of semilinear pseudo-parabolic equations in Hilbert space, *Funkcialaj Ekvacioj,* **22**(1979), 51–66.

[20] A. Quarteroni, Fourier spectral methods for pseudo-parabolic equations, *SIAM J. Numer. Anal.*, **2**(24)(1987), 323–335.

[21] T. B. Benjamin, J. L. Bona, J. J. Mahong, Model equations for long waves in non-linear dispersive systems, *Philos. R. Soc. Lond. Ser. A,* **272**(1972), 47–78.

[22] J. L. Bona, R. Smith, The initial-value problem for the Korteweg-de Vries equation, *Philos. Trans. Roy. Soc., London, Ser. A,* **278**(1975), 555–601.

[23] L. A. Medeiros, G. Perla Menzala, Existence and uniqueness for periodic solutions of the Benjamin-Bona-Mahony equation, *SIAM J. Math. Anal.,* **8**(1977), 792–799.

[24] Y. L. Zhou, H. Y. Fu, Periodic solutions (boundary problems) and initial value problems for nonlinear pseudo-parabolic systems of higher order, *Scientia Sinica (Series A),* **XXV**(10)(1982), 1021–1031.

[25] E. R. Atamanov, Uniqueness and estimation of the stability of the solution of an ill-posed problem for a pseudoparabolic equation, (Russian) *Studies in integro-differential equations*, **261**(1979), 218–223.

[26] G. M. Amiraliev, Investigation of some difference schemes for a pseudoparabolic equation, (Russian) *Azerbaĭdzhan. Gos. Univ. Uchen. Zap.*, (1)(1979), 29–39.

[27] M. Stecher, W. Rundell, Maximum principle for pseudo-parabolic partial differential equations, *J. Math. Analysis Applic.,* **57**(1977), 110–118.

[28] O. Ladyzenskaja, V. Solonnikov and N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, vol. **23**, American Mathematical Society, Providence, 1968.

[29] E. Mitidieri and S. I. Pohozaev, Nonexistence of positive solutions for quasilinear elliptic problems on $\mathbb{R}^N$, *Proc. Steklov Inst. Math.*, **227**(1999), 186–216.

[30] A. L. Gladkov, Unique solvability of the Cauchy problem for certain quasilinear pseudoparabolic equations, *Mathematical Notes,* **60**3(1996), 264–268.

[31] Z. Q. Wu, J. X. Yin, C. P. Wang, Elliptic & Parabolic Equations, World Scientific, 2006.

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