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OSCILLATION THEOREMS FOR SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper new oscillation criteria for the second order neutral differential equations of the form

$$(E) \qquad \left(r(t)\left[x(t)+p(t)x(\tau(t))\right]'\right)'+q(t)x(\sigma(t))+v(t)x(\eta(t))=0$$

are presented. Gained results are based on the new comparison theorems, that enable us to reduce the problem of the oscillation of the second order equation to the oscillation of the first order equation. Obtained comparison principles essentially simplify the examination of the studied equations. We cover all possible cases when arguments are delayed, advanced or mixed.

1. INTRODUCTION

This paper is concerned with the oscillation behavior of the solutions of the second order neutral differential equations

$$(E) \qquad \left(r(t)\left[x(t)+p(t)x(\tau(t))\right]'\right)'+q(t)x(\sigma(t))+v(t)x(\eta(t))=0,$$

where $q(t), v(t) \in C([t_0,\infty))$, $r(t), p(t), \tau(t), \eta(t), \sigma(t) \in C^1([t_0,\infty))$ and

(H_1) $r(t)>0$, $q(t)>0$, $v(t)>0$, $0 \le p(t) \le p_0 < \infty$;

(H_2) $\lim_{t\to\infty} \sigma(t)=\infty$, $\lim_{t\to\infty} \eta(t)=\infty$;

(H_3) $\tau'(t) \ge \tau_0 > 0$, $\tau\circ\sigma=\sigma\circ\tau$, $\tau\circ\eta=\eta\circ\tau$.

Throughout the paper we shall assume that

$$R(t)=\int_{t_0}^{t}\frac{1}{r(s)}\,\mathrm{d}s \to \infty \text{ as } t\to\infty. \tag{1.1}$$

We set $z(t)=x(t)+p(t)x(\tau(t))$. By a solution of Eq. (E) we mean a function $x(t) \in C([T_x,\infty))$, $T_x \ge t_0$, which has the property $r(t)z'(t) \in C^1([T_x,\infty))$ and satisfies (E) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$ and otherwise, it is said to be nonoscillatory. Equation (E) itself is said to be oscillatory if all its solutions are oscillatory.

Since the second order equations have the applied applications there is the permanent interest in obtaining new sufficient conditions for the oscillation or nonoscillation of the solutions of varietal types of the second order equations. We refer the reader to the papers [1–6, 8, 9, 12–19] and the books [7, 10, 11], and the references cited therein. The authors mainly studied delay equations.

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Grammatikopoulos et al. [9] have showed that $0 \le p(t) \le 1$ together with $\int^\infty q(s)\big(1 - p(s-\sigma)\big)\,\mathrm{d}s = \infty$ guarantee the oscillation of the neutral equation

$$\big(x(t) + p(t)x(t-\tau)\big)'' + q(t)x(t-\sigma) = 0.$$

For the same equation Erbe et al. [7] established the oscillation criterion that requires

$$q(t) \ge q_0 > 0, \quad p_1 \le p(t) \le p_2, \quad p(t) \quad \textit{not eventually negative.}$$

This result has been improved and generalized by other authors. We mention Grace and Lalli [8] who studied the oscillation of

$$\big(r(t)\left[x(t) + p(t)x(t-\tau)\right]'\big)' + q(t)f(x(t-\sigma)) = 0,$$

under the conditions

$$\frac{f(x)}{x} \ge k > 0, \quad \int^\infty \frac{\mathrm{d}s}{r(s)} = \infty,$$

and

$$\int^\infty \rho(s)q(s)\big(1 - p(s-\sigma)\big) - \frac{(\rho'(s))^2 r(s-\sigma)}{4k\rho(s)}\,\mathrm{d}s = \infty,$$

where $\rho(t)$ is an optional function.

Xu and Xia [17] established the oscillation of

$$\big(x(t) + p(t)x(t-\tau)\big)'' + q(t)f(x(t-\sigma)) = 0,$$

provided that

$$0 \le p(t) < \infty, \quad q(t) \ge M > 0.$$

Li at al. [12] studied the neutral differential equation

$$\big(r(t)\left[x(t) + p(t)x(\tau(t))\right]'\big)' + q(t)f(x(\sigma(t))) = 0.$$

They presented new oscillation criteria, where they required $0 \le p(t) \le p_0 < \infty$, $\int^\infty \frac{\mathrm{d}t}{r(t)} = \infty$, $f(x)/x \ge k > 0$, $\sigma'(t) \ge 0$, $\tau \circ \sigma = \sigma \circ \tau$, and inter alia $\sigma(t) \le \tau(t) \le t$, and

$$\int^\infty \rho(s)q(s)\min\{q(s), q(\tau(s))\} - \left(1 + \frac{p_0}{\tau_0}\right)\frac{(\rho'(s))^2 r(\sigma(s))}{4\rho(s)\sigma'(s)}\,\mathrm{d}s = \infty,$$

where $\rho(t)$ is an optional function.

The present authors tried [4] to eliminate the above-mentioned restrictions for the delay equation

$$\big(r(t)\left[x(t) + p(t)x(\tau(t))\right]'\big)' + q(t)x(\sigma(t)) = 0.$$

In this paper we shall investigate the properties of delayed, advanced, and mixed equations. We shall establish new comparison theorems in which we compare the second order equation (E) with the first order differential inequalities in the sense that the absence of the positive solutions of these first order inequalities yields the oscillation of (E). Our technique permits us to eliminate some restrictions that are usually imposed on the coefficients of the studied neutral differential equations.

Remark 1. *The conditions* $\tau \circ \sigma = \sigma \circ \tau$ *and* $\tau \circ \eta = \eta \circ \tau$ *contained in the hypothesis* (H_3) *are satisfied for instance if* $\tau(t)$, $\sigma(t)$, *and* $\eta(t)$ *are of the same form that is if e.g.,* $\tau(t) = \alpha t$, *then at the same time* $\sigma(t) = \beta t$, *and* $\eta(t) = \gamma t$.

Remark 2. *All functional inequalities considered in this paper are assumed to hold eventually, that is they are satisfied for all* t *large enough.*

Remark 3. *Without loss of generality we can deal only with the positive solutions of* (E).

2. Main Results

It follows from (1.1) that the positive solutions of (E) have the following property.

Lemma 1. *If* $x(t)$ *is a positive solution of* (E), *then the corresponding function* $z(t) = x(t) + p(t)x(\tau(t))$ *satisfies*

$$z(t) > 0, \quad r(t)z'(t) > 0, \quad \big(r(t)z'(t)\big)' < 0, \tag{2.1}$$

eventually.

Proof. Assume that $x(t)$ is a positive solution of (E). Then it follows from (E) that

$$\big(r(t)z'(t)\big)' = -q(t)x(\sigma(t)) - v(t)x(\eta(t)) < 0.$$

Consequently, $r(t)z'(t)$ is decreasing and thus either $z'(t) > 0$ or $z'(t) < 0$ eventually. If we let $z'(t) < 0$, then also $r(t)z'(t) < -c < 0$ and integrating this from t_1 to t, we obtain

$$z(t) \le z(t_1) - c\int_{t_1}^{t} \frac{1}{r(s)}\,\mathrm{d}s \to -\infty \text{ as } t \to \infty.$$

This contradicts the positivity of $z(t)$ and the proof is complete. $\square$

For our intended references, let us denote

$$Q(t) = \min\{q(t), q(\tau(t))\}, \quad V(t) = \min\{v(t), v(\tau(t))\}, \tag{2.2}$$

and

$$Q_1(t) = Q(t)\big(R(\sigma(t)) - R(t_1)\big), \quad V_1(t) = V(t)\big(R(\eta(t)) - R(t_1)\big), \tag{2.3}$$

where $t \ge t_1$ and t_1 is large enough.

Theorem 1. *Assume that the first order neutral differential inequality*

$$\left(y(t) + \frac{p_0}{\tau_0}\,y(\tau(t))\right)' + Q_1(t)y(\sigma(t)) + V_1(t)y(\eta(t)) \le 0 \tag{E_2}$$

has no positive solution. Then (E) *is oscillatory.*

Proof. Assume that $x(t)$ is a positive solution of (E). Then the corresponding function $z(t)$ satisfies

$$\begin{aligned} z(\sigma(t)) &= x(\sigma(t)) + p(\sigma(t))x(\tau(\sigma(t))) \\ &\le x(\sigma(t)) + p_0 x(\sigma(\tau(t))), \end{aligned} \tag{2.4}$$

where we have used the hypothesis (H_3) and similarly

$$z(\eta(t)) \leq x(\eta(t)) + p_0 x(\eta(\tau(t))). \tag{2.5}$$

On the other hand, it follows from (E) that

$$\big(r(t)z'(t)\big)' + q(t)x(\sigma(t)) + v(t)x(\eta(t)) = 0 \tag{2.6}$$

and moreover taking (H_1) and (H_3) into account, we have

$$\begin{aligned} 0 &= \frac{p_0}{\tau'(t)} \big(r(\tau(t))z'(\tau(t))\big)' + p_0 q(\tau(t))x(\sigma(\tau(t))) + p_0 v(\tau(t))x(\eta(\tau(t))) \\ &\geq \frac{p_0}{\tau_0} \big(r(\tau(t))z'(\tau(t))\big)' + p_0 q(\tau(t))x(\sigma(\tau(t))) + p_0 v(\tau(t))x(\eta(\tau(t))). \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we are led to

$$\begin{aligned} \big(r(t)z'(t)\big)' &+ \frac{p_0}{\tau_0} \big(r(\tau(t))z'(\tau(t))\big)' + q(t)x(\sigma(t)) + p_0 q(\tau(t))x(\sigma(\tau(t))) \\ &+ v(t)x(\eta(t)) + p_0 v(\tau(t))x(\eta(\tau(t))) \leq 0, \end{aligned}$$

which in view of (2.4), (2.5) and (2.2) provides

$$\big(r(t)z'(t)\big)' + \frac{p_0}{\tau_0} \big(r(\tau(t))z'(\tau(t))\big)' + Q(t)z(\sigma(t)) + V(t)z(\eta(t)) \leq 0. \tag{2.8}$$

It follows from Lemma 1 that $y(t) = r(t)z'(t) > 0$ is decreasing and then

$$\begin{aligned} z(t) &\geq \int_{t_1}^{t} \frac{1}{r(s)} \big(r(s)z'(s)\big)\, \mathrm{d}s \geq y(t) \int_{t_1}^{t} \frac{1}{r(s)}\, \mathrm{d}s \\ &= y(t)\big(R(t) - R(t_1)\big). \end{aligned} \tag{2.9}$$

Therefore, setting $r(t)z'(t) = y(t)$ in (2.8) and utilizing (2.9), one can see that $y(t)$ is a positive solution of (E_2). This contradicts our assumptions and the proof is complete. □

Remark 4. *In the comparison principle in Theorem 1 we do not stipulate whether (E) is equation with delay, advanced or mixed arguments, so that the obtained results are applicable to all three types of equations. Moreover, our results hold also for both cases when $\tau(t) \leq t$ or $\tau(t) \geq t$. On the other hand, the comparison theorem established in Theorem 1 reduces oscillation of (E) to the research of the first order neutral differential inequality (E_2). Therefore, applying the conditions for (E_2) to have no positive solution, we immediately get oscillation criteria for (E).*

Employing the additional conditions on the coefficients of (E), we can deduce from Theorem 1 various oscillation criteria for (E). We shall discuss separately the following two cases:

$$\tau(t) \geq t, \tag{2.10}$$

$$\tau(t) \leq t. \tag{2.11}$$

Theorem 2. *Assume that (2.10) holds. If the first order differential inequality*

$$w'(t) + \frac{\tau_0}{\tau_0 + p_0} Q_1(t)w(\sigma(t)) + \frac{\tau_0}{\tau_0 + p_0} V_1(t)w(\eta(t)) \leq 0 \tag{E_3}$$

has no positive solution, then (E) is oscillatory.

Proof. We assume that $x(t)$ is a positive solution of (E). Then Lemma 1 and the proof of Theorem 1 imply that $y(t) = r(t)z'(t) > 0$ is decreasing and it satisfies (E_2). Let us denote $w(t) = y(t) + \frac{p_0}{\tau_0} y(\tau(t))$. It follows from (2.10) that

$$w(t) \le y(t) \left(1 + \frac{p_0}{\tau_0}\right).$$

Substituting these terms into (E_2), we get that $w(t)$ is a positive solution of (E_3). A contradiction. □

Adding the restriction that both $\sigma(t)$ and $\eta(t)$ are delay arguments, we get easily verifiable oscillation criterion for the delay equation (E).

Corollary 1. *Assume that* (2.10) *holds and*

$$\sigma(t) < t, \quad \eta(t) < t. \tag{2.12}$$

If $\sigma(t) \le \eta(t)$ *and also*

$$\liminf_{t\to\infty} \int_{\eta(t)}^{t} (V_1(s) + Q_1(s))\,\mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0\,\mathrm{e}}, \tag{2.13}$$

or $\sigma(t) \ge \eta(t)$ *and also*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} (V_1(s) + Q_1(s))\,\mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0\,\mathrm{e}}, \tag{2.14}$$

then (E) is oscillatory.

Proof. Theorem 2 ensures the oscillation of (E) provided that (E_3) has no positive solution. Assume that $w(t)$ is a positive solution of (E_3). Then $w(t)$ is decreasing and if $\sigma(t) \le \eta(t)$, then $w(\sigma(t)) \ge w(\eta(t))$. Setting the last inequality to (E_3), we see that $w(t)$ is a positive solution of the differential inequality

$$w'(t) + \frac{\tau_0}{\tau_0 + p_0} \big(Q_1(t) + V_1(t)\big)\, w(\eta(t)) \le 0. \tag{E_3^*}$$

But according to Theorem 2.1.1 from [10] the condition (2.13) guarantees that (E_3^*) has no positive solution. This contradiction finishes the proof of the first part of the corollary. The second part can be verify similarly and so the rest of the proof can be omitted. □

For our incoming references, let us denote

$$Q_2(t) = Q(t)\big(R(t) - R(t_1)\big), \quad V_2(t) = V(t)\big(R(t) - R(t_1)\big), \tag{2.15}$$

where $t \ge t_1$, t_1 is large enough and $Q(t)$ and $V(t)$ are defined in (2.2).

Putting on the constraint that both $\sigma(t)$ and $\eta(t)$ are the advanced arguments, we get the following oscillation criterion for the advanced equation (E).

Theorem 3. *Assume that* (2.10) *holds and*

$$\sigma(t) > t, \quad \eta(t) > t. \tag{2.16}$$

If the first order advanced differential inequality

$$w'(t) - \frac{\tau_0}{\tau_0 + p_0} Q_2(t) w(\sigma(t)) - \frac{\tau_0}{\tau_0 + p_0} V_2(t) w(\eta(t)) \ge 0 \tag{E_4}$$

has no positive solution, then (E) *is oscillatory.*

Proof. We assume that $x(t)$ is a positive solution of (E). Then proceeding exactly as in the proof of Theorem 1, we verify that the corresponding $z(t)$ satisfies (2.8). An integration of (2.8) from t to ∞ provides

$$r(t)z'(t) + \frac{p_0}{\tau_0} r(\tau(t))z'(\tau(t)) \geq \int_t^\infty (Q(s)z(\sigma(s)) + V(s)z(\eta(s)))\,\mathrm{d}s \tag{2.17}$$

Since $r(t)z'(t)$ is decreasing and (2.10) holds, then

$$r(t)z'(t) + \frac{p_0}{\tau_0} r(\tau(t))z'(\tau(t)) \leq r(t)z'(t)\left(1 + \frac{p_0}{\tau_0}\right). \tag{2.18}$$

Combining (2.17) together with (2.18), we are led to

$$r(t)z'(t)\left(1 + \frac{p_0}{\tau_0}\right) \geq \int_t^\infty (Q(s)z(\sigma(s)) + V(s)z(\eta(s)))\,\mathrm{d}s. \tag{2.19}$$

Integrating the last inequality from t_1 to t, we get

$$\begin{aligned} z(t) &\geq \frac{\tau_0}{\tau_0 + p_0} \int_{t_1}^t \frac{1}{r(u)} \int_u^\infty \big(Q(s)z(\sigma(s)) + V(s)z(\eta(s))\big)\,\mathrm{d}s\,\mathrm{d}u \\ &\geq \frac{\tau_0}{\tau_0 + p_0} \int_{t_1}^t \big(Q(s)z(\sigma(s)) + V(s)z(\eta(s))\big) \int_{t_1}^s \frac{1}{r(u)}\,\mathrm{d}u\,\mathrm{d}s. \end{aligned}$$

Hence,

$$z(t) \geq \frac{\tau_0}{\tau_0 + p_0} \int_{t_1}^t \big(Q_2(s)z(\sigma(s)) + V_2(s)z(\eta(s))\big)\,\mathrm{d}s. \tag{2.20}$$

Let us denote the right hand side of (2.20) by $w(t)$. Since $z(t) \geq w(t)$, we see that $w(t)$ is a positive solution of (E_4). This contradicts our assumption and the proof is complete now. $\square$

Corollary 2. *Assume that* (2.10) *and* (2.16) *holds. If* $\sigma(t) \leq \eta(t)$ *and also*

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} (Q_2(s) + V_2(s))\,\mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0\,\mathrm{e}}, \tag{2.21}$$

or $\sigma(t) \geq \eta(t)$ *and also*

$$\liminf_{t\to\infty} \int_t^{\eta(t)} (Q_2(s) + V_2(s))\,\mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0\,\mathrm{e}}, \tag{2.22}$$

then (E) is oscillatory.

Proof. It follows from Theorem 3 that (E) is oscillatory provided that (E_4) has no positive solution. Assume that $w(t)$ is a positive solution of (E_4). Then $w(t)$ is increasing and if $\sigma(t) \leq \eta(t)$, then $w(\sigma(t)) \leq w(\eta(t))$. Setting the last inequality to (E_4), we see that $w(t)$ is a positive solution of the differential inequality

$$w'(t) - \frac{\tau_0}{\tau_0 + p_0}\big(Q_2(t) + V_2(t)\big)\,w(\sigma(t)) \geq 0. \tag{E_4^*}$$

But according to Theorem 2.4.1 from [10] the condition (2.21) guarantees that (E_4^*) has no positive solution. This contradiction finishes the proof of the first part of the corollary. The second part can be verify similarly and so the rest of the proof can be omitted. $\square$

For our ultimate references, let us denote

$$Q_3(t) = \frac{Q(\sigma^{-1}(t))}{\sigma'(\sigma^{-1}(t))}\big(R(t)-R(t_1)\big), \tag{2.23}$$

$$V_3(t) = V_2(t)\exp\left\{\frac{\tau_0}{\tau_0+p_0}\int_t^{\eta(t)} Q_3(s)\,\mathrm{d}s\right\}, \tag{2.24}$$

where $t \ge t_1$, t_1 is large enough, $Q(t)$ is defined in (2.2), while $Q_2(t)$ and $V_2(t)$ are defined by (2.15) and $\sigma^{-1}(t)$ is the inverse function to $\sigma(t)$.

Imposing the assumption that $\sigma(t)$ is the delay and $\eta(t)$ is the advanced argument, we establish the following oscillation criterion for equation (E) with mixed arguments.

Theorem 4. *Assume that* (2.10) *holds and*

$$\sigma'(t) > 0, \quad \sigma(t) \le t, \quad \eta(t) > t. \tag{2.25}$$

If the first order advanced differential inequality

$$w'(t) - \frac{\tau_0}{\tau_0+p_0}V_3(t)w(\eta(t)) \ge 0 \tag{E_5}$$

has no positive solution, then (E) *is oscillatory.*

Proof. We assume that $x(t)$ is a positive solution of (E). Then proceeding exactly as in the proof of Theorem 3, we verify that the corresponding $z(t)$ satisfies (2.19). On the other hand, using the substitution $\sigma(s) = u$, we see that

$$\begin{aligned}\int_t^\infty Q(s)z(\sigma(s))\,\mathrm{d}s &= \int_{\sigma(t)}^\infty \frac{Q(\sigma^{-1}(u))}{\sigma'(\sigma^{-1}(u))}z(u)\,\mathrm{d}u\\ &\ge \int_t^\infty \frac{Q(\sigma^{-1}(u))}{\sigma'(\sigma^{-1}(u))}z(u)\,\mathrm{d}u.\end{aligned} \tag{2.26}$$

Combining (2.19) together with (2.26), one gets

$$r(t)z'(t)\left(1+\frac{p_0}{\tau_0}\right) \ge \int_t^\infty \left(\frac{Q(\sigma^{-1}(s))}{\sigma'(\sigma^{-1}(s))}z(s) + V(s)z(\eta(s))\right)\mathrm{d}s. \tag{2.27}$$

Integrating the last inequality from t_1 to t with applying the similar process as in the proof of Theorem 3, we get

$$z(t) \ge \frac{\tau_0}{\tau_0+p_0}\int_{t_1}^t \big(Q_3(s)z(s) + V_2(s)z(\eta(s))\big)\,\mathrm{d}s. \tag{2.28}$$

Let us denote the right hand side of (2.28) by $y(t)$. Since $z(t) \ge y(t)$, we see that $y(t)$ is a positive solution of

$$y'(t) - \frac{\tau_0}{\tau_0+p_0}Q_3(t)y(t) - \frac{\tau_0}{\tau_0+p_0}V_2(t)y(\eta(t)) \ge 0. \tag{E_6}$$

Now, we set

$$y(t) = \exp\left\{\frac{\tau_0}{\tau_0+p_0}\int_{t_1}^t Q_3(s)\,\mathrm{d}s\right\}w(t).$$

Then in the view of (E_6) it is easy to verify that $w(t)$ is a positive solution of (E_5). This is a contradiction and the proof is complete. $\square$

Corollary 3. *Assume that* (2.10) *and* (2.25) *hold. If*

$$\liminf_{t\to\infty} \int_t^{\eta(t)} V_3(s)\,\mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0\,\mathrm{e}}, \tag{2.29}$$

then (E) is oscillatory.

Proof. Theorem 2.4.1 in [10] implies that the condition (2.29) guarantees that (E_5) has no positive solution and the assertion now follows from Theorem 4. $\square$

Now, we turn our attention to the case when $\tau(t)$ is the delay argument. We shall provide the results analogous to Theorems 2-4.

Theorem 5. *Assume that* (2.11) *holds. If the first order differential inequality*

$$w'(t) + \frac{\tau_0}{\tau_0 + p_0}\, Q_1(t) w(\tau^{-1}(\sigma(t))) + \frac{\tau_0}{\tau_0 + p_0}\, V_1(t) w(\tau^{-1}(\eta(t))) \le 0 \tag{E_7}$$

has no positive solution, then (E) *is oscillatory.*

Proof. We assume that $x(t)$ is a positive solution of (E). Then $y(t) = r(t)z'(t) > 0$ is a decreasing solution of (E_2). We denote $w(t) = y(t) + \frac{p_0}{\tau_0}\, y(\tau(t))$. What is more (2.11) implies

$$w(t) \le y(\tau(t)) \left(1 + \frac{p_0}{\tau_0}\right).$$

Substituting this into (E_2), we get that $w(t)$ is a positive solution of (E_7). A contradiction. $\square$

Corollary 4. *Assume that* (2.11) *holds and*

$$\sigma(t) < \tau(t), \quad \eta(t) < \tau(t). \tag{2.30}$$

If $\sigma(t) \le \eta(t)$ *and also*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\eta(t))}^{t} \big(Q_1(s) + V_1(s)\big)\,\mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0\,\mathrm{e}}, \tag{2.31}$$

or $\sigma(t) \ge \eta(t)$ *and also*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} \big(Q_1(s) + V_1(s)\big)\,\mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0\,\mathrm{e}}, \tag{2.32}$$

then (E) is oscillatory.

Proof. We admit that $w(t)$ is a positive solution of (E_7). If $\sigma(t) \le \eta(t)$, then $w(\tau^{-1}(\sigma(t))) \ge w(\tau^{-1}(\eta(t)))$ and (E_7) gives that $w(t)$ is a solution of the differential inequality

$$w'(t) + \frac{\tau_0}{\tau_0 + p_0}\, \big(Q_1(t) + V_1(t)\big)\, w(\tau^{-1}(\eta(t))) \le 0. \tag{E_7^*}$$

But according to Theorem 2.1.1 from [10] the condition (2.31) guarantees that (E_7^*) has no positive solution. Therefore (E_7) has no positive solution and Theorem 5 provides the oscillation of (E). The case $\sigma(t) \ge \eta(t)$ can be treated similarly. $\square$

For our future references, let us denote

$$Q_4(t) = Q(t)\big(R(\tau(t)) - R(\tau(t_1))\big), \quad V_4(t) = V(t)\big(R(\tau(t)) - R(\tau(t_1))\big), \tag{2.33}$$

where $t \geq t_1$, t_1 is large enough and $Q(t)$ and $V(t)$ are defined in (2.2).

Theorem 6. *Assume that* (2.11) *holds and*

$$\sigma(t) > \tau(t), \quad \eta(t) > \tau(t). \tag{2.34}$$

If the first order advanced differential inequality

$$w'(t) - \frac{\tau_0}{\tau_0 + p_0} Q_4(t) w(\tau^{-1}(\sigma(t))) - \frac{\tau_0}{\tau_0 + p_0} V_4(t) w(\tau^{-1}(\eta(t))) \geq 0 \tag{E_8}$$

has no positive solution, then (E) *is oscillatory.*

Proof. We assume that $x(t)$ is a positive solution of (E). Then it follows from the proof of Theorem 1, that the corresponding function $z(t)$ satisfies (2.17). Since $r(t)z'(t)$ is decreasing and (2.11) holds, then

$$r(t)z'(t) + \frac{p_0}{\tau_0} r(\tau(t))z'(\tau(t)) \leq r(\tau(t))z'(\tau(t)) \left(1 + \frac{p_0}{\tau_0}\right). \tag{2.35}$$

Combining (2.17) together with (2.35), we obtain

$$r(\tau(t))z'(\tau(t)) \left(1 + \frac{p_0}{\tau_0}\right) \geq \int_t^\infty (Q(s)z(\sigma(s)) + V(s)z(\eta(s)))\,\mathrm{d}s. \tag{2.36}$$

Multiplying the last inequality by $\tau'(t)/r(\tau(t))$ and then integrating the result from t_1 to t, we get

$$\begin{aligned} z(\tau(t)) &\geq \frac{\tau_0}{\tau_0 + p_0} \int_{t_1}^t \frac{\tau'(u)}{r(\tau(u))} \int_u^\infty \big(Q(s)z(\sigma(s)) + V(s)z(\eta(s))\big)\,\mathrm{d}s\,\mathrm{d}u \\ &\geq \frac{\tau_0}{\tau_0 + p_0} \int_{t_1}^t \big(Q(s)z(\sigma(s)) + V(s)z(\eta(s))\big) \int_{t_1}^s \frac{\tau'(u)}{r(\tau(u))}\,\mathrm{d}u\,\mathrm{d}s. \end{aligned}$$

Hence,

$$z(\tau(t)) \geq \frac{\tau_0}{\tau_0 + p_0} \int_{t_1}^t \big(Q_4(s)z(\sigma(s)) + V_4(s)z(\eta(s))\big)\,\mathrm{d}s. \tag{2.37}$$

Let us denote the right hand side of (2.37) by $w(t)$. Since $z(t) \geq w(t)$, we see that $w(t)$ is a positive solution of (E_8). This contradicts our assumption and the proof is complete now. □

Remark 5. *The assumptions imposed in Theorem 6 do not require for $\sigma(t)$ and $\eta(t)$ to be advanced arguments. We only need for $\tau^{-1}(\sigma(t))$ and $\tau^{-1}(\eta(t))$ to be advanced arguments. So the conclusions of Theorem 6 hold for all types of equations i.e., advanced, delay, with mixed arguments and even if $t - \sigma(t)$ or $t - \eta(t)$ oscillates.*

Corollary 5. *Assume that* (2.11) *and* (2.34) *hold. If $\sigma(t) \leq \eta(t)$ and also*

$$\liminf_{t\to\infty} \int_t^{\tau^{-1}(\sigma(t))} \big(Q_4(s) + V_4(s)\big)\,\mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0\,\mathrm{e}}, \tag{2.38}$$

or $\sigma(t) \geq \eta(t)$ and also

$$\liminf_{t\to\infty} \int_t^{\tau^{-1}(\eta(t))} \big(Q_4(s) + V_4(s)\big)\,\mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0\,\mathrm{e}}, \tag{2.39}$$

then (E) is oscillatory.

Proof. We let $w(t)$ to be a positive solution of (E_8). If $\sigma(t) \leq \eta(t)$, then $w(\tau^{-1}(\sigma(t))) \leq w(\tau^{-1}(\eta(t)))$ and (E_8) implies that $w(t)$ satisfies

$$w'(t) - \frac{\tau_0}{\tau_0 + p_0}\big(Q_4(t) + V_4(t)\big)\, w(\tau^{-1}(\sigma(t))) \geq 0. \tag{E_8^*}$$

But according to Theorem 2.4.1 from [10] the condition (2.38) guarantees that (E_8^*) has no positive solution. This contradiction ensures that (E_8) has no positive solution and taking Theorem 6 into account, we see that (E) is oscillatory. The case $\sigma(t) \geq \eta(t)$ is left to the reader. □

For our incoming references, let us denote

$$Q_5(t) = \frac{Q(\sigma^{-1}(\tau(t)))}{\sigma'(\sigma^{-1}(\tau(t)))}\,\tau'(t)\big(R(\tau(t)) - R(\tau(t_1))\big), \tag{2.40}$$

$$V_5(t) = V_4(t)\exp\left\{\frac{\tau_0}{\tau_0 + p_0}\int_t^{\tau^{-1}(\eta(t))} Q_5(s)\,\mathrm{d}s\right\}, \tag{2.41}$$

where $t \geq t_1$, t_1 is large enough, $Q(t)$ is defined in (2.2), while $V_4(t)$ is defined by (2.33).

Theorem 7. *Assume that* (2.11) *holds and*

$$\sigma'(t) > 0, \quad \sigma(t) \leq \tau(t), \quad \eta(t) > \tau(t). \tag{2.42}$$

If the first order advanced differential inequality

$$w'(t) - \frac{\tau_0}{\tau_0 + p_0}\, V_5(t) w(\tau^{-1}(\eta(t))) \geq 0 \tag{E_9}$$

has no positive solution, then (E) *is oscillatory.*

Proof. We assume that $x(t)$ is a positive solution of (E). Then proceeding exactly as in the proof of Theorem 6, we verify that the corresponding $z(t)$ satisfies (2.36). On the other hand, using the substitution $\sigma(s) = \tau(u)$, we see that

$$\begin{aligned}\int_t^\infty Q(s) z(\sigma(s))\,\mathrm{d}s &= \int_{\tau^{-1}(\sigma(t))}^\infty \frac{Q(\sigma^{-1}(\tau(u)))}{\sigma'(\sigma^{-1}(\tau(u)))}\,\tau'(u) z(\tau(u))\,\mathrm{d}u \\ &\geq \int_t^\infty \frac{Q(\sigma^{-1}(\tau(u)))}{\sigma'(\sigma^{-1}(\tau(u)))}\,\tau'(u) z(\tau(u))\,\mathrm{d}u. \end{aligned} \tag{2.43}$$

Combining (2.36) together with (2.43), one gets

$$\begin{aligned} & r(\tau(t)) z'(\tau(t))\left(1 + \frac{p_0}{\tau_0}\right) \geq \\ & \int_t^\infty \left(\frac{Q(\sigma^{-1}(\tau(u)))}{\sigma'(\sigma^{-1}(\tau(u)))}\,\tau'(u) z(\tau(u)) + V(s) z(\eta(s))\right)\mathrm{d}s. \end{aligned} \tag{2.44}$$

Multiplying the last inequality by $\tau'(t)/r(\tau(t))$ and then integrating the resulting inequality from t_1 to t, and using the similar process as in the proof of Theorem 6, we get

$$z(\tau(t)) \geq \frac{\tau_0}{\tau_0 + p_0} \int_{t_1}^{t} \big(Q_5(s) z(\tau(s)) + V_4(s) z(\eta(s))\big)\,\mathrm{d}s. \tag{2.45}$$

Let us denote the right hand side of (2.45) by $y(t)$. Since $z(\tau(t)) \geq y(t)$, we see that $y(t)$ is a positive solution of

$$y'(t) - \frac{\tau_0}{\tau_0 + p_0} Q_5(t) y(t) - \frac{\tau_0}{\tau_0 + p_0} V_4(t) y(\tau^{-1}(\eta(t))) \geq 0. \tag{E_{10}}$$

Now setting

$$y(t) = \exp\left\{ \frac{\tau_0}{\tau_0 + p_0} \int_{t_1}^{t} Q_5(s)\,\mathrm{d}s \right\} w(t),$$

we see in the view of (E_{10}) that $w(t)$ is a positive solution of (E_9). This is a contradiction and the proof is complete. $\square$

Corollary 6. *Assume that* (2.11) *and* (2.42) *hold. If*

$$\liminf_{t\to\infty} \int_{t}^{\tau^{-1}(\eta(t))} V_5(s)\,\mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0\,\mathrm{e}}, \tag{2.46}$$

then (E) is oscillatory.

Proof. According to Theorem 2.4.1 from [10] the condition (2.46) guarantees that (E_9) has no positive solution and the assertion now follows from Theorem 7. $\square$

Example 1. *We consider the second order neutral differential equation*

$$\left(t^{1/2}\left[x(t) + p_0 x(\tau_0 t)\right]'\right)' + \frac{a}{t^{3/2}} x(\alpha t) + \frac{b}{t^{3/2}} x(\beta t) = 0, \tag{E_{11}}$$

where $p_0, \tau_0, \alpha, \beta, a, b$ are positive constants.

If $\tau_0 \geq 1$, then $Q(t) = q(\tau(t)) = a(\tau_0 t)^{-3/2}$ and $V(t) = v(\tau(t)) = b(\tau_0 t)^{-3/2}$. It follows from Corollaries 1-3 that (E_{11}) is oscillatory provided that at least one of the following conditions is satisfied:

$$\alpha, \beta < 1 \qquad \text{and} \qquad -(a\sqrt{\alpha} + b\sqrt{\beta}) \ln\big(\max\{\alpha,\beta\}\big) > \frac{(\tau_0 + p_0)\sqrt{\tau_0}}{2\,\mathrm{e}},$$

$$\alpha, \beta > 1 \qquad \text{and} \qquad (a + b) \ln\big(\min\{\alpha,\beta\}\big) > \frac{(\tau_0 + p_0)\sqrt{\tau_0}}{2\,\mathrm{e}},$$

$$\alpha \leq 1 < \beta \qquad \text{and} \qquad b\beta^{\overline{\frac{2a\sqrt{\alpha}}{(\tau_0 + p_0)\sqrt{\tau_0}}}} \ln \beta > \frac{(\tau_0 + p_0)\sqrt{\tau_0}}{2\,\mathrm{e}}.$$

If $\tau_0 \leq 1$, then $Q(t) = q(t) = at^{-3/2}$ and $V(t) = q(t) = bt^{-3/2}$. It follows from Corollaries 4-6 that (E_{11}) is oscillatory provided that at least one of the following *conditions is satisfied:*

$$\alpha,\beta<\tau_0 \qquad \textit{and} \qquad (a\sqrt{\alpha}+b\sqrt{\beta})\ln\left(\frac{\tau_0}{\max\{\alpha,\beta\}}\right)>\frac{\tau_0+p_0}{2\tau_0\mathrm{e}},$$

$$\alpha,\beta>\tau_0 \qquad \textit{and} \qquad (a+b)\ln\left(\frac{\min\{\alpha,\beta\}}{\tau_0}\right)>\frac{\tau_0+p_0}{2\tau_0^{3/2}\mathrm{e}},$$

$$\alpha\le\tau_0<\beta \qquad \textit{and} \qquad b\left(\frac{\beta}{\tau_0}\right)^{\frac{2\tau_0 a\sqrt{\alpha}}{\tau_0+p_0}}\ln\frac{\beta}{\tau_0}>\frac{\tau_0+p_0}{2\tau_0^{3/2}\mathrm{e}}.$$

Consequently, we have covered oscillation of (E_{11}) *for all* $\tau_0\in(0,\infty)$ *that is for* $\tau(t)=\tau_0 t$ *to be delay or advanced argument. Note that all above mentioned known oscillatory criteria fail for* (E_{11})*.*

3. Summary

In this paper we have introduced new comparison theorems for investigation of the oscillation of (E). The established comparison principles reduce oscillation of the second order neutral equations to studying properties of various types of the first order differential inequalities, which essentially simplifies examination of (E). Our technique permits to relax restrictions usually imposed on the coefficients of (E). So that our results are of high generality. Obtained results are easily applicable and are illustrated on a suitable example.

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