# On the superlinear problem involving the $p(x)$-Laplacian

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**Abstract**

This paper deals with the superlinear elliptic problem without Ambrosetti and Rabinowitz type growth condition of the form:

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset R^N (N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter. Existence of nontrivial solution is established for arbitrary $\lambda > 0$. Firstly, by using the mountain pass theorem a nontrivial solution is constructed for almost every parameter $\lambda > 0$. Then, it is considered the continuation of the solutions. Our results are a generalization of Miyagaki and Souto.

**2000 Mathematics Subject Classification:** 35J60, 58E30

**Keywords**: Superlinear problem; $p(x)$-Laplacian; Variational method; Variable exponent spaces

## 1 Introduction

In this paper we consider the following nonlinear eigenvalue problem involving the $p(x)$-Laplacian:

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N (N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $1 < p(x) \in C(\overline{\Omega})$, $f \in C(\overline{\Omega} \times R)$ is superlinear and don't satisfy Ambrosetti and Rabinowitz type growth condition, $\lambda > 0$ is a parameter.

Fan and Zhang in [1] established an existence of nontrivial solution for problem (1.1), by assuming the following conditions:

$(f_0)$ $f : \Omega \times R \to R$ satisfies Caratheodory condition and

$$|f(x,t)| \leq C_1 + C_2|t|^{\alpha(x)-1}, \qquad \forall (x,t) \in \Omega \times R,$$

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where $\alpha(x) \in C_+(\overline{\Omega}) = \{h | h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\}$ and $\alpha(x) < p^*(x)$, $p^*(x)$ is the Sobolev critical exponent and

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)}, & p(x) < N, \\ \infty, & p(x) \geq N. \end{cases}$$

$(f_1)$ $\exists M > 0$, $\theta > p^+ := \max_{\overline{\Omega}} p(x)$ such that

$$0 < \theta F(x,t) \leq t f(x,t), \qquad |t| \geq M, x \in \Omega,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

$(f_2)$ $f(x,t) = o(|t|^{p^+-1})$, $t \to 0$, for $x \in \Omega$ uniformly and $\alpha^- := \min_{\overline{\Omega}} \alpha(x) > p^+$.

When $p(x) \equiv 2$, several researchers that studied problem (1.1) tried to drop above condition $(f_1)$(see [2, 3, 4, 5]), that is
$(f_1')$ $\exists M > 0$, $\theta > 2$ such that

$$0 < \theta F(x,t) \leq t f(x,t), \qquad |t| \geq M, x \in \Omega,$$

where $F(x,t) = \int_0^t f(x,s)ds$.
$(f_1')$ is the famous Ambrosetti and Rabinowitz growth condition and $(f_1)$ is a generalization of $(f_1')$ to problem involving the $p(x)$-Laplacian, here we call it Ambrosetti and Rabinowitz type grow condition. For the case $p(x) \equiv p$, we may refer [6]. It's well known (see [1]) that $(f_1)$ is quite important not only to ensure that the Euler-lagrange functional associated to problem (1.1) has a mountain pass geometry, but also to guarantee that Palais-Smale sequence of the Euler-Lagrange functional is bounded. But this condition is very restrictive eliminating many nonlinearities. We recall that $(f_1)$ implies a weaker condition

$$F(x,t) \geq c_1|t|^\theta - c_2, \qquad c_1, c_2 > 0, x \in \Omega, t \in R \text{ and } \theta > p^+.$$

The above condition implies another much weaker condition, which is a consequence of the superlinearity of $f$ at infinity:
$(f_3)$

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{p^+}} = +\infty, \quad \text{uniformly } a.e.\ x \in \Omega.$$

When $p(x) \equiv 2$, under conditions $(f_0)$, $(f_2)$, $(f_3)$ and the following condition:
$(f_4')$ There is $t_0 > 0$ such that

$$\frac{f(x,t)}{t} \text{ is increasing in } t \geq t_0 \text{ and decreasing in } t \leq -t_0,\ \forall x \in \Omega,$$

if $f \in C(\overline{\Omega} \times R)$, Miyagaki and Souto in [3] got a nontrivial solution of problem (1.1), for all $\lambda > 0$. Here we will generalize results in [3] to the variable exponent case. Because the $p(x)$-Laplacian possesses more complicated nonlinearities than Laplacian and $p$-laplacian, for example, it is inhomogeneous, thus our problem is the more difficult.

The following is our main result, namely,
**Theorem 1.1.** *Under hypotheses* $(f_0)$, $(f_2)$, $(f_3)$ *and*
$(f_4)$ There is $t_0 > 0$ such that

$$\frac{f(x,t)}{t^{p^+-1}} \text{ is increasing in } t \geq t_0 \text{ and decreasing in } t \leq -t_0, \forall x \in \Omega.$$

*Moreover,* $f \in C(\overline{\Omega} \times R)$, *then problem* (1.1) *has a nontrivial weak solution, for all* $\lambda > 0$.

**Example 1.1.** *Function* $f(x,t) = t^{\alpha(x)-1}(\alpha(x)\ln t + 1)(F(x,t) = t^{\alpha(x)}\ln t)$ *where* $\alpha(x) \in C_+(\overline{\Omega})$ *satisfies condition* $(f_4)$ , *but it does not satisfy* $(f_1)$ *if* $2\alpha^- > p^+ > \alpha^+$.

**Remark 1.1.** *Actually our result still holds if we consider a weaker condition than* $(f_4)$, *namely*
$(f_4')$ *There is* $C_* > 0$ *such that*

$$tf(x,t) - p^+F(x,t) \leq sf(x,s) - p^+F(x,s) + C_*$$

*for all* $0 < t < s$ *or* $s < t < 0$.

The variational problems and differential equations with nonstandard growth conditions have been a very attractive topic in recent years. We refer to [7, 8] for applied background, to [9, 10] for the variable exponent Lebesgue-Sobolev spaces and to [1, 11, 12, 13, 14] for the $p(x)$-Laplacian equations and the corresponding variational problems.

The paper is divided into three sections. In Section 2 we present some preliminary knowledge on the variable exponent spaces. In Section 3, we give some preliminary lemmas and the proof of Theorem 1.1.

# 2 Preliminary

Throughout this paper, we always assume $p(x) \in C_+(\overline{\Omega})$ and $f \in C(\overline{\Omega} \times R)$. Set

$$L^{p(x)}(\Omega) = \{u \mid u \text{ is a measurable real-valued function} : \int_\Omega |u|^{p(x)}dx < \infty\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u}{\lambda}|^{p(x)}dx \leq 1\}$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space, that is generalized Lebesgue space.

**Proposition 2.1**([1])**.**
**(1)** *The space* $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ *is separable, uniform convex Banach space, and its conjugate space is* $L^{q(x)}(\Omega)$ *where* $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. *For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have*

$$\Big|\int_\Omega uvdx\Big| \leq (\frac{1}{p^-} + \frac{1}{q^-})|u|_{p(x)}|v|_{q(x)}.$$

**(2)** *If* $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$ *for any* $x \in \overline{\Omega}$, *then* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ *and the imbedding is continuous.*

**Proposition 2.2**([1], [9], [10])**.** *Set* $\rho(u) = \int_\Omega |u(x)|^{p(x)}dx$. *If* $u, u_k \in L^{p(x)}(\Omega)$, *we have*
**(1)** *For* $u \neq 0$, $|u|_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$.
**(2)** $|u|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$.
**(3)** *If* $|u|_{p(x)} > 1$, *then* $|u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$.
**(4)** *If* $|u|_{p(x)} < 1$, *then* $|u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$.
**(5)** $\lim_{k\to\infty} |u_k|_{p(x)} = 0 \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = 0$.
**(6)** $\lim_{k\to\infty} |u_k|_{p(x)} = \infty \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = \infty$.

The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\}$$

and it can be equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Moreover, we have
**Proposition 2.3**([1])**.**
**(1)** $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable, reflexive Banach spaces;*
**(2)** *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$, *then the imbedding from* $W^{1,p(x)}(\Omega)$ *to* $L^{q(x)}(\Omega)$ *is compact and continuous;*
**(3)** *There is constant* $C > 0$, *such that*

$$|u|_{p(x)} \le C|\nabla u|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

By (3) of Proposition 2.3, we know that $|\nabla u|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. We will use $|\nabla u|_{p(x)}$ to replace $\|u\|$ in the following discussions.

# 3 Main Results

Now we introduce the energy functional $I_\lambda : W_0^{1,p(x)}(\Omega) \to R$ associated with problem (1.1), defined by

$$I_\lambda(u) = \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx - \lambda \int_\Omega F(x,u)dx.$$

From the hypotheses on $f$, it is standard to check that $I_\lambda \in C^1(W_0^{1,p(x)}(\Omega), R)$ and its Gateaux derivative is

$$I'_\lambda(u) \cdot v = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v - \lambda \int_\Omega f(x,u) v dx, \ u, v \in W_0^{1,p(x)}(\Omega).$$

Thus the critical points of $I_\lambda$ are precisely the weak solutions of problem (1.1).

First of all, notice that $I_\lambda$ verifies the mountain pass geometry, in a uniform way on compact sets:

**Lemma 3.1.**
**(1)** *Under the condition* $(f_3)$*, the functional* $I_\lambda$ *is unbounded from below;*
**(2)** *Under the conditions* $(f_0)$ *and* $(f_2)$*,* $u = 0$ *is a strict local minimum for the functional* $I_\lambda$ *.*

**Proof of (1).** From $(f_3)$ follows that, for all $M > 0$ there exists $C_M > 0$, such that

$$F(x,t) \geq M|t|^{p^+} - C_M, \qquad \forall x \in \Omega, \forall t > 0. \tag{3.1}$$

Take $\phi \in W_0^{1,p(x)}(\Omega)$ with $\phi > 0$, from (3.1) we obtain

$$I_\lambda(t\phi) \leq t^{p^+}\Big(\int_\Omega \frac{|\nabla\phi|^{p(x)}}{p(x)} - \lambda M \int_\Omega |\phi|^{p^+}\Big) + C_M|\Omega|,$$

where $t \geq 1$ and $|\Omega|$ denotes the Lebesgue measure of $\Omega$. If $M$ is large, then

$$\lim_{t\to\infty} I_\lambda(t\phi) = -\infty.$$

This proves (1).

**Proof of (2).** From $(f_0)$ and $(f_2)$, we have

$$F(x,t) \leq \epsilon|t|^{p^+} + C(\epsilon)|t|^{\alpha(x)}, \ \forall (x,t) \in \Omega \times R.$$

Then

$$\begin{aligned} I_\lambda(u) &\geq \int_\Omega \frac{1}{p^+}|\nabla u|^{p^+}dx - \epsilon\lambda\int_\Omega |u|^{p^+}dx - C(\epsilon)\lambda\int_\Omega |u|^{\alpha(x)}dx \\ &\geq \frac{1}{p^+}\|u\|^{p^+} - \epsilon\lambda C_0^{p^+}\|u\|^{p^+} - C(\epsilon)\lambda\|u\|^{\alpha^-} \\ &\geq \frac{1}{2p^+}\|u\|^{p^+} - \lambda C(\epsilon)\|u\|^{\alpha^-}, \ \text{when } \|u\| \leq 1, \end{aligned}$$

there exist $r > 0$ and $\delta > 0$ such that $I_\lambda(u) \geq \delta > 0$ for every $u \in W_0^{1,p(x)}(\Omega)$ and $\|u\| = r$. The proof is complete.

Fix $0 < \lambda_0 < \mu_0$. Now, we can see that the geometry on $I_\lambda$ works uniformly on $[\lambda_0, \mu_0]$. From the proof of Lemma 3.1 (2), we obtain

$$I_\lambda(u) \geq \frac{1}{2p^+}\|u\|^{p^+} - \mu_0 C(\epsilon)\|u\|^{\alpha^-}, \text{ when } \|u\| \leq 1, 0 < \lambda \leq \mu_0.$$

That is, there exist $r > 0$ and $\delta > 0$ such that $I_\lambda(u) \geq \delta > 0$ for every $u \in W_0^{1,p(x)}(\Omega)$, $\|u\| = r$ and $\forall\lambda \leq \mu_0$.

By choosing $e \in W_0^{1,p(x)}(\Omega)$ such that $I_{\lambda_0}(e) < 0$, we infer that

$$\frac{I_\lambda(e)}{\lambda} \leq \frac{I_{\lambda_0}(e)}{\lambda_0} < 0, \quad \lambda_0 \leq \lambda \leq \mu_0.$$

We also have

$$\frac{I_\lambda(u)}{\lambda} \leq \frac{I_\mu(u)}{\mu}, \quad \forall u \in W_0^{1,p(x)}(\Omega), \mu < \lambda. \tag{3.2}$$

Define

$$P = \{\gamma : [0,1] \to W_0^{1,p(x)}(\Omega) : \gamma \,\text{is continuous and}\, \gamma(0) = 0 \,\text{and}\, \gamma(1) = e\},$$

and for $\lambda_0 \leq \lambda \leq \mu_0$, let

$$c_\lambda = \inf_{\gamma \in P} \max_{t \in [0,1]} I_\lambda(\gamma(t)).$$

We recall that the map $c : [\lambda_0, \mu_0] \to R_+$, given by $c(\lambda) = c_\lambda$, is such that $\frac{c_\lambda}{\lambda}$ is decreasing, left semi-continuous and bounded from below by $c_{\mu_0} > 0$.

In fact, from (3.2) follows the monotonicity. While the estimate in Lemma 3.1 (2) implies that $c_\lambda \geq \delta > 0$.

Now, we check the left semi-continuous of $\frac{c_\lambda}{\lambda}$. Fix $\mu \in [\lambda_0, \mu_0]$ and $\epsilon > 0$. Then fix $\gamma \in P$ such that

$$c(\mu) \leq \max_{t \in [0,1]} I_\mu(\gamma(t)) \leq c(\mu) + \frac{\epsilon\mu}{4}.$$

Let $R_0 = \max_{t \in [0,1]} \int_\Omega F(x, \gamma(t))dx$. Then, for $\lambda > \frac{\mu}{2}$ and such that $\frac{1}{\lambda} < \frac{1}{\mu} + \frac{\epsilon}{2\mu}$,

$$\begin{aligned} I_\lambda(\gamma(t)) &= (I_\lambda(\gamma(t)) - I_\mu(\gamma(t))) + I_\mu(\gamma(t)) \\ &= I_\mu(\gamma(t)) + (\mu - \lambda) \int_\Omega F(x, \gamma(t))dx \\ &\leq R_0|\lambda - \mu| + c_\mu + \frac{\epsilon\mu}{4}, \ \forall t \in [0,1], \end{aligned}$$

that is,

$$c(\lambda) \leq c(\mu) + \frac{\epsilon\mu}{2}, \text{ if } |\lambda - \mu| < \frac{\epsilon\mu}{4R_0}.$$

Hence, if $\mu > \lambda$, it follows that

$$\frac{c_\mu}{\mu} - \epsilon < \frac{c_\mu}{\mu} \leq \frac{c_\lambda}{\lambda} \leq \frac{c_\mu}{\lambda} + \frac{2\epsilon}{3} \leq \frac{c_\mu}{\mu} + \epsilon.$$

This proves the left semi-continuity of $\frac{c_\lambda}{\lambda}$ and $c_\lambda$.

**Lemma 3.2.** *There exists $d > 0$, such that*

$$\|I'_\mu(u) - I'_\lambda(u)\|_* \leq d(1 + \|u\|^{\alpha^+ - 1})|\mu - \lambda|, \forall \lambda, \mu > 0.$$

**Proof.** For $\alpha(x) \in C_+(\overline{\Omega})$, define $\alpha'(x)$ such that $\frac{1}{\alpha(x)} + \frac{1}{\alpha'(x)} = 1$ for $\forall x \in \overline{\Omega}$. From condition $(f_0)$, one has

$$|f(x,t)|^{\alpha'(x)} = |f(x,t)|^{\frac{\alpha(x)}{\alpha(x)-1}} \leq d_1 + d_2|t|^{\alpha(x)}, \forall x \in \Omega, \forall t \in R,$$

for some constants $d_1$, $d_2 > 0$ and then

$$\int_\Omega |f(x,u)|^{\alpha'(x)} \leq d_1|\Omega| + d_2 \int_\Omega |u|^{\alpha(x)}dx.$$

Therefore, there exist positive constants $d_3$ and $d_4 > 0$, such that

$$\int_\Omega |f(x,u)|^{\alpha'(x)} \leq d_3 + d_4\|u\|^{\alpha^+}, \ \forall u \in W_0^{1,p(x)}(\Omega).$$

Now, for all $v \in W_0^{1,p(x)}(\Omega)$ with $\|v\| \le 1$, we have

$$I'_\mu(u)v - I'_\lambda(u)v = (\lambda - \mu)\int_\Omega f(x,u)vdx.$$

Moreover, one has

$$\begin{aligned} |I'_\mu(u)v - I'_\lambda(u)v| &\le |\lambda-\mu|\int_\Omega |f(x,u)v|dx \\ &\le 2|\lambda-\mu||f(x,u)|_{\alpha'(x)}|v|_{\alpha(x)} \\ &\le 2C_0|\lambda-\mu|(d_3 + d_4\|u\|^{\alpha^+})^{\frac{\alpha^+-1}{\alpha^+}}\|v\|. \end{aligned}$$

So there exists constant $d > 0$ such that

$$\|I'_\mu(u) - I'_\lambda(u)\|_* \le d(1 + \|u\|^{\alpha^+-1})|\mu-\lambda|, \ \forall \lambda, \mu > 0.$$

**Remark 3.1.** *We recall that the map $b : [\lambda_0, \mu_0] \to R_+$, given by $b(\lambda) = \frac{c_\lambda}{\lambda}$, is monotone decreasing. Thus $b_\lambda$ and $c_\lambda$ are differentiable at almost all values $\lambda \in (\lambda_0, \mu_0)$.*

**Lemma 3.3.** *Suppose the map $c : [\lambda_0, \mu_0] \to R_+$, given by $c(\lambda) = c_\lambda$, is differentiable in $\mu$, then there exists a sequence $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ such that*

$$I_\mu(u_n) \to c_\mu, \qquad I'_\mu(u_n) \to 0, \quad \text{and} \quad \|u_n\|^{p^-} \le C',$$

*as $n \to \infty$ and actually $C' = p^+c_\mu + p^+\mu(2 - c'(\mu)) + 1$.*

The proof of the Lemma is similar to the proof of Lemma 2.3 in [3], so omit it.

The next lemma follows directly Lemma 3.3.

**Lemma 3.4.** *For almost all $\lambda > 0$, $c_\lambda$ is a critical value for $I_\lambda$.*

Combining above Lemmas and arguments, now we give the proof of Theorem 1.1.

**Proof.** As $c_\lambda$ is left semi-continuous, from Lemma 3.4, for each $\mu > 0$ we can fix sequence $\{u_n\}$ in $W_0^{1,p(x)}(\Omega)$ and $\{\lambda_n\} \subset R$ such that $\lambda_n \to \mu$, $c_{\lambda_n} \to c_\mu$ as $n \to \infty$,

$$I_{\lambda_n}(u_n) = c_{\lambda_n} \quad \text{and} \quad I'_{\lambda_n}(u_n) = 0.$$

For the proof of Theorem, it is enough that one can prove that the sequence $\{u_n\}$ is bounded. If it is unbounded we define $\omega_n = \frac{u_n}{\|u_n\|}$. Without loss of generality, suppose that there is $\omega \in W_0^{1,p(x)}(\Omega)$ such that

$$\omega_n(x) \rightharpoonup \omega(x) \quad \text{in } W_0^{1,p(x)}(\Omega),\ n \to \infty,$$

$$\omega_n(x) \to \omega(x) \quad \text{in } L^{\alpha(x)}(\Omega),\ n \to \infty,$$

$$\omega_n(x) \to \omega(x) \quad \text{for } a.e. x \in \Omega,\ n \to \infty.$$

Let $\Omega_{\neq} = \{x \in \Omega : \omega(x) \neq 0\}$. If $x \in \Omega_{\neq}$, then

$$\lim_{n\to\infty} \frac{F(x, u_n(x))}{|u_n(x)|^{p^+}}|\omega_n(x)|^{p^+} = \infty.$$

Applying the Fatou Lemma and the limit

$$\lim_{n\to\infty}\int_\Omega \frac{F(x,u_n(x))}{|u_n(x)|^{p^+}}|\omega_n(x)|^{p^+} \le \frac{1}{\mu p^-}.$$

These two last limits are incompatible if $|\Omega_{\neq}| > 0$, so $\Omega_{\neq}$ has zero measure, that is $\omega = 0$ a.e. in $\Omega$.
Let $t_n \in [0,1]$ such that

$$I_{\lambda_n}(t_n u_n) = \max_{t\in[0,1]} I_{\lambda_n}(tu_n).$$

If $t_n = 1$, $I_{\lambda_n}(tu_n)$ is bounded for all $t \in [0,1]$. If $t_n < 1$, $I'_{\lambda_n}(t_n u_n)u_n = 0$. Since $I'_{\lambda_n}(t_n u_n)(t_n u_n) = 0$, from $(f_4')$, we have

$$\begin{aligned}
I_{\lambda_n}(tu_n) &\le I_{\lambda_n}(t_n u_n) - \frac{1}{p^+}I'_{\lambda_n}(t_n u_n)(t_n u_n)\\
&= \int_\Omega (\frac{1}{p(x)} - \frac{1}{p^+})|\nabla t_n u_n|^{p(x)}dx\\
&+ \lambda_n \int_\Omega (\frac{1}{p^+}t_n u_n f(x,t_n u_n) - F(x,t_n u_n))dx\\
&\le \int_\Omega (\frac{1}{p(x)} - \frac{1}{p^+})|\nabla u_n|^{p(x)}dx\\
&+ \lambda_n \int_\Omega (\frac{1}{p^+}u_n f(x,u_n) - F(x,u_n) + \frac{C_*}{p^+})dx\\
&= c_{\lambda_n} + \frac{C_*\lambda_n}{p^+}|\Omega|
\end{aligned}$$

for all $t \in [0,1]$.
On the other hand, for all $R > 1$, set $R' = (2p^+R)^{\frac{1}{p^-}}$

$$I_{\lambda_n}(R'\omega_n) \ge 2R - \lambda_n \int_\Omega F(x,R'\omega_n)dx \ge R.$$

which contradicts $I_{\lambda_n}(R'\omega_n) \le c_{\lambda_n} + \frac{C_*\lambda_n}{p^+}|\Omega|$, for $n$ large.
Now we have a bounded sequence $\{u_n\}$ such that

$$I_\mu(u_n) \to c_\mu \quad \text{and} \quad I'_\mu(u_n) \to 0, \quad \text{as}\, n \to \infty.$$

The proof is complete.

**Acknowledgement**

The author is grateful to the reviewers for useful comments.

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(Received January 16, 2011)