Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **68**, 1-13; **http://www.math.u-szeged.hu/ejqtde/**

OSCILLATION RESULTS ON MEROMORPHIC SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS IN THE COMPLEX PLANE

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Abstract. The main purpose of this paper is to consider the oscillation theory on meromorphic solutions of second order linear differential equations of the form $f^{''} + A(z)f = 0$ where A is meromorphic in the complex plane. We improve and extend some oscillation results due to Bank and Laine, Kinnunen, Liang and Liu, and others.

1. Introduction and main results

Let us define inductively, for $r \in [0, +\infty)$, $\exp_1 r = e^r$ and $\exp_{n+1} r = \exp(\exp_n r)$, $n \in \mathbb{N}$. For all r sufficiently large, we define $\log_1 r = \log^+ r = \max\{\log x, 0\}$ and $\log_{n+1} r = \log(\log_n r)$, $n \in \mathbb{N}$. We also denote $\exp_0 r = r = \log_0 r$, $\log_{-1} r = \exp_1 r$ and $\exp_{-1} r = \log_1 r$. We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (e.g. see [8, 21]), such as $T(r, f)$, $m(r, f)$, and $N(r, f)$. Throughout the paper, a meromorphic function f means meromorphic in the complex plane $\mathbb{C}$. To express the rate of fast growth of meromorphic functions, we recall the following definitions (e.g. see [4, 6, 12, 14, 15, 19]).

Definition 1.1. *The iterated n-order $\sigma_n(f)$ of a meromorphic function f is defined by*

$$\sigma_n(f) = \limsup_{r\to\infty} \frac{\log_n T(r, f)}{\log r} \quad (n \in \mathbb{N}).$$

Remark 1.1. *If f is an entire function, then*

$$\sigma_n(f) = \limsup_{r\to\infty} \frac{\log_{n+1} M(r, f)}{\log r}.$$

Definition 1.2. *The growth index (or the finiteness degree) of the iterated order of a meromorphic function f is defined by*

$$i(f) = \begin{cases} 0 & \text{if } f \text{ is rantional,} \\ \min\{n \in \mathbb{N} : \sigma_n(f) < \infty\} & \text{if } f \text{ is transendental and } \sigma_n(f) < \infty \\ & \text{for some } n \in \mathbb{N}, \\ \infty & \text{if } f \text{ with } \sigma_n(f) = \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

2000 *Mathematics Subject Classification.* Primary 34M10; Secondary 30D35, 34M05.
Key words and phrases. meromorphic function, differential equation, complex oscillation, iterated order.
This work was supported by the NSF of Jiangxi (No. 2008GQS0075) and the YFED of Jiangxi (No. GJJ10050) of China.

Definition 1.3. *The iterated convergence exponent of the sequence of a-points of a meromorphic function f is defined by*

$$\lambda_n(f-a) = \limsup_{r\to\infty} \frac{\log_n N(r, \frac{1}{f-a})}{\log r} \quad (n \in \mathbb{N}).$$

Definition 1.4. *The growth index (or the finiteness degree) of the iterated convergence exponent of the sequence of a-points of a meromorphic function f with iterated order is defined by*

$$i_\lambda(f-a) = \begin{cases} 0 & \text{if } n(r, \frac{1}{f-a}) = O(\log r), \\ \min\{n \in \mathbb{N} : \lambda_n(f) < \infty\} & \text{if } \lambda_n(f-a) < \infty \text{ for some } n \in \mathbb{N}\ , \\ \infty & \text{if } \lambda_n(f-a) = \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

Remark 1.2. *Similarly, we can use the notation $\overline{\lambda}_n(f-a)$ to denote the iterated convergence exponent of the sequence of distinct a-points, and use the notation $i_{\overline{\lambda}}(f-a)$ to denote the growth index of $\overline{\lambda}_n(f-a)$.*

It is well-known that Nevanlinna theory has appeared to be a powerful tool in the field of complex differential equations (see [1-7, 11-18, 20], for example) . The active research of the complex oscillation theory of linear differential equations in the complex plane $\mathbb{C}$ was started to investigate the second order differential equation

$$f'' + A(z)f = 0 \tag{1}$$

by Bank and Laine [1, 2]. They investigated this question in the case where A is an entire function, mainly from the point of determining the distribution of zeros of solutions. In this case all solutions of Eq.(1) are entire. When A is meromorphic, there are some immediate difficulties. For example, it is possible that no solution of Eq.(1) except the zero solution is single-valued on the plane. This obstacle was handled since Bank and Laine [2] gave necessary and sufficient conditions for all solutions of Eq.(1) to be meromorphic, and hence single-valued, in a simply-connected region. To consider poles as well as zeros, they obtained the following theorems.

Theorem 1.1. *([2], Theorem 5) Let A be a transcendental meromorphic function of order $\sigma_1(A)$, where $0 < \sigma_1(A) \le \infty$, and assume that $\overline{\lambda}_1(A) < \sigma_1(A)$. Then, if $f \not\equiv 0$ is a meromorphic solution of Eq.(1), we have*

$$\sigma_1(A) \le \max\{\overline{\lambda}_1(f), \overline{\lambda}_1(\frac{1}{f})\}.$$

Theorem 1.2. *([2], Theorem 6) Let A be a transcendental meromorphic function, and assume that Eq.(1) possesses two linearly independent meromorphic solutions f_1 and f_2 satisfying $\overline{\lambda}_1(f_1) < \infty$, $\overline{\lambda}_1(f_2) < \infty$. Then, any solution $f \not\equiv 0$ of Eq.(1) which is not a constant multiple of either f_1 or f_2 satisfies,*

$$\max\{\overline{\lambda}_1(f), \overline{\lambda}_1(1/f)\} = \infty,$$

unless all solutions of Eq.(1) are of finite order (namely, finite iterated 1-order). In the special case where $\overline{\lambda}_1(\frac{1}{A}) < \infty$ (e.g. A is of finite order), we can conclude that $\overline{\lambda}_1(f) = \infty$ unless all solutions of Eq.(1) are finite order.

In [14], Kinnunen obtained some results on the Eq. (1) with entire solutions by using the idea of iterated n-order.

Theorem 1.3. *([14], Theorem 3.2) Let A be an entire function with $i(A) = n$, assuming $0 < n < \infty$. Let f_1 and f_2 be two linearly independent solutions of Eq.(1), and denote $E := f_1 f_2$. Then $i_\lambda(E) \leq n+1$ and*

$$\lambda_{n+1}(E) = \sigma_{n+1}(E) = \max\{\lambda_{n+1}(f_1), \lambda_{n+1}(f_2)\} \leq \sigma_n(A).$$

If $i_\lambda(E) < n$, then $i_\lambda(f) = n+1$ holds for all solutions of type of $f = c_1 f_1 + c_2 f_2$, where $c_1 \neq 0$ and $c_2 \neq 0$.

Theorem 1.4. *([14], Theorem 3.3) Let A be an entire function with $0 < i(A) = n < \infty$, let f be any non-trivial solution of Eq.(1), and assume $\overline{\lambda}_n(A) < \sigma_n(A) \neq 0$. Then $\lambda_{n+1}(f) \leq \sigma_n(A) \leq \lambda_n(f)$.*

It is conjectured that a situation where $\max\{\lambda(f_1), \lambda(f_2)\} < \infty$ for an equation $f'' + A(z)f = 0$ implies that $\max\{\lambda(g_1), \lambda(g_2)\} = \infty$ is true for the equation $g'' + B(z)g = 0$ where $B \neq A$ is sufficiently close to A in some sense (see [15], p.109). Kinnunen obtained the following result, of this type corresponding to Theorem 3.1 in [3].

Theorem 1.5. *([14], Theorem 3.6) Let A be an entire function with $i(A) = n$ and the iterated order $\sigma_n(A) = \sigma$, where $1 < n < \infty$. Let f_1 and f_2 be two linearly independent solutions of Eq.(1) such that $\max\{\lambda_n(f_1), \lambda_n(f_2)\} < \sigma$. Let $\Pi \neq 0$ be any entire function for which either $i(\Pi) < n$ or $i(\Pi) = n$ and $\sigma_n(\Pi) < \sigma$. Then any two linearly independent solutions g_1 and g_2 of the differential equation $g'' + (A(z) + \Pi(z))g = 0$ satisfy $\max\{\lambda_n(g_1), \lambda_n(g_2)\} \geq \sigma$.*

Thus it is interesting to consider the complex oscillation on the meromorphic solutions of the Eq. (1) for the case where A is meromorphic function in the terms of the idea of iterated order. In 2007, Liang and Liu[17] considered the complex oscillation on the Eq. (1) when A is a meromorphic function with finite many poles. By using the Wiman-Valiron theory (for an entire function [9, 11], for a meromorphic function [7, 20]), they obtained some results which extend Theorems 1.3 and 1.4. There arises naturally a question:

Question 1.1. *What can be said if A has infinitely many poles?*

Although the Wiman-Valiron theory is a powerful tool to investigate entire solutions, it is only useful for the meromorphic function A with $\lambda_1(\frac{1}{A}) < \sigma_1(A)$ if considering the Eq. (1). In this paper we shall make use of a recent result due to Chiang and Hayman (see Lemma 2.3 in the next section) instead of the Wiman-Valiron theory, and thus answer the above question. In fact, we obtain the following results which improve and extend some oscillation results due to Bank & Laine [2], Liang & Liu [17] and others. Furthermore, considering the deficiencies of poles of the coefficient A and solutions f of Eq. (1), we obtain some special results. For $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the deficiency of a with respect to a meromorphic function g in $\mathbb{C}$ is defined by

$$\delta(a, g) = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{g-a})}{T(r, g)},$$

provided that g has unbounded characteristic. The first result is the following

Theorem 1.6. *Let A be a meromorphic function with $0 < i(A) = n < \infty$, and assume that $\overline{\lambda}_n(A) < \sigma_n(A) \neq 0$. Then, if f is a nonzero meromorphic solution of the Eq. (1) we have*

$$\sigma_n(A) \leq \max\{\overline{\lambda}_n(f), \overline{\lambda}_n(\frac{1}{f})\}. \tag{2}$$

In the special case where either $\delta(\infty, f) > 0$ or the poles of f are of uniformly bounded multiplicities, we can conclude that

$$\max\{\lambda_{n+1}(f), \lambda_{n+1}(\frac{1}{f})\} \leq \sigma_n(A) \leq \max\{\overline{\lambda}_n(f), \overline{\lambda}_n(\frac{1}{f})\}. \tag{3}$$

Theorem 1.6 improves and extends Theorem 3.3 in [14] and Theorem 5 in [2]. It is obvious from Theorem 1.6 that the following corollary is true, which improves and extends Corollary 3.4 in [14].

Corollary 1.1. *Let A be a meromorphic function with $1 < i(A) = n < \infty$. If $i_{\overline{\lambda}}(A) < n$, then any nonzero meromorphic solution f of the Eq. (1) satisfies $\max\{i_{\overline{\lambda}}(f), i_{\overline{\lambda}}(\frac{1}{f})\} \geq n$.*

The next result improves and extends Theorem 6 and Corollary 7 in [2].

Theorem 1.7. *Let A be a meromorphic function with $0 < i(A) = n < \infty$. Assume that the Eq. (1) possesses two linearly independent meromorphic solutions f_1 and f_2. Denote $E := f_1 f_2$. If $\overline{\lambda}_n(E) < \infty$, then any nonzero solution f of (1) which is not a constant multiple of either f_1 or f_2 satisfies, $\overline{\lambda}_n(f) = \infty$, unless all solutions of (1) are of finite iterated n-order. In the special case where $\delta(\infty, A) > 0$, or $i_\lambda(\frac{1}{A}) < n$, or $\lambda_n(\frac{1}{A}) < \sigma_n(A)$, (e.g. A is an entire function), we can conclude that $\overline{\lambda}_n(f) = \infty$.*

We remark that Theorem 1.7 and the following theorem are the improvement and extension of Theorem 3.2 in [14].

Theorem 1.8. *Let A be a meromorphic function satisfying $0 < i(A) = n < \infty$. Assume that the Eq. (1) possesses two linearly independent meromorphic solutions f_1 and f_2. Denote $E := f_1 f_2$. If $\delta(\infty, A) > 0$, or $i_\lambda(\frac{1}{A}) < n$, or $\lambda_n(\frac{1}{A}) < \sigma_n(A)$, and if either $\delta(\infty, f) > 0$ or the poles of f are of uniformly bounded multiplicities, then we have $i_\lambda(E) \leq n + 1$ and have*

$$\begin{aligned} \overline{\lambda}_{n+1}(E) &= \lambda_{n+1}(E) = \sigma_{n+1}(E) = \max\{\overline{\lambda}_{n+1}(f_1), \overline{\lambda}_{n+1}(f_2)\} \\ &\leq \sigma_{n+1}(f_1) = \sigma_{n+1}(f_1) = \sigma_n(A). \end{aligned}$$

From the proof of Theorem 1.8 one can get the following result.

Corollary 1.2. *Let A be a meromorphic function satisfying $0 < i(A) = n < \infty$. Assume that the Eq. (1) possesses two linearly independent meromorphic solutions f_1 and f_2. Denote $E := f_1 f_2$. If $\delta(\infty, A) > 0$, then we have $i_\lambda(E) \leq n+1$ and have*

$$\begin{aligned} \overline{\lambda}_{n+1}(E) &= \lambda_{n+1}(E) = \sigma_{n+1}(E) = \max\{\overline{\lambda}_{n+1}(f_1), \overline{\lambda}_{n+1}(f_2)\} \\ &\leq \sigma_{n+1}(f_1) = \sigma_{n+1}(f_1). \end{aligned}$$

The following corollary is immediately obtained from Theorem 1.8 which is an improvement of Theorem 1.3.

Corollary 1.3. *Let A be an entire function with $0 < i(A) = n < \infty$. Let f_1 and f_2 be two linear independent solutions of Eq.(1), and denote $E := f_1 f_2$. Then $i_\lambda(E) \leq n+1$ and*

$$\begin{aligned}\overline{\lambda}_{n+1}(E) &= \lambda_{n+1}(E) = \sigma_{n+1}(E) = \max\{\overline{\lambda}_{n+1}(f_1), \overline{\lambda}_{n+1}(f_2)\} \\ &\leq \sigma_{n+1}(f_1) = \sigma_{n+1}(f_1) = \sigma_n(A).\end{aligned}$$

Finally, we show the following result which extends and improves Theorem 1.5.

Theorem 1.9. *Let A be a meromorphic function with $1 < i(A) = n < \infty$. Assume that f_1 and f_2 are two linearly independent meromorphic solutions of the Eq. (1) such that*

$$\max\{\lambda_n(f_1), \lambda_n(f_2)\} < \sigma_n(A). \tag{4}$$

Let $\Pi \neq 0$ be any meromorphic function for which either $i(\Pi) < n$ or $\sigma_n(\Pi) < \sigma_n(A)$. Let g_1 and g_2 be two linearly independent solutions of the differential equation $g^{''} + (A(z) + \Pi(z))g = 0$. Denote $E := f_1 f_2$ and $F := g_1 g_2$. If either

$$\max\{i_\lambda(1/E), i_\lambda(1/F)\} < n \quad \textit{or} \quad \max\{\lambda_n(1/E), \lambda_n(1/F)\} < \sigma_n(A),$$

then $\max\{\lambda_n(g_1), \lambda_n(g_2)\} \geq \sigma_n(A)$.

The remainder of this paper is organized as follows. Section 2 is for some lemmas and the other sections are for the proofs of our main results. The idea and formulations of our main results come from [1, 2, 14]. The proof of Theorem 1.6 is from the proof of Theorem 5 in [2], the proof of Theorem 1.7 is essentially from the proof of Theorem 6 in [2], and the proof of Theorem 1.9 is a parallel to a corresponding reasoning in the proof of Theorem 3.6 in [14].

2. Some lemmas

To prove our results, we need the following lemmas.

Lemma 2.1. *(*[10]*, Theorem 4) Let f be a transcendental meromorphic function not of the form $e^{\alpha z+\beta}$. Then*

$$T\left(\frac{f}{f'}\right) \leq 3\overline{N}(r,f) + 7\overline{N}\left(\frac{1}{f}\right) + 4\overline{N}\left(\frac{1}{f''}\right) + S\left(r, \frac{f}{f'}\right),$$

where $S(r,f) := o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure.

Lemma 2.2. *(*[14]*, Remark 1.3) If f is a transcendental meromorphic function, then $\sigma_n(f) = \sigma_n(f^{'})$.*

Lemma 2.3. *(*[13]*, Theorem 6.2) Let f be a meromorphic solution of*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \ldots + A_1(z) f^{'} + A_0(z) f = 0 \tag{5}$$

where $A_0, \ldots, A_{k-1}$ are meromorphic functions in the plane $\mathbb{C}$. Assume that not all coefficients A_j are constants. Given a real constant $\gamma > 1$, and denoting $T(r) := \sum_{j=0}^{k-1} T(r, A_j)$, we have

$$\log m(r,f) < T(r)\left\{(\log r)\log T(r)\right\}^{\gamma}, \quad \text{if } p = 0;$$

and

$$\log m(r,f) < r^{2p+\gamma-1}T(r)\left\{\log T(r)\right\}^{\gamma}, \quad \text{if } p > 0;$$

outside of an exceptional set E_p with $\int_{E_p} t^{p-1}dt < +\infty$.

We note that in the above lemma, $p = 1$ corresponds to Euclidean measure and $p = 0$ to logarithmic measure. Using logarithmic measure not Euclidean measure, we correct here the proof of Theorem 3.2 in [6].

Lemma 2.4. *([6], Theorem 3.2) Let $A_0, A_1, \ldots, A_{k-1}$ be meromorphic functions such that $0 < \max\{i(A_j) : j = 0, 1, \ldots, k-1\} = n < \infty$. If f is a meromorphic solution of (5) whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$, then $\sigma_{n+1}(f) \leq \max\{\sigma_n(A_j) : j = 0, 1, \ldots, k-1\}$.*

Proof. It obvious that if $\sigma_n(f) < \infty$, then $\sigma_{n+1}(f) = 0 \leq \sigma := \max\{\sigma_n(A_j) : j = 0, 1, \ldots, k-1\}$. Now we assume $\sigma_n(f) = \infty$. By (5) we get that the poles of $f(z)$ can only occur at the poles of $A_0, A_1, \ldots, A_{k-1}$. Note that the multiplicities of poles of f are uniformly bounded, and thus we have

$$N(r,f) \leq M_1\overline{N}(r,f) \leq M_1\sum_{j=0}^{k-1}\overline{N}(r,A_j) \leq M\max\{N(r,A_j) : j = 0,1,\ldots,k-1\},$$

where M_1 and M are some suitable positive constants. This gives

$$T(r,f) = m(r,f) + O(\max\{N(r,A_j) : j = 0,1,\ldots,k-1\}). \tag{6}$$

If $\delta(\infty, f) := \delta_1 > 0$, then for sufficiently large r,

$$m(r,f) \geq \frac{\delta_1}{2}T(r,f). \tag{7}$$

Applying now (6) or (7) with Lemma 2.3, we obtain

$$\log T(r,f) \leq \log m(r,f) + O(\log T(r)) \leq O\left\{T(r)\left\{(\log r)\log T(r)\right\}^{\gamma}\right\}$$

or

$$\log T(r,f) \leq \log(\frac{2}{\delta_1}m(r,f)) \leq O\left\{T(r)\left\{(\log r)\log T(r)\right\}^{\gamma}\right\}$$

outside of an exceptional set E_0 with finite logarithmic measure. Using a standard method to deal with the finite logarithmic measure set, one immediately gets from above inequalities that $\sigma_{n+1}(f) \leq \max\{\sigma_n(A_j) : j = 0, 1, \ldots, k-1\}$. □

Lemma 2.5. *([6], Lemma 3.6) Let $\Phi(r)$ be a continuous and positive increasing function, defined for r on $(0, +\infty)$, with $\sigma_n(\Phi) = \limsup_{r\to\infty} \frac{\log_n \Phi(r)}{\log r}$. Then for any subset E of $[0, +\infty)$ that has finite linear measure, there exists a sequence $\{r_m\}, r_m \notin E$ such that*

$$\sigma_n(\Phi) = \lim_{r_m\to\infty}\frac{\log_n \Phi(r_m)}{\log r_m}.$$

Replacing the notation $n(r, f)$ by $\overline{n}(r, f)$ and following the reasoning of the proof of Lemma 1.7 in [14], one can easily obtain the following lemma.

Lemma 2.6. *Let g_1 and g_2 be two entire functions. Then*

$$i_{\overline{\lambda}}(g_1g_2) = \max\{i_{\overline{\lambda}}(g_1), i_{\overline{\lambda}}(g_2)\}.$$

If $i_{\overline{\lambda}}(g_1g_2) := n > 0$, then

$$\overline{\lambda}_n(g_1g_2) = \max\{\overline{\lambda}_n(g_1), \overline{\lambda}_n(g_2)\}.$$

We recall here the essential part of the factorization theorem for meromorphic functions of finite iterated order.

Lemma 2.7. *([12], Satz 12.4) A meromorphic function f for which $i(f) = n$ can be represented by the form*

$$f(z) = \frac{U(z)e^{g(z)}}{V(z)},$$

where U, V and g are entire functions such that

$$\lambda_n(f) = \lambda_n(U) = \sigma_n(U), \quad \lambda_n(\frac{1}{f}) = \lambda_n(V) = \sigma_n(V)$$

and

$$\sigma_n(f) = \max\{\sigma_n(U), \sigma_n(V), \sigma_n(e^g)\}.$$

The following result plays a key role in the present paper, which is an improvement and extension of Theorem 3.1 in [14] and Theorem 1 in [17].

Lemma 2.8. *Let A be a meromorphic function with $i(A) = n$ $(0 < n < \infty)$, and let f be a nonzero meromorphic solution of the Eq.* (1)*. Then*

(i) if either $\delta(\infty, f) > 0$ or the poles of f are of uniformly bounded multiplicities, then $i(f) \leq n + 1$ and $\sigma_{n+1}(f) \leq \sigma_n(A)$.

(ii) if $\delta(\infty, A) > 0$, or $i_\lambda(\frac{1}{A}) < n$, or $\lambda_n(\frac{1}{A}) < \sigma_n(A)$, then $i(f) \geq n + 1$ and $\sigma_{n+1}(f) \geq \sigma_n(A)$.

Proof. Assume that f is a nonzero meromorphic solution of the Eq. (1). It is obvious that (i) is just a special case of Lemma 2.4.

We now assume that A satisfies $\delta(\infty, A) > 0$, or $i_\lambda(\frac{1}{A}) < n$, or $\lambda_n(\frac{1}{A}) < \sigma_n(A)$. Then we shall prove $i(f) \geq n + 1$ and $\sigma_{n+1}(f) \geq \sigma_n(A)$. By (1), we get

$$-A(z) = \frac{f''}{f}. \tag{8}$$

By the lemma of the logarithmic derivative and (8), we get that

$$m(r, A) \leq m(r, \frac{f''}{f}) = O\{\log(rT(r, f))\}$$

holds for all sufficiently large $r \notin E$, where $E \subset (0, \infty)$ has finite linear measure. Hence

$$T(r, A) = m(r, A) + N(r, A) \leq N(r, A) + O\{\log(rT(r, f))\} \tag{9}$$

holds for all sufficiently large $|z| = r \notin E$.

If $\sigma_n(A)=0$, and hence $i_\lambda(\frac{1}{A})<n$, then by Lemma 2.5 there exists a sequence $\{r_m\}$ such that for all $r_m \notin E_6$,

$$T(r_m, A) \geq \exp_{n-2}\{r_m^M\} \tag{10}$$

holds for any sufficiently large constant $M>0$. If $\sigma_n(A_0)>0$, then again by Lemma 2.5 there exists a sequence $\{r_m\}$ such that for all $r_m \notin E_6$,

$$T(r_m, A) \geq \exp_{n-1}\{r_m^{\sigma-\varepsilon}\} \tag{11}$$

holds for any given ε $(0<\varepsilon<\sigma)$.

Now we consider three cases below.

Case 1. Assume that $\delta(\infty, A) := \delta_2 > 0$. Then for sufficiently large r,

$$\frac{\delta_2}{2} T(r, A) \leq m(r, A) = O\left\{\log\left(rT(r, f)\right)\right\}. \tag{12}$$

If $\sigma_n(A)=0$, then from (10) and (12) we get that $i(f) \geq n+1$ and $\sigma_{n+1}(f) \geq \sigma_n(A)=0$. If $\sigma_n(A)>0$, then from (11) and (12) we get that $i(f) \geq n+1$ and $\sigma_{n+1}(f) \geq \sigma_n(A)$.

Case 2. Assume that $i_\lambda(\frac{1}{A})<n$. Then

$$N(r, A) \leq \exp_{n-2}(r^{\alpha_1}) \tag{13}$$

holds for a positive constant $\alpha_1 < M$. By (13), (9) and either (10) or (11), we get that $i(f) \geq n+1$ and $\sigma_{n+1}(f) \geq \sigma_n(A)$.

Case 3. Assume that $\lambda_n(\frac{1}{A}) < \sigma_n(A) = \sigma$. Then there holds

$$N(r, A) \leq \exp_{n-1}(r^{\lambda_n(\frac{1}{A})+\varepsilon}), \tag{14}$$

where $(0<2\varepsilon<\sigma-\lambda_n(\frac{1}{A}))$. Thus by (9), (14) and (10), we get that $i(f) \geq n+1$ and $\sigma_{n+1}(f) \geq \sigma_n(A)$.

□

Following Hayman [10], we shall use the abbreviation "n. e." (nearly everywhere) to mean "everywhere in $(0,\infty)$ except in a set of finite measure" in the proofs of our main theorems, see the following sections.

3. Proof of Theorem 1.6

Since f is a solution of (1) where $\sigma_n(A)>0$, it is obvious that f can not be rational, nor be of the form e^{az+b} for constants a and b. Hence, by Lemma 2.1 we have

$$T\left(r, \frac{f}{f'}\right) = O\left(\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f''}\right)\right) \quad n.e. \quad \text{as} \quad r \to \infty. \tag{15}$$

In addition, by (1) we have

$$\overline{N}\left(r, \frac{1}{f''}\right) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{A}\right). \tag{16}$$

By assumption, $\overline{\lambda}_n(A) < \sigma_n(A)$. Hence, if we assume that (2) fails to hold, then we deduce by (15) and (16) that $\sigma_n\left(\frac{f}{f'}\right) < \sigma_n(A)$. By the first main theorem, we then see that if $\varphi = \frac{f'}{f}$, then $\sigma_n(\varphi) < \sigma_n(A)$. However, from (1) it easily follows that

$-A = \varphi^{'} + \varphi^2$, and so we obtain $\sigma_n(A) \le \sigma_n(\varphi) < \sigma_n(A)$, a contradiction. Hence, (2) is true.

In the special case where either $\delta(\infty, f) > 0$ or the poles of f are of uniformly bounded multiplicities, by Lemma 2.8 we have

$$\max\{\lambda_{n+1}(\frac{1}{f}), \lambda_{n+1}(f)\} \le \sigma_{n+1}(f) \le \sigma_n(A).$$

Hence, we obtain (3).

4. Proof of Corollary 1.1

Let f be a nonzero meromorphic solution of Eq.(1). Assume that

$$\max\{\overline{\lambda}_{n-1}(f), \overline{\lambda}_{n-1}(\frac{1}{f})\} < \infty,$$

where $n = i(A) > 1$. Then we obtain

$$\overline{N}(r, \frac{1}{f}) = O(\exp_{n-2}(\frac{1}{1-r})^{\alpha_1}) \quad \text{and} \quad \overline{N}(r, f) = O(\exp_{n-2}(\frac{1}{1-r})^{\alpha_2})$$

for some finite constants α_1 and α_2. Similarly, we get from the assumption $i_{\overline{\lambda}}(A) < n$ that

$$\overline{N}(r, \frac{1}{A}) = O(\exp_{n-2}(\frac{1}{1-r})^{\alpha_3})$$

for some finite constant α_3. The relations (15) and (16) now yield

$$T(r, \frac{f}{f'}) = O(\exp_{n-2}(\frac{1}{1-r})^{\alpha})$$

for some finite constant $\alpha \ge \max\{\alpha_1, \alpha_2, \alpha_3\}$. Hence we have

$$\sigma_{n-1}(\frac{f^{'}}{f}) = \sigma_{n-1}(\frac{f}{f'}) < \infty.$$

By the relation $-A = \varphi^{'} + \varphi^2$ where $\varphi = \frac{f^{'}}{f}$, we now obtain $\sigma_{n-1}(A) < \infty$, a contradiction. Therefore, we obtain the conclusion $\max\{i_{\overline{\lambda}}(f), i_{\overline{\lambda}}(\frac{1}{f})\} \ge n$.

5. Proof of Theorem 1.7

Assume that the Eq. (1) possesses two linearly independent meromorphic solutions f_1 and f_2 such that $\overline{\lambda}_n(E) < \infty$, where $E_1 := E = f_1 f_2$. Let $f = af_1 + bf_2$ where a and b are nonzero constants, and set $E_2 := ff_1$. It is easy to see that any pole of f is a pole of A. Since $\overline{\lambda}_n(\frac{1}{A}) \le \sigma_n(A) < \infty$, we thus have $\overline{\lambda}_n(\frac{1}{f}) \le \overline{\lambda}_n(\frac{1}{A}) < \infty$. Assume that $\overline{\lambda}_n(f) = \infty$ fails to hold, so that $\overline{\lambda}_n(f) < \infty$ and $\overline{\lambda}_n(\frac{1}{f}) < \infty$. From these relations we easily see that $\overline{\lambda}_n(E_1) < \infty$ and $\overline{\lambda}_n(E_2) < \infty$. By Lemma D(e) in [2], there is a constant $c > 0$ such that n. e. as $r \to \infty$,

$$T(r, E_j) = O\left(\overline{N}\left(r, \frac{1}{E_j}\right) + T(r, A) + \log r\right) = O\left(\exp_{n-1}(r^c) + T(r, A)\right) \tag{17}$$

for $j = 1, 2$. Since $E_2 = af_1^2 + bE_1$, we thus obtain that n. e. as $r \to \infty$,

$$T(r, f_1) = O\left(\exp_{n-1}(r^c) + T(r, A)\right). \tag{18}$$

Since $A = -\frac{f''}{f}$, we deduce by the lemma of logarithmic derivative that n. e. as $r \to \infty$,

$$m(r, A) = O\left(\log T\left(r, f_1\right) + \log r\right). \tag{19}$$

From (1), we see that any pole of A is at most double and is either a zero or pole of f, we thus have

$$N(r, A) \leq 2\left(\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f)\right).$$

Hence by assumption, $N(r, A) = O\left(\exp_{n-1}(r^d)\right)$ as $r \to \infty$ for some $d > 0$. Together with (18) and (19), we obtain $N(r, A) = O\left(\exp_{n-1}(r^d)\right)$ n. e. as $r \to \infty$, from which it follows by standard reasoning that f_1 is of finite iterated n-order. By the identity of Abel, we have

$$\left(\frac{f_2}{f_1}\right)' = \frac{\beta}{f_1^2}, \tag{20}$$

where β is equal to the Wronskian of f_1 and f_2. Hence, by Lemma 2.2 and (20), we obtain

$$\sigma_n(f_2) = \sigma_n\left(f_1 \frac{f_2}{f_1}\right) \leq \max\left\{\sigma_n\left(\frac{f_2}{f_1}\right), \sigma_n(f_1)\right\} = \sigma_n(f_1).$$

Reversing the roles of f_1 and f_2, we can conclude that $\sigma_n(f_1) = \sigma_n(f_2)$. Hence, all solutions of (1) are of finite iterated n-order if $\overline{\lambda}_n(f) < \infty$.

In special case where $\delta(\infty, A) > 0$, or $i_\lambda(\frac{1}{A}) < n$, or $\lambda_n(\frac{1}{A}) < \sigma_n(A)$, by Lemma 2.8 that all meromorphic solutions $f \not\equiv 0$ of (1) satisfy $i(f) \geq n+1$ and $\sigma_{n+1}(f) \geq \sigma_n(A)$. Therefore, we can conclude that $\overline{\lambda}_n(f) = \infty$ holds for any solution $f \not\equiv 0$ of (1) which is not a constant multiple of either f_1 or f_2.

6. Proof of Theorem 1.8

It is obvious that (see page 664 in [2]) $\sigma_{n+1}(f_1) = \sigma_{n+1}(f_2)$. Assume that $\delta(\infty, A) > 0$, or $i_\lambda(\frac{1}{A}) < n$, or $\lambda_n(\frac{1}{A}) < \sigma_n(A)$, and that either $\delta(\infty, f) > 0$ or the poles of f are of uniformly bounded multiplicities. Then by Lemma 2.8 we obtain

$$\sigma_{n+1}(E) \leq \max\left\{\sigma_{n+1}(f_1), \sigma_{n+1}(f_2)\right\} \leq \sigma_{n+1}(f_1) = \sigma_{n+1}(f_2) = \sigma_n(A) < \infty.$$

By Lemma D(e) in [2], there is a constant $c > 0$ such that n. e. as $r \to \infty$,

$$T(r, E) = O\left(\overline{N}\left(r, \frac{1}{E}\right) + T(r, A) + \log r\right). \tag{21}$$

By the lemma of logarithmic derivative and Lemma 2.8, we have

$$m(r, A) = m(r, \frac{f_1''}{f_1}) = O\left(\log r T(r, f_1)\right) = O(\exp_{n-1}(r^{a_1}))$$

for some $a_1 < \infty$ outside of a possible exceptional set $G \subset [0, \infty)$ with finite linear measure. If $\delta(\infty, A) := \delta_3 > 0$. Then for sufficiently large r,

$$\frac{\delta_3}{2} T(r, A) \leq m(r, A) = O(\exp_{n-1}(r^{a_1})), \quad r \notin G. \tag{22}$$

If either $i_\lambda(\frac{1}{A}) < n$ or $\lambda_n(\frac{1}{A}) < \sigma_n(A) < \infty$, we have

$$N(r, A) = O(\exp_{n-1}(r^{a_2}))$$

for some $a_2 < \infty$. Thus

$$T(r, A) = m(r, A) + N(r, A) = O(\exp_{n-1}(r^a)), \quad r \notin G, \tag{23}$$

where $a = \max\{a_1, a_2\}$. Hence, together with (21) and either (22) or (23) we obtain

$$T(r, E) = O\left(\overline{N}\left(r, \frac{1}{E}\right) + \exp_{n-1}(r^a)\right), \quad r \notin G. \tag{24}$$

Suppose that $\overline{\lambda}_{n+1}(E) < \sigma_{n+1}(E)$, then we have $\overline{N}\left(r, \frac{1}{E}\right) = O\left(\exp_n(r^b)\right)$ for some $b < \sigma_{n+1}(E)$. Together with (24), $T(r, E) = O\left(\exp_n(r^b)\right)$, $r \notin G$, and then by standard reasoning, we obtain $\sigma_{n+1}(E) \le b < \sigma_{n+1}(E)$. This is a contradiction. Hence, we have $\overline{\lambda}_{n+1}(E) \ge \sigma_{n+1}(E)$. Noting that $\overline{\lambda}_{n+1}(E) \le \lambda_{n+1}(E) \le \sigma_{n+1}(E)$, we obtain $\overline{\lambda}_{n+1}(E) = \lambda_{n+1}(E) = \sigma_{n+1}(E)$.

By Lemma D(a) in [2], f_1 and f_2 have no common zeros. Let $f_j = \frac{g_j}{d_j}$, where g_j and d_j have no common zeros, $j = 1, 2$. This implies that g_1 and g_2 have no common zeros, that $\overline{\lambda}_n(f_j) = \overline{\lambda}_n(g_j)$ for $j = 1, 2$, and that $\overline{\lambda}_n(E) = \overline{\lambda}_n(g_1 g_2)$. Hence, by Lemma 2.6, we have $\overline{\lambda}_{n+1}(E) = \max\{\overline{\lambda}_{n+1}(f_1), \overline{\lambda}_{n+1}(f_2)\}$.

Therefore, we obtain the conclusion

$$\begin{aligned} \overline{\lambda}_{n+1}(E) &= \lambda_{n+1}(E) = \sigma_{n+1}(E) = \max\{\overline{\lambda}_{n+1}(f_1), \overline{\lambda}_{n+1}(f_2)\} \\ &\le \sigma_{n+1}(f_1) = \sigma_{n+1}(f_1) = \sigma_n(A) < \infty. \end{aligned}$$

7. Proof of Theorem 1.9

We denote $E := f_1 f_2$ and $F := g_1 g_2$. By a similar argument, by Lemma 1.7 in [14] we obtain $\lambda_n(F) = \max\{\lambda_n(g_1), \lambda_n(g_2)\}$. We assume that $\lambda_n(F) < \sigma_n(A)$.

By the assumption (4), we have

$$\overline{N}(r, \frac{1}{E}) = O\left(\exp_{n-1}(r^\beta)\right)$$

for some $\beta < \sigma_n(A)$ and the iterated order of the function A implies that

$$T(r, A) = O\left(\exp_{n-1}(r^{\sigma_n(A)+\varepsilon})\right).$$

Again by Lemma D(e) in [2], we have also the Eq. (21), and thus we obtain

$$T(r, E) = O\left(\exp_{n-1}(r^\beta)\right).$$

So, we obtain $\sigma_n(E) \le \sigma_n(A)$. On the other hand, by Lemma B(iv) in [2] we have

$$4A = \left(\frac{E'}{E}\right)^2 - 2\frac{E''}{E} - \frac{1}{E^2}, \tag{25}$$

which implies that $\sigma_n(A) \le \sigma_n(E)$. Noting that either $i(\Pi) < n$ or $\sigma_n(\Pi) < \sigma_n(A)$. The same reasoning is valid for the function F, and hence, we have $\sigma_n(E) = \sigma_n(F) = \sigma_n(A)$.

By the assumption (4) and Lemma 2.7, we can write

$$E = \frac{Qe^P}{U}, F = \frac{Re^S}{V}, \tag{26}$$

where $\sigma_n(Q) = \lambda_n(E) < \sigma_n(A)$, $\sigma_n(R) = \lambda_n(F) < \sigma_n(A)$. Together with the assumption that $\max\{i_\lambda(1/E), i_\lambda(1/F)\} < n$ or $\max\{\lambda_n(1/E), \lambda_n(1/F)\} < \sigma_n(A)$, we have

$$\sigma_n(e^P) = \sigma_n(e^S) = \sigma_n(A).$$

Substituting (26) into (25) and following the similar reasoning step by step as in the proof of Theorem 3.1 in [3], one may derive the fact that

$$ce^{2(P-S)} = -\frac{U^2R^2}{V^2Q^2},$$

where $c \neq 0$. Hence,

$$\frac{E^2}{F^2} = \frac{V^2Q^2}{U^2R^2}e^{2(P-S)} = -\frac{1}{c}. \tag{27}$$

From the equations (25), (27) and the similar equation for F,

$$4(A+\Pi) = \left(\frac{F'}{F}\right)^2 - 2\frac{F''}{F} - \frac{1}{F^2}, \tag{28}$$

we obtain

$$4\left(A + \Pi + \frac{1}{c}A\right) = \left(\frac{F'}{F}\right)^2 - 2\frac{F''}{F} + \frac{1}{c}\left(\frac{E'}{E}\right)^2 - \frac{2}{c}\frac{E''}{E}.$$

Since $\infty > i(A) = n > 1$, then by the lemma of logarithmic derivative, we obtain

$$\begin{aligned} T\left(r, A(1+\frac{1}{c}) + \Pi\right) &= m\left(r, A(1+\frac{1}{c}) + \Pi\right) + N\left(r, A(1+\frac{1}{c}) + \Pi\right) \\ &= O\left(\exp_{n-2}(r^{\sigma_n(A)+\varepsilon})\right) \end{aligned}$$

n.e. as $r \to \infty$. This implies that $i(A\left(1+\frac{1}{c}\right)+\Pi\Big) < n$ or $\sigma_n(A\left(1+\frac{1}{c}\right)+\Pi\Big) < \sigma_n(A)$. Hence, c must be -1. Thus $E^2 = F^2$, and so we have $\frac{E'}{E} = \frac{F'}{F}$ and $\frac{E''}{E} = \frac{F''}{F}$. we can see from the equations (25) and (28) that $\Pi = 0$. This is a contradiction.

Acknowledgement. The authors would like to thank the referees for making valuable suggestions and comments to improve the present paper.

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(Received April 14, 2010)

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