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Existence and uniqueness of positive solutions for Neumann problems of second order impulsive differential equations

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Abstract This work is concerned with the existence and uniqueness of positive solutions for Neumann boundary value problems of second order impulsive differential equations. The result is obtained by using a fixed point theorem of generalized concave operators.

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1 Introduction

Impulsive differential equations have been studied extensively in recent years. The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. Second order impulsive differential equations have been studied by many authors with much of the attention given to positive solutions. For a small sample of such work, we refer the reader to works [1-11]. The results of these papers are based on Schauder fixed point theorem, Leggett-Williams theorem, fixed point index theorems in cones, Krasnoselskii's fixed point theorem, the method of upper-lower solutions, fixed point theorems in cones and so on. In this paper,we consider the existence and uniqueness of positive solutions for the following Neumann boundary value problems of second order impulsive differential equations:

$$\begin{cases} -u''(t)+\gamma^2 u(t)=f(t,u(t)), t\neq t_k, k=1,2,\cdots m,\\ \triangle u' \mid_{t=t_k}=I_k(u(t_k)), k=1,2,\cdots m,\\ u'(0)=u'(1)=0, \end{cases} \tag{1.1}$$

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where γ is a positive constant, $f \in C[J \times \mathbf{R}, \mathbf{R}], J = [0,1], 0 < t_1 < t_2 < \cdots < t_m < 1, \triangle u' \mid_{t=t_k} = u'(t_k^+) - u'(t_k^-), u'(t_k^+), u'(t_k^-)$ denote the right limit (left limit)of $u'(t)$ at $t = t_k$, respectively. $I_k \in C[\mathbf{R}, \mathbf{R}], k = 1, 2, \cdots m$.

Neumann boundary value problem for the ordinary differential equations and elliptic equations is an important kind of boundary value problems. During the last two decades, Neumann boundary value problems have deserved the attention of many researchers [12-21]. However, few papers can be found in the literature on the existence of positive solutions for Neumann boundary value problems for second-order impulsive differential equations. In this paper, we shall study the problem (1.1) and not suppose the existence of upper-lower solutions. Different from the above works mentioned, in this paper we will use a fixed point theorem of generalized concave operators to show the existence and uniqueness of positive solutions for the problem (1.1).

2 Preliminaries

Suppose that E is a real Banach space which is partially ordered by a cone $P \subset E, i.e., x \leq y$ if and only if $y - x \in P$. By θ we denote the zero element of E. A non-empty closed convex set $P \subset E$ is called a cone if it satisfies (i) $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P, -x \in P \Rightarrow x = \theta$.

Moreover, P is called normal if there exists a constant $N > 0$ such that, for $x, y \in E,\ \theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$; in this case N is called the normality constant of P. We say that an operator $A : E \to E$ is increasing(decreasing) if $x \leq y$ implies $Ax \leq Ay (Ax \geq Ay)$.

For $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta (i.e., h \geq \theta$ and $h \neq \theta)$, we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. Clearly, $P_h \subset P$ is convex and $\lambda P_h = P_h$ for $\lambda > 0$.

We now present a fixed point theorem of generalized concave operators which will be used in the latter proof. See [22] for further information.

Theorem 2.1(from the Lemma 2.1 and Theorem 2.1 in [22]). Let $h > \theta$ and P be a normal cone. Assume that: (D_1) $A : P \to P$ is increasing and $Ah \in P_h$; (D_2) For any $x \in P$ and $t \in (0,1)$, there exists $\alpha(t) \in (t, 1]$ such that $A(tx) \geq \alpha(t)Ax$. Then (i) there are $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 < v_0,\ u_0 \leq Au_0 \leq Av_0 \leq v_0$; (ii) operator equation $x = Ax$ has a unique solution in P_h.

Remark 2.2. An operator A is said to be **generalized concave** if A satisfies condition (D_2).

In what follows, for the sake of convenience, let $J' = J \backslash \{t_1, t_2, \cdots, t_m\},\ C[J, \mathbf{R}] = \{u | u : J \to \mathbf{R}$ is continuous$\}$, $PC^1[J, \mathbf{R}] = \{u \in C[J, \mathbf{R}] | u'(t)$ is continuous at $t \neq t_k$ and left continuous at $t = t_k,\ u'(t_k^+)$ exists, $k = 1, 2, \cdots, m\}$. Evidently, $C[J, \mathbf{R}]$ is a Banach space with the norm $\|u\|_C = \sup\{|u(t)| : t \in J\}$ and $PC^1[J, \mathbf{R}]$ is a Banach space with the norm $\|u\|_{PC^1} = \sup\{\|u\|_C, \|u'\|_C\}$.

Definition 2.3. A function $u \in PC^1[J, \mathbf{R}] \bigcap C^2[J', \mathbf{R}]$ is called a solution of the problem (1.1), if it satisfies the problem (1.1).

Lemma 2.4. $u \in PC^1[J, \mathbf{R}] \cap C^2[J^1, \mathbf{R}]$ is a solution of the problem (1.1) if only and if $u \in$

$PC^1[J,\mathbf{R}]$ is the solution of the following integral equation:

$$u(t) = \int_0^1 G(t,s)f(s,u(s))ds - \sum_{k=1}^{m} G(t,t_k)I_k(u(t_k)), \tag{2.1}$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases} \psi(s)\psi(1-t), 0 \le s \le t \le 1, \\ \psi(t)\psi(1-s), 0 \le t \le s \le 1, \end{cases} \tag{2.2}$$

with

$$\rho = \frac{1}{2}\gamma(e^{\gamma} - e^{-\gamma}), \psi(t) = \frac{1}{2}(e^{\gamma t} + e^{-\gamma t}).$$

Proof. First suppose that $u \in PC^1[J,\mathbf{R}] \cap C^2[J^1,\mathbf{R}]$ is a solution of the problem (1.1). Then

$$-u''(t) + \gamma^2 u(t) = f(t,u(t)), t \ne t_k.$$

That is, $(e^{-2\gamma t}(e^{\gamma t}u(t))')' = -f(t,u(t))e^{-\gamma t}, t \ne t_k$. Let

$$y(t) = e^{-2\gamma t}(e^{\gamma t}u(t))' = e^{-\gamma t}(u'(t) + \gamma u(t)). \tag{2.3}$$

Then

$$y'(t) = -f(t,u(t))e^{-\gamma t}, t \ne t_k. \tag{2.4}$$

It is easy to see by integration of (2.4) that

$$y(t_1) - y(0) = -\int_0^{t_1} f(s,u(s))e^{-\gamma s}ds,$$

$$y(t) - y(t_1^+) = -\int_{t_1}^{t} f(s,u(s))e^{-\gamma s}ds, t_1 < t \le t_2.$$

So

$$y(t) = y(0) - \int_0^t f(s,u(s))e^{-\gamma s}ds + y(t_1^+) - y(t_1), t_1 < t \le t_2.$$

In the same way,we can show that

$$y(t) = y(0) - \int_0^t f(s,u(s))e^{-\gamma s}ds + \sum_{0<t_k<t} e^{-\gamma t_k}I_k(u(t_k)). \tag{2.5}$$

In view of (2.3),we have

$$(e^{\gamma t}u(t))' = e^{2\gamma t}(y(0) - \int_0^t f(s,u(s))e^{-\gamma s}ds + \sum_{0<t_k<t} e^{-\gamma t_k}I_k(u(t_k))).$$

Let

$$z(t) = e^{\gamma t}u(t), l(t) = e^{2\gamma t}(y(0) - \int_0^t f(s,u(s))e^{-\gamma s}ds + \sum_{0<t_k<t} e^{-\gamma t_k}I_k(u(t_k))). \tag{2.6}$$

Then $z(t) = z(0) + \int_0^t l(s)ds, t \in J$. Further,

$$u(t) = z(t)e^{-\gamma t} = e^{-\gamma t}(u(0) + \int_0^t l(s)ds), t \in J. \tag{2.7}$$

By calculation,we can get

$$\begin{aligned} \int_0^t l(s)ds &= \frac{1}{2\gamma}\left\{y(0)(e^{2\gamma t}-1) + \int_0^t e^{\gamma s}f(s,u(s))ds \; - e^{2\gamma t}\int_0^t e^{-\gamma s}f(s,u(s))ds \right. \\ &+ \left. \sum_{0<t_k<t}(e^{2\gamma t} - e^{2\gamma t_k})e^{-\gamma t_k}I_k(u(t_k))\right\}. \end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.7),we obtain

$$\begin{aligned} u(t) &= \frac{1}{2\gamma}\left\{(\gamma u(0) - u'(0))e^{-\gamma t} + (\gamma u(0) + u'(0))e^{\gamma t} + e^{-\gamma t}\int_0^t e^{\gamma s}f(s,u(s))ds \right. \\ &- e^{\gamma t}\int_0^t e^{-\gamma s}f(s,u(s))ds + \sum_{0<t_k<t} e^{\gamma(t-t_k)}I_k(u(t_k)) \\ &- \left. \sum_{0<t_k<t} e^{-\gamma(t-t_k)}I_k(u(t_k))\right\}, t \in J, \end{aligned} \tag{2.9}$$

$$\begin{aligned} u'(t) &= \frac{1}{2}\left\{-(\gamma u(0) - u'(0))e^{-\gamma t} + (\gamma u(0) + u'(0))e^{\gamma t} - e^{-\gamma t}\int_0^t e^{\gamma s}f(s,u(s))ds \right. \\ &- \left. e^{\gamma t}\int_0^t e^{-\gamma s}f(s,u(s))ds + \sum_{0<t_k<t}(e^{\gamma(t-t_k)} + e^{-\gamma(t-t_k)})I_k(u(t_k))\right\}, t \in J. \end{aligned} \tag{2.10}$$

In view of $u'(0) = u'(1) = 0$, we have

$$\begin{aligned} u(0) &= \frac{1}{\gamma(e^{\gamma} - e^{-\gamma})}\left\{\int_0^1 (e^{-\gamma(1-s)} + e^{\gamma(1-s)})f(s,u(s))ds \right. \\ &- \left. \sum_{k=1}^m (e^{\gamma(1-t_k)} + e^{-\gamma(1-t_k)})I_k(u(t_k))\right\}. \end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.9) and making use of the fact that

$$\sum_{k=1}^m I_k(u(t_k)) = \sum_{0<t_k<1} I_k(u(t_k)) = \sum_{0<t_k<t} I_k(u(t_k)) + \sum_{t\le t_k<1} I_k(u(t_k)),$$

we obtain

$$\begin{aligned} u(t) &= \frac{e^{\gamma t} + e^{-\gamma t}}{2\gamma(e^{\gamma} - e^{-\gamma})}\left\{\int_0^1 (e^{-\gamma(1-s)} + e^{\gamma(1-s)})f(s,u(s))ds - \sum_{k=1}^m (e^{\gamma(1-t_k)} + e^{-\gamma(1-t_k)})I_k(u(t_k))\right\} \\ &+ \frac{1}{2\gamma}\int_0^t (e^{\gamma(s-t)} - e^{\gamma(t-s)})f(s,u(s))ds + \frac{1}{2\gamma}\sum_{0<t_k<t}(e^{\gamma(t-t_k)} - e^{-\gamma(t-t_k)})I_k(u(t_k)) \\ &= \frac{1}{\rho}\int_0^t \psi(s)\psi(1-t)f(s,u(s))ds + \frac{1}{\rho}\int_t^1 \psi(t)\psi(1-s)f(s,u(s))ds \\ &- \sum_{k=1}^m \frac{1}{\rho}\psi(t)\psi(1-t_k)I_k(u(t_k)) + \frac{1}{2\gamma}\sum_{0<t_k<t}(e^{\gamma(t-t_k)} - e^{-\gamma(t-t_k)})I_k(u(t_k)) \\ &= \int_0^1 G(t,s)f(s,u(s))ds - \sum_{k=1}^m G(t,t_k)I_k(u(t_k)). \end{aligned}$$

That is, $u(t)$ is a solution of the equation (2.1).

Conversely,assume that $u \in C[J, \mathbf{R}]$ is a solution of the equation (2.1). Direct differentiation of (2.1) implies, for $t \neq t_k$

$$\begin{aligned} u'(t) &= \int_0^1 G_t(t,s)f(s,u(s))ds - \sum_{k=1}^{m} G_t(t,t_k)I_k(u(t_k)) \\ &= \frac{1}{\rho}\left[(\psi(1-t))' \int_0^t \psi(s)f(s,u(s))ds + \psi'(t)\int_t^1 \psi(1-s)f(s,u(s))ds\right] \\ &- \frac{1}{\rho}\psi'(t)\sum_{k=1}^{m}\psi(1-t_k)I_k(u(t_k)) + \frac{1}{2}\sum_{0<t_k<t}(e^{\gamma(t-t_k)} + e^{-\gamma(t-t_k)})I_k(u(t_k)), \end{aligned}$$

$$\begin{aligned} u''(t) &= \frac{1}{\rho}\left[(\psi(1-t))'' \int_0^t \psi(s)f(s,u(s))ds + (\psi(1-t))'\psi(t)f(t,u(t))\right. \\ &+ \left.\psi''(t)\int_t^1 \psi(1-s)f(s,u(s))ds - \psi'(t)\psi(1-t)f(t,u(t))\right] \\ &- \frac{1}{\rho}\psi''(t)\sum_{k=1}^{m}\psi(1-t_k)I_k(u(t_k)) + \frac{\gamma}{2}\sum_{0<t_k<t}(e^{\gamma(t-t_k)} - e^{-\gamma(t-t_k)})I_k(u(t_k)). \end{aligned}$$

Making use of the facts

$$\psi''(1-t) = \gamma^2\psi(1-t), \psi''(t) = \gamma^2\psi(t), (\psi(1-t))'\psi(t) - \psi'(t)\psi(1-t) = -\rho,$$

we can easily obtain $u''(t) = \gamma^2 u(t) - f(t,u(t)), t \neq t_k$. Moreover,

$$\triangle u' \mid_{t=t_k} = u'(t_k^+) - u'(t_k^-) = I_k(u(t_k)).$$

So $u \in C^2[J', \mathbf{R}]$ and it is easy to verify that $u'(0) = u'(1) = 0$ and the lemma is proved. □

Remark 2.5. To the best of our knowledge, the expression (2.1) is new for the Neumann problem. Similar expressions have been obtained for periodic problems of first order and for higher order ordinary differential equations with impulses, see Theorem 2.2 in [23] and Lemma 2.1 in [24,25].

Lemma 2.6. (i) $0 < G(t,s) \leq G(t,t), G(t,s) \leq G(s,s), \quad 0 \leq t,s \leq 1$;

(ii) $G(t,s) \geq C\psi(t)\psi(1-t)G(t_0,s), \quad t,t_0,s \in [0,1]$, where $C = 1/\psi^2(1)$.(See [20])

3 Existence and uniqueness of positive solutions for problem (1.1)

In this section, we apply Theorem 2.1 to study the problem (1.1) and we obtain a new result on the existence and uniqueness of positive solutions. The method used here is new to the literature and so is the existence and uniqueness result to the second-order impulsive differential equations.

Set $\tilde{P} = \{x \in C[J, \mathbf{R}] | x(t) \geq 0, \ t \in J\}$, the standard cone. It is clear that $\tilde{P}$ is a normal cone in $C[J, \mathbf{R}]$ and the normality constant is 1. Our main result is summarized in the following theorem.

Theorem 3.1. Assume that

(H_1) $f(t,0) \geq 0, f(t,a) > 0, t \in [0,1]$ and $f(t,x)$ is increasing in $x \in [0,\infty)$ for each $t \in [0,1]$, where $a = \frac{1}{4}(e^\gamma + e^{-\gamma} + 2)$;

(H_2) $I_k(0) \le 0$ and $I_k(x)$ is decreasing in $x \in [0,\infty), k = 1,2,\cdots,m$;
(H_3) for any $\lambda \in (0,1)$ and $x \ge 0$, there exist $\alpha_1(\lambda), \alpha_2(\lambda) \in (\lambda, 1]$ such that

$$f(t,\lambda x) \ge \alpha_1(\lambda) f(t,x), I_k(\lambda x) \le \alpha_2(\lambda) I_k(x), k = 1,2,\cdots,m.$$

(H_4) $\sum\limits_{k=1}^{m} G(t_k,t_k) I_k(b) < 0$, where $b = \frac{1}{2}(e^{\gamma} + e^{-\gamma})$.
Then (i) there exist $u_0, v_0 \in \tilde{P}_h$ such that

$$u_0(t) \le \int_0^1 G(t,s) f(s,u_0(s))ds - \sum_{k=1}^{m} G(t,t_k) I_k(u_0(t_k)), t \in J,$$

$$v_0(t) \ge \int_0^1 G(t,s) f(s,v_0(s))ds - \sum_{k=1}^{m} G(t,t_k) I_k(v_0(t_k)), t \in J;$$

(ii) the problem (1.1) has a unique positive solution x^* in $\tilde{P}_h \bigcap PC^1[J,\mathbf{R}]$, where

$$h(t) = \psi(t)\psi(1-t), t \in [0,1].$$

Remark 3.2. Some examples of $\alpha_i(\lambda), i = 1,2$ which satisfy the condition (H_3) are:
(1) $\alpha_i(\lambda) = \lambda^{\delta_i}, i = 1,2, \forall\ \lambda \in (0,1)$, where $\delta_i \in (0,1)$.
(2) $\alpha_i(\lambda) = \lambda(1+\eta_i(\lambda))$ with $0 < \eta_i(\lambda) \le \frac{1}{\lambda} - 1, \forall\ \lambda \in (0,1), i = 1,2$.
Remark 3.3. It is easy to check that $a = \min\{h(t) : t \in [0,1]\}, b = \max\{h(t) : t \in [0,1]\}$, where a, b are given as in $(H_1), (H_4)$.
Proof of Theorem 3.1 Define an operator $A : C[J,\mathbf{R}] \to C[J,\mathbf{R}]$ by

$$Au(t) = \int_0^1 G(t,s) f(s,u(s))ds - \sum_{k=1}^{m} G(t,t_k) I_k(u(t_k)).$$

Then from Lemma 2.4, $u \in PC^1[J,\mathbf{R}] \bigcap C^2[J',\mathbf{R}]$ is a solution of the problem (1.1) if and only if $u \in PC^1[J,\mathbf{R}]$ is a fixed point of the operator A. Firstly, we show that $A : \tilde{P} \to \tilde{P}$ is increasing, generalized concave. For any $u \in \tilde{P}$, from $(H_1), (H_2)$, we obtain $Au(t) \ge 0,\ t \in [0,1]$. Further, also from $(H_1), (H_2)$, we can easily prove that $A : \tilde{P} \to \tilde{P}$ is increasing. For any $\lambda \in (0,1)$ and $u \in \tilde{P}$, from $(H_1) - (H_3)$, we have

$$\begin{aligned} A(\lambda u)(t) &= \int_0^1 G(t,s) f(s,\lambda u(s))ds - \sum_{k=1}^{m} G(t,t_k) I_k(\lambda u(t_k)) \\ &\ge \int_0^1 G(t,s) \alpha_1(\lambda) f(s,u(s))ds - \sum_{k=1}^{m} G(t,t_k) \alpha_2(\lambda) I_k(u(t_k)). \end{aligned}$$

Set $\alpha(\lambda) = \min\{\alpha_1(\lambda), \alpha_2(\lambda)\}, \lambda \in (0,1)$. Then $\alpha(\lambda) \in (\lambda, 1]$. We have

$$A(\lambda u)(t) \ge \alpha(\lambda) \left[\int_0^1 G(t,s) f(s,u(s))ds - \sum_{k=1}^{m} G(t,t_k) I_k(u(t_k)) \right] = \alpha(\lambda) Au(t).$$

That is, $A(\lambda u) \ge \alpha(\lambda) Au, u \in \tilde{P}, \lambda \in (0,1)$. So $A : \tilde{P} \to \tilde{P}$ is generalized concave.

Secondly, we prove $Ah \in \tilde{P}_h$. To illuminate this, set

$$r_1 = \min_{t\in[0,1]} f(t,a),\ \ r_2 = \max_{t\in[0,1]} f(t,b).$$

Then from (H_1), we have $r_2 \geq r_1 > 0$. Further, from $(H_1), (H_2)$ and Lemma 2.6,

$$\begin{aligned} Ah(t) &= \int_0^1 G(t,s)f(s,h(s))ds - \sum_{k=1}^m G(t,t_k)I_k(h(t_k)) \\ &\geq \int_0^1 G(t,s)f(s,h(s))ds \geq \int_0^1 C\psi(t)\psi(1-t)G(t_0,s)f(s,a)ds \\ &\geq Cr_1 \int_0^1 G(t_0,s)ds \cdot h(t). \end{aligned}$$

Note that

$$\int_0^1 G(t_0,s)ds = \frac{1}{\rho}\int_0^{t_0} \psi(s)\psi(1-t_0)ds + \frac{1}{\rho}\int_{t_0}^1 \psi(t_0)\psi(1-s)ds = \frac{1}{\gamma^2},$$

we have $Ah(t) \geq \frac{Cr_1}{\gamma^2}h(t)$. It follows from Lemma 2.6 and (H_4) that

$$\begin{aligned} Ah(t) &= \int_0^1 G(t,s)f(s,h(s))ds - \sum_{k=1}^m G(t,t_k)I_k(h(t_k)) \\ &\leq \int_0^1 G(t,t)f(s,b)ds - \sum_{k=1}^m G(t_k,t_k)I_k(b) \\ &\leq \frac{r_2}{\rho}h(t) - \frac{4}{e^\gamma + e^{-\gamma}+2}h(t)\sum_{k=1}^m G(t_k,t_k)I_k(b) \\ &= \left[\frac{r_2}{\rho} - \frac{4}{e^\gamma + e^{-\gamma}+2}\sum_{k=1}^m G(t_k,t_k)I_k(b)\right]h(t). \end{aligned}$$

Hence,

$$\frac{Cr_1}{\gamma^2}h \leq Ah \leq \left[\frac{r_2}{\rho} - \frac{4}{e^\gamma + e^{-\gamma}+2}\sum_{k=1}^m G(t_k,t_k)I_k(b)\right]h.$$

That is, $Ah \in \tilde{P}_h$. Finally, an application of Theorem 2.1 implies that (i) there are $u_0, v_0 \in \tilde{P}_h$ such that $u_0 \leq Au_0, Av_0 \leq v_0$; (ii) operator equation $u = Au$ has a unique solution x^* in $\tilde{P}_h$. That is,

$$u_0(t) \leq \int_0^1 G(t,s)f(s,u_0(s))ds - \sum_{k=1}^m G(t,t_k)I_k(u_0(t_k)), t \in J,$$

$$v_0(t) \geq \int_0^1 G(t,s)f(s,v_0(s))ds - \sum_{k=1}^m G(t,t_k)I_k(v_0(t_k)), t \in J;$$

and the problem (1.1) has a unique solution x^* in $\tilde{P}_h$. Moreover, from Lemma 2.4 we know that $x^* \in PC^1[J,\mathbf{R}]$. Evidently, x^* is a positive solution of the problem (1.1).□

Remark 3.4. For the case of $I_k = 0, k = 1,2,\ldots,m$, the problem(1.1) reduces to the following Neumann boundary value problem for ordinary differential equations:

$$\begin{cases} -u''(t) + \gamma^2 u(t) = f(t,u(t)),\ \ 0 < t < 1, \\ u'(0) = u'(1) = 0, \end{cases} \tag{3.1}$$

We can establish the existence and uniqueness of positive solutions for the problem (3.1) by using the same method used in this paper, which is new to the literature. So the method used in this paper is different from previous ones in literature and the result obtained in this paper is new.

4 An example

To illustrate how our main result can be used in practice we present an example.

Example 4.1. Consider the following boundary value problem

$$\begin{cases} -u''(t) + (\ln 2)^2 u(t) = u^{\beta}(t) + q(t),\ t \in J, t \neq \frac{1}{2}, \\ \Delta u'|_{t=\frac{1}{2}} = -\sqrt[4]{u(\frac{1}{2})}, \\ u'(0) = u'(1) = 0, \end{cases} \tag{4.1}$$

where $\beta \in (0,1)$ and $q : [0,1] \to [0,+\infty)$ is a continuous function.

Conclusion. The impulsive problem (4.1) has a unique positive solution in $\tilde{P}_h \bigcap PC^1[J,\mathbf{R}]$, where

$$h(t) = \frac{5}{8} + \frac{1}{4}(2^{1-2t} + 2^{2t-1}), t \in [0,1].$$

Proof. The problem (4.1) can be regarded as a boundary value problem of the form (1.1), where $\gamma = \ln 2,\ t_1 = \frac{1}{2}, f(t,x) := x^{\beta} + q(t), I_1(x) := -x^{\frac{1}{4}}$. After a simple calculation, we get $a = \frac{9}{8},\ b = \frac{5}{4}$ and

$$f(t,a) = (\frac{9}{8})^{\beta} + q(t) > 0, G(t_1,t_1)I_1(b) = G(\frac{1}{2},\frac{1}{2})I_1(\frac{5}{4}) = -\frac{e^2+1}{4(e^2-1)}(\frac{5}{4})^{\frac{1}{4}} < 0.$$

It is not difficult to see that the conditions $(H_1),(H_2)$ and (H_4) hold. In addition, let $\alpha_1(\lambda) = \lambda^{\beta}, \alpha_2(\lambda) = \lambda^{\frac{1}{4}}$. Then, the condition (H_3) of Theorem 3.1 holds. Hence, by Theorem 3.1, the conclusion follows, and the proof is complete.

Remark 4.1. Example 4.1 implies that there is a large number of functions that satisfy the conditions of Theorem 3.1. In addition, the conditions of Theorem 3.1 are also easy to check.

References

[1] R. Agarwal, D. O'Regan, A multiplicity result for second order impulsive differential equations via the Leggett Williams fixed point theorem, Appl. Math. Comput. 161 (2005) 433-439.

[2] M.Q. Feng, D.X. Xie, Multiple positive solutions of multi-point boundary value problem forsecond-order impulsive differential equations, J. Comput. Appl. Math. 223(2009)438-448.

[3] T. Jankowski, Existence of positive solutions to second order four-point impulsive differential problems with deviating arguments, Comput. Math. Appl. 58 (2009) 805-817.

[4] E.K. Lee, Y.H. Lee, Multiple positive solutions of singular two point boundary value problems for second order impulsive differential equations, Appl. Math. Comput. 158 (2004) 745-759.

[5] X.N. Lin, D.Q. Jiang, Multiple positive solutions of Dirichlet boundary value problems for second order impulsive differential equations, J. Math.Anal.Appl. 321 (2006) 501-514.

[6] L.S. Liu, L.L. Hu, Y.H. Wu, Positive solutions of two-point boundary value problems for systems of nonlinear second-order singular and impulsive differential equations, Nonlinear Anal. 69(2008) 3774-3789.

[7] E. Liz, J.J. Nieto, The monotone iterative technique for periodic boundary value problems of second order impulsive differential equations, Comment. Math. Univ. Carolin. 34 (1993) 405-411.

[8] J.J. Nieto, R. Rodriguez-Lopez, Periodic boundary value problem for non-Lipschitzian impulsive functional differential equations, J. Math. Anal. Appl. 318 (2006) 593-610.

[9] I. Rachůnková, M. Tvrdý, Existence results for impulsive second order periodic problems, Nonlinear Anal. 59(2004) 133-146.

[10] W.B. Wang, X.L. Fu, X.X. Yang, Positive solutions of periodic boundary value problems for impulsive differential equations, Comput. Math. Appl. 58(2009) 1623-1630.

[11] X.Q. Zhang, Existence of positive solution for second-order nonlinear impulsive singular differential equations of mixed type in Banach spaces, Nonlinear Anal. 70(2009)1620-1628.

[12] A. Bensedik, M. Bouchekif, Symmetry and uniqueness of positive solutions for a Neumann boundary value problem, Appl. Math. Lett. 20(2007) 419-426.

[13] A. Cabada, R.R.L. Pouse, Existence result for the problem $(\varphi(u'))' = f(t, u, u')$ with periodic and Neumann boundary conditions, Nonlinear Anal. 30 (1997) 1733-1742.

[14] A. Cabada, L. Sanchez, A positive operator approach to the Neumann problem for a second order ordinary differential equation, J. Math. Anal. Appl. 204 (1996) 774-785.

[15] J. Chu, X. Lin, D. Jiang, D. O'Regan, R.P. Agarwal, Positive solutions for second-order superlinear repulsive singular Neumann boundary value problems, Positivity 12 (2008) 555-569.

[16] D. Jiang, H. Liu, Existence of positive solutions to second order Neumann boundary value problem, J. Math. Res. Exposition 20 (2000) 360-364.

[17] F. Li, Y. Zhang, Y. Li, Sign-changing solutions on a kind of fourth-order Neumann boundary value problem, J. Math. Anal. Appl. 344 (2008) 417-428.

[18] J. Sun, W. Li, Multiple positive solutions to second-order Neumann boundary value problems, Appl. Math. Comput. 146 (2003) 187-194.

[19] J. Sun, W. Li, S. Cheng, Three positive solutions for second-order Neumann boundary value problems, Appl. Math. Lett. 17 (2004) 1079-1084.

[20] Y.P. Sun, Y. Sun, Positive solutions for singular semi-positone Neumann boundary value problems, Electron. J. Diff. Eqns. 133(2004) 1-8.

[21] N. Yazidi, Monotone method for singular Neumann problem, Nonlinear Anal. 49 (2002) 589-602.

[22] C.B. Zhai, C.Yang, X.Q. Zhang, Positive solutions for nonlinear operator equations and several classes of applications, Math. Z. 266(2010) 43-63.

[23] D. Franco, J.J. Nieto, A new maximum principle for impulsive first-order problems, Int. J. Theor. Phys. 37 (1998) 1607-1616.

[24] A. Cabada, E. Liz, S. Lois, Green's function and maximum principle for higher order ordinary differential equations with impulses, Rocky MT. J. Math. 30 (2000) 435-446.

[25] A. Cabada, The method of lower and upper solutions for second, third, fourth, and higher order boundary value problems, J. Math. Anal. Appl. 185 (1994) 302-320.

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