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CONNECTIONS BETWEEN THE STABILITY OF A POINCARE MAP AND BOUNDEDNESS OF CERTAIN ASSOCIATE SEQUENCES

SADIA ARSHAD, CONSTANTIN BUŞE, AMMARA NOSHEEN AND AKBAR ZADA

Abstract. Let $m \geq 1$ and $N \geq 2$ be two natural numbers and let $\mathcal{U} = \{U(p,q)\}_{p\geq q\geq 0}$ be the N-periodic discrete evolution family of $m \times m$ matrices, having complex scalars as entries, generated by $\mathcal{L}(\mathbb{C}^m)$-valued, N-periodic sequence of $m \times m$ matrices (A_n). We prove that the solution of the following discrete problem

$$y_{n+1} = A_n y_n + e^{i\mu n} b, \quad n \in \mathbb{Z}_+, \quad y_0 = 0$$

is bounded for each $\mu \in \mathbb{R}$ and each m-vector b if the Poincare map $U(N,0)$ is stable. The converse statement is also true if we add a new assumption to the boundedness condition. This new assumption refers to the invertibility for each $\mu \in \mathbb{R}$ of the matrix $V_\mu := \sum_{\nu=1}^{N} U(N,\nu)e^{i\mu\nu}$. By an example it is shown that the assumption on invertibility cannot be removed. Finally, a strong variant of Barbashin's type theorem is proved.

1. Introduction

It is well-known, see [4], that the matrix A is dichotomic, i.e. its spectrum does not intersect the unit circle if and only if there exists a projector, i.e. an $m \times m$ matrix P verifying $P^2 = P$, which commutes with A and has the property that for each real number μ and each vector $b \in \mathbb{C}^m$, the following two discrete Cauchy problems

$$\left\{ \begin{array}{rcl} x_{n+1} & = & Ax_n + e^{i\mu n} Pb, \quad n \in \mathbb{Z}_+ \\ x_0 & = & 0 \end{array} \right. \qquad (A, \mu, Pb, 0)$$

and

$$\left\{ \begin{array}{rcl} y_{n+1} & = & A^{-1} y_n + e^{i\mu n}(I-P)b, \quad n \in \mathbb{Z}_+ \\ y_0 & = & 0 \end{array} \right.$$

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have bounded solutions. In particular, the spectrum of A lies in the interior of the unit circle if and only if for each real number μ and each m-vector b, the solution of the Cauchy problem $(A, \mu, b, 0)$ is bounded.

On the other hand in [7], it is shown that an N-periodic evolution family $\mathcal{U} = \{U(p, q)\}_{p\geq q\geq 0}$ of bounded linear operators acting on a complex space X, is uniformly exponentially stable, i.e. the spectral radius of the Poincare map $U(N, 0)$ is less than one, if and only if for each real number μ and each N-periodic sequence (z_n) decaying to $n = 0$, have that

$$\sup_{n\geq 1}\left\|\sum_{k=1}^{n} e^{i\mu k}U(n, k)z_{k-1}\right\| = M(\mu, b) < \infty. \tag{1}$$

A consequence of (1), which is also pointed out in [7], is a variant of the theorem of Datko. Having in mind these two results, it is natural to raise the question is it possible to preserve the result from [7], whenever the class of all sequences (z_n) in (1) is replaced by the class of all constants, m-vector valued sequences. The answer of this question is NO. However, we prove such result adding to the assumption like (1) a new one. More exactly, we prove that the spectral radius of the matrix $U(N, 0)$ is less than one, if for each real μ and each m-vector b the operator $V_\mu := \sum_{\nu=1}^{N} e^{i\mu\nu}U(N, \nu)$ is invertible and

$$\sup_{k\geq 1}\left\|\sum_{j=1}^{kN} e^{i\mu(j-1)}U(kN, j)b\right\| < \infty. \tag{2}$$

Moreover, we prove that the assumption on invertibility of V_μ, for each real number μ, cannot be removed. This condition seems to be difficult to verify, but however, in the case $N = 2$, it reduces to the fact that the matrix $U(2, 1)$ is a dichotomic one.

It is clear that the boundedness condition (2) is implied by the following one which seems to tracked back by an old result of Barbashin, see [1]. In our framework, the Barbashin condition can be written as:

$$\sup_{k\geq 1}\sum_{j=1}^{kN}\|U(kN, j)b\| = M(b) < \infty. \tag{3}$$

In (3), the vector b is taken in a complex Banach space X and $U(p, q)$ are bounded linear operators acting on X. The estimation in (3) is made with respect to the strong operator topology in $\mathcal{L}(X)$ while that the original one is related to the topology generated by the operatorial norm in $\mathcal{L}(\mathbb{C}^m)$. At least in the continuous case, the problem if a strong Barbashin condition like (3) implies the uniform exponential stability of the family $\mathcal{U}$, seems to be an open one. In this direction some progress was made in [6], [5], where the dual family of $\mathcal{U}$ is involved, and the estimation like (3), was made with respect to the strong operator topology in $\mathcal{L}(X^*)$. We prove that the boundedness condition (3) implies the uniform exponential stability of the family $\mathcal{U}$ without any other assumption.

2. Notations and Preliminary Results

By $\mathcal{X}$ we denote the Banach algebra of all $m \times m$ matrices with complex entries endowed with the usual operatorial norm. An eigenvalue of a matrix $A \in \mathcal{X}$ is any complex scalar λ having the property that there exists a nonzero vector $v \in \mathbb{C}^m$ such that $Av = \lambda v$. The spectrum of the matrix A, denoted by $\sigma(A)$, consists of all its eigenvalues. The resolvent set of A, denoted by $\rho(A)$, is the complement in $\mathbb{C}$ of $\sigma(A)$. Denote $\Gamma_1 = \{z \in \mathbb{C} : |z| = 1\}$, $\Gamma_1^+ := \{z \in \mathbb{C} : \quad |z| > 1\}$ and $\Gamma_1^- := \{z \in \mathbb{C} : \quad |z| < 1\}$. Clearly $\mathbb{C} = \Gamma_1^- \cup \Gamma_1 \cup \Gamma_1^+$.

Recall that an $m \times m$ matrix A is stable if its spectrum lies in Γ_1^-. Via the Spectral Decomposition Theorem (see e.g. [4]) this is equivalent with the fact that there exist two positive constants N and ν such that $\|A^n\| \leq Ne^{-\nu n}$ for all $n = 0, 1, 2 \dots$.

We begin with few lemmas which would be useful later.

Lemma 1. *Let A be a square matrix of order m having complex entries. If for a given real number μ, have that*

$$\sup_{n \in \{1,2,3\dots\}} \|I + e^{i\mu}A + \dots + (e^{i\mu}A)^n\| = K(\mu) < \infty \tag{4}$$

then $e^{-i\mu}$ belongs to the resolvent set of A.

Proof. See [3]. □

Lemma 2. *Let $A \in \mathcal{X}$. If for each real number μ the inequality* (4) *is fulfilled then A is stable.*

Proof. We use the identity

$$(I - e^{i\mu}A)(I + e^{i\mu}A + \dots + (e^{i\mu}A)^{n-1}) = I - (e^{i\mu}A)^n.$$

Passing to the norm we get :

$$
\begin{aligned}
||A^n|| = \|(e^{i\mu}A)^n\| &\le 1 + \|(I - e^{i\mu}A)\|\|(I + e^{i\mu}A + \cdots + (e^{i\mu}A)^{n-1})\| \\
&\le 1 + (1 + \|A\|)K(\mu),
\end{aligned}
$$

that is, the matrix A is power bounded. Then the spectral radius $r(A)$, of A, is less than or equal to one. Recall that

$$
r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\} = \lim_{n\to\infty} ||A^n||^{\frac{1}{n}}.
$$

As consequence $\sigma(A) \subset \Gamma_1 \cup \Gamma_1^-$. On the other hand, from Lemma 1, follows that each complex number $z = e^{-i\mu}$ is in the resolvent set of A. Combining these two facts, it follows that $\sigma(A)$ is a subset of Γ_1^-. $\square$

The infinite dimensional version of Lemma 2 has been stated in [7]. See also [11] and [9] for other variants or different proofs.

3. Stability and Boundedness

Let $\mathbb{Z}_+$ be the set of all nonnegative integer numbers. A family $\mathcal{U} = \{U(p,q) : (p,q) \in \mathbb{Z}_+ \times \mathbb{Z}_+\}$ of $m \times m$ matrices having complex scalars as entries is called N-periodic discrete evolution family if it satisfies the following properties.

$U(p,q)U(q,r) = U(p,r)$ for all nonnegative integers $p \ge q \ge r$.

$U(p,p) = I$ for all $p \in \mathbb{Z}_+$.

$U(p+N, q+N) = U(p,q)$ for all nonnegative integers $p \ge q$.

Such families lead naturally to the solutions of the following discrete Cauchy problems.

$$
\begin{cases} y_{n+1} = A_n y_n + e^{i\mu n} b, & n \in \mathbb{Z}_+ \\ y_0 = 0\,. \end{cases} \qquad (A_n, \mu, b)_0
$$

Indeed, if the sequence of $m \times m$ matrices (A_n) is N-Periodic, i.e. $A_{n+N} = A_n$ for all $n \in \mathbb{Z}_+$, and define

$$
U(n,j) := \begin{cases} A_{n-1}A_{n-2}\ldots A_j, & j \le n-1 \\ I, & j = n, \end{cases}
$$

then the family $\{U(n,j)\}_{n\ge j\ge 0}$ is a discrete N-periodic evolution family and the solution $(y_n(\mu,b))$ of the Cauchy Problem $(A_n,\mu,b)_0$ is given by:

$$
y_n(\mu, b) = \sum_{j=1}^{n} U(n,j) e^{i\mu(j-1)} b. \tag{5}
$$

For further details related to the general theory of difference equations we refer to [8].

The continuous systems of type $(A_n, \mu, b, 0)$ appears for example as mathematical models for the classical **RLC circuits**. We will look at the case of one inductor of inductance $L > 0$, one capacitor of capacitance $C > 0$, and one resistor of resistance $R > 0$ arranged in a loop together with an external power source $V(t) = V_0 e^{i\omega t}$. By Kirchhoff's first law the current through the inductor is the same with that through the capacitor or resistor. Denote by $I(t)$ the common value of the current at the time t. By the Kirchhoff's second law the $I(t)$ and $V(t)$ are connected by

$$\dot{x}(t) = Ax(t) + e^{i\omega t}b, \quad t \geq 0$$

where

$$x(t) = \begin{pmatrix} I(t) \\ I'(t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ \frac{-1}{LC} & \frac{-R}{L} \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 \\ \frac{i\omega V_0}{L} \end{pmatrix}.$$

The discrete case, i.e.

$$x(n+1) = Ax(n) + e^{i\omega n}b, \quad n \in \mathbb{Z}_+,$$

may be obtained from the continuous one, by replacing $\dot{x}(t)$ by $x(n+1) - x(n)$ and A by $A - I$.

Our first result is stated as follows:

Theorem 1. *The sequence $(y_n(\mu, b))$ given in (5) is bounded for any real number μ and any m-vector b if the matrix $U(N, 0)$ is stable.*

Proof. Let $r \in \{0, 1, 2, \ldots, N-1\}$ and $n = kN + r$. From (5) follows:

$$y_{Nk+r}(\mu, b) = \sum_{j=1}^{Nk+r} U(Nk+r, j)e^{i\mu(j-1)}b.$$

For each $\nu \in \{1, 2, \ldots, N\}$ consider the set

$$\mathcal{A}_\nu = \{\nu, \nu + N, \ldots, \nu + (k-1)N\}$$

and

$$\mathcal{R} = \{kN + 1, kN + 2, \ldots, kN + r\}.$$

Then

$$\mathcal{R} \cup (\cup_{\nu=1}^{N} A_\nu) = \{1, 2, \ldots, n\}.$$

Thus

$$\begin{aligned}
y_{Nk+r}(\mu,b) &= e^{-i\mu}\sum_{\nu=1}^{N}\sum_{j\in\mathcal{A}_\nu} U(Nk+r,j)e^{i\mu j}b \\
&+ e^{-i\mu}\sum_{j\in\mathcal{R}} U(Nk+r,j)e^{i\mu j}b \\
&= e^{-i\mu}\sum_{\nu=1}^{N}\sum_{s=0}^{k-1} U(Nk+r,\nu+sN)e^{i\mu(\nu+sN)}b + \\
&\quad e^{-i\mu}\sum_{\rho=1}^{r} U(Nk+r,Nk+\rho)e^{i\mu(kN+\rho)}b \\
&= e^{-i\mu}\sum_{\nu=1}^{N}\sum_{s=0}^{k-1} U(r,0)U(N,0)^{(k-s-1)}U(N,\nu)e^{i\mu(\nu+sN)}b \\
&+ e^{-i\mu}\sum_{\rho=1}^{r} U(r,\rho)e^{i\mu(kN+\rho)}b.
\end{aligned}$$

Let $z_\mu := e^{i\mu N}$ and $L = U(N,0)$. Denoting $\sum_{\nu=1}^{N} U(N,\nu)e^{i\mu\nu}$ by V_μ, we get

$$\begin{aligned}
y_{Nk+r}(\mu,b) &= e^{-i\mu}U(r,0)\big(L^{k-1}z_\mu^0 + L^{k-2}z_\mu^1 + \cdots + L^0 z_\mu^{k-1}\big)V_\mu b + \\
&\quad e^{-i\mu}z_\mu^k\sum_{\rho=1}^{r} U(r,\rho)e^{i\mu\rho}b.
\end{aligned}$$

By the assumption $\sigma(L)$ lies in Γ_1^- so z_μ belongs to the resolvent set of L. Therefore the matrix $(z_\mu I - L)$ is invertible and the previous equality may be shortened to

$$y_{Nk+r}(\mu,b) = e^{-i\mu}U(r,0)(z_\mu I-L)^{-1}(z_\mu^k I-L^k)V_\mu b + e^{-i\mu}z_\mu^k\sum_{\rho=1}^{r} U(r,\rho)e^{i\mu\rho}b.$$

Taking norm of both sides, we get

$$\begin{aligned}\|y_{Nk+r}(\mu,b)\| &\le \|U(r,0)(z_\mu I-L)^{-1}z_\mu^k V_\mu b\|+\\ &+ \|U(r,0)(z_\mu I-L)^{-1}L^k V_\mu b\|+\sum_{\rho=1}^{r}\|U(r,\rho)b\|\\ &\le \|U(r,0)\|\|(z_\mu I-L)^{-1}\|\|V_\mu b\|+\\ &+ \|U(r,0)\|\|(z_\mu I-L)^{-1}\|\|L^k V_\mu b\|+\sum_{\rho=1}^{r}\|U(r,\rho)b\|.\end{aligned}$$

Let $\sigma(L)=\{\lambda_1,\lambda_2,\dots,\lambda_\xi\}$ and let $P_L(\lambda)=(\lambda-\lambda_1)^{m_1}\dots(\lambda-\lambda_\xi)^{m_\xi}$ be the characteristic polynomial of L. Here each m_j is a natural number and $m_1+m_2+\cdots+m_\xi=m$. By the Spectral Decomposition Theorem, see e.g. [4], we have that

$$L^kV_\mu b=\lambda_1^k p_1(k)+\lambda_2^k p_2(k)+\cdots+\lambda_\xi^k p_\xi(k),$$

where each $p_j(k)$ is $\mathbb{C}^m$-valued polynomial having degree at most (m_j-1) for any $j\in\{1,2,\dots,\rho\}$. By assumption $|\lambda_j|<1$ for each $j\in\{1,2,\dots,\xi\}$. Thus $\|L^kV_\mu b\|\to 0$ as $k\to\infty$. Therefore the subsequence $(y_{Nk+r}(\mu,b))_k$ is bounded for any $r=0,1,2,\dots,N-1$, that is, the sequence $(y_n(\mu,b))_n$ is bounded. □

¿From the proof we also realize that the sequence $(y_n(\mu,b))_n$ is bounded if and only if its subsequence $(y_{Nk}(\mu,b))_k$ is bounded.

A partial converse of Theorem 1 may be read as follows:

Theorem 2. *If for each $\mu\in\mathbb{R}$ and each non zero $b\in\mathbb{C}^m$ the sequence $(y_{Nk}(\mu,b))_k$ is bounded and the matrix $\sum_{\nu=1}^{N}U(N,\nu)e^{i\mu\nu}$ is invertible then the matrix $U(N,0)$ is stable.*

Proof. Suppose on contrary that the spectrum of the operator L is not contained in the interior of the unite circle. When $\sigma(L)\cap\Gamma_1$ is a non empty set, let $\omega\in\sigma(L)\cap\Gamma_1$ and let $y\in\mathbb{C}^m$ be a nonzero vector such that $Ly=\omega y$. Then for each natural number k we have that $L^ky=\omega^ky$. Choose $\mu_0\in\mathbb{R}$ such that $e^{i\mu_0N}=\omega$. Given that V_{μ_0} is invertible therefore there exists $b_0\in\mathbb{C}^m$ such that $y=V_{\mu_0}b_0$. Then

$$\begin{aligned}y_{kN}(\mu_0,b_0) &= e^{-i\mu_0}\big(L^{k-1}z_{\mu_0}^0+L^{k-2}z_{\mu_0}+\dots+L^0z_{\mu_0}^{k-1}\big)V_{\mu_0}b_0\\ &= e^{-i\mu_0}\sum_{j=0}^{k-1}z_{\mu_0}^{k-j-1}(L^jV_{\mu_0}b_0)=e^{-i\mu_0}\sum_{j=0}^{k-1}z_{\mu_0}^{k-1}(V_{\mu_0}b_0)\\ &= ke^{-i\mu_0}z_{\mu_0}^{k-1}y.\end{aligned}$$

Taking norm of both sides, we get

$$\|y_{kN}(\mu_0, b_0)\| = k||y|| \to \infty \text{ as } k \to \infty$$

and a contradiction arises.

Now if L is dichotomic and the spectrum of L contains a complex number ω such that $|\omega| > 1$ then the matrix $(z_\mu I - L)$ is invertible for each $\mu \in \mathbb{R}$ and there exists a nonzero vector y such that $Ly = \omega y$. Thus

$$\begin{aligned} y_{kN}(\mu, b) &= e^{-i\mu}(z_\mu I - L)^{-1}(z_\mu^k - L^k)V_\mu b \\ &= A_k(\mu, b) + B_k(\mu, b) \end{aligned}$$

where

$$A_k(\mu, b) = e^{-i\mu}(z_\mu I - L)^{-1} z_\mu^k V_\mu b$$

and

$$B_k(\mu, b) = e^{-i\mu}(z_\mu I - L)^{-1} L^k V_\mu b.$$

Clearly the sequence $(A_k(\mu, b))_k$ is bounded for each real number μ and any m-vector b. Now let $\mu \in \mathbb{R}$ be fixed and let $b_1 \in \mathbb{C}^m$ such that $V_\mu b_1 = y$. Then

$$||B_k(\mu, b_1)|| = |\omega^k|||(z_\mu I - L)^{-1}y|| \to \infty \text{ as } k \to \infty,$$

which is a contradiction. $\square$

Now we give an example that shows that the assumption on invertibility of V_μ, for each real number μ, cannot be removed.

Example 1. *Let $N = 2$. Then $V_\mu = e^{i\mu}\big(U(2, 1) + e^{i\mu}I\big)$ and it is invertible for each $\mu \in \mathbb{R}$ if and only if the matrix $U(2, 1)$ is dichotomic, or equivalently if $\sigma(U(2, 1)) \cap \Gamma_1$ is the empty set. Take*

$$U(2, 1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } U(2, 0) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly $\sigma(U(2, 1)) \cap \Gamma_1$ is a non-empty set, thus V_μ is not invertible for some real number μ. Moreover,

$$\begin{aligned} y_{2k}(\mu, b) &= \sum_{s=0}^{k-1} L^{k-s-1} z_\mu^s \big(U(2, 1) + z_\mu I\big) b \\ &= \begin{pmatrix} \sum_{s=0}^{k-1} (\frac{1}{2})^{k-s-1} z_\mu^s & 0 \\ 0 & \sum_{s=0}^{k-1} z_\mu^s \end{pmatrix} b \end{aligned}$$

When $z_\mu \neq 1$, *we have that*

$$y_{2k}(\mu, b) = \begin{pmatrix} a_k & 0 \\ 0 & b_k \end{pmatrix} b$$

where

$$|a_k| = \frac{\left|(\frac{1}{2})^k - z_\mu^k\right|}{\left|\frac{1}{2} - z_\mu\right|} \leq \frac{2}{\left|\frac{1}{2} - z_\mu\right|} \leq 4$$

and

$$|b_k| = \frac{\left|z_\mu^k - 1\right|}{\left|1 - z_\mu\right|} \leq \frac{2}{|1 - z_\mu|}.$$

So for the corresponding values of $\mu \in \mathbb{R}$ *and each nonzero* $b \in \mathbb{C}^2$ *the sequence* $(y_{2k}(\mu, b))_k$, *is bounded. If* $z_\mu = 1$ *then*

$$\begin{aligned} y_{2k}(\mu, b) &= \sum_{s=0}^{k-1} L^{k-s-1} z_\mu^s \big(U(2,1) + z_\mu I\big) b \\ &= \begin{pmatrix} 2\left(1 - \frac{1}{2^k}\right) & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} b = \begin{pmatrix} 4\left(1 - \frac{1}{2^k}\right) & 0 \\ 0 & 0 \end{pmatrix} b. \end{aligned}$$

We have again that $(y_{2k}(\mu, b))_k$ *is bounded. On the other hand* $1 \in \sigma(U(2,0))$, *i.e. the matrix* $U(2,0)$ *is unstable.*

The above Theorem 2 may be of the some interest because it provides a criteria for stability in terms of boundedness of solutions for certain discrete Cauchy problems. However, the assumption of invertibility on V_μ may be difficult to verify, except for small values of N. On the other hand after a careful inspection of the above example we can see that the sequence $(y_{2k}(\mu, b))$ is not uniformly bounded with respect to μ, that is, for each nonzero b, we have that

$$\sup_{\mu \in \mathbb{R}} \sup_{k \geq 1} ||y_{2k}(\mu, b)|| = \infty.$$

Then it is natural to ask can the assumption on invertibility be removed whenever the boundedness assumption on the subsequence $(y_{Nk}(\mu, b))_k$ in Theorem 2 take a stronger one? A positive answer is given in the next theorem. This is, in the same time, a strong variant of a Barbashin's type theorem.

Theorem 3. *Let $(U(n,k))_{n\geq k}$ be an N-periodic evolution family. If for each vector $b \in \mathbb{C}^m$ the following inequality*

$$\sup_{k\geq 1} \sum_{j=1}^{Nk} \|U(Nk,k)b\| = M(b) < \infty$$

holds then the matrix $U(N,0)$ is stable.

Proof. We know that

$$\begin{aligned} \sum_{j=1}^{Nk} \|U(Nk,j)b\| &= \sum_{\nu=1}^{N} \sum_{j\in A_\nu} \|U(Nk,j)b\| \\ &= \sum_{\nu=1}^{N} \sum_{s=0}^{k-1} \|U(N,0)^{k-s-1}U(N,\nu)b\| \leq M(b). \end{aligned}$$

As a consequence

$$\sup_{k\geq 1} \sum_{s=0}^{k-1} \|U(N,0)^{k-s-1}b\| \leq M(b) < \infty.$$

The assertion follows now from Lemma 2.

□

Having in mind that Lemma 2 has an infinite dimensional version, the proof of Theorem 3 is also an argument for the same result in the more general framework of bounded linear operators acting on an arbitrary Banach space.

An interesting problem is if the following uniform inequality

$$\sup_{\mu\in\mathbb{R}} \sup_{k\geq 1} ||y_{2k}(\mu,b)|| = K(b) < \infty, \tag{6}$$

holds for all $b \in \mathbb{C}^m$ then the matrix $U(N,0)$ is stable. Under an assumption like (6)*, Jan van Neerven proved in* [10] *that a strongly continuous semigroup acting on a complex Banach space is exponentially stable. Moreover, when the semigroup acts in a complex Hilbert space it is uniformly exponentially stable. A transparent proof of this later result can be found in* [11]*. In connection with* (6) *we also mention the paper* [2] *where it is proved that if a vector valued function has a bounded holomorphic extension to the open right half plane then its primitive grows like $M(1+t)$ for $t \geq 0$.*

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Government College University, Abdus Salam School of Mathematical Sciences, (ASSMS), Lahore, Pakistan
E-mail address: sadia_735@yahoo.com

West University of Timisoara, Department of Mathematics, Bd. V. Parvan No. 4, 300223-Timisoara, Romania and

Government College University, Abdus Salam School of Mathematical Sciences, (ASSMS), Lahore, Pakistan
E-mail address: buse@math.uvt.ro

Government College University, Abdus Salam School of Mathematical Sciences, (ASSMS), Lahore, Pakistan
E-mail address: hafiza_amara@yahoo.com

Government College University, Abdus Salam School of Mathematical Sciences, (ASSMS), Lahore, Pakistan
E-mail address: zadababo@yahoo.com