Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **62**, 1-14; **http://www.math.u-szeged.hu/ejqtde/**

On the Step-type Contrast Structure of a Second-order Semilinear Differential Equation with Integral Boundary Conditions

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Abstract

In this paper we investigate the step-type contrast structure of a second-order semilinear differential equation with integral boundary conditions. The asymptotic solution is constructed by the boundary function method, and the uniform validity of the formal solution is proved by the theory of differential equalities.

2000 Mathematics Subject Classification. 34B15; 34E05; 34E20.

Key words and phrases. Singular perturbation; contrast structure; integral boundary conditions.

1. Introduction

We shall consider the existence of contrast structure for the following singularly perturbed differential equation with integral boundary conditions

$$\mu^2 \frac{d^2y}{dt^2} = f(t, y), \quad 0 < t < 1, \tag{1.1}$$

$$y(0, \mu) = \int_0^1 h_1(y(s, \mu))ds, \quad y(1, \mu) = \int_0^1 h_2(y(s, \mu))ds, \tag{1.2}$$

where μ is a small and positive parameter, and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$, $h_i : \mathbb{R} \to \mathbb{R}\,(i = 1, 2)$ are $C^{(2)}$-functions.

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Singularly perturbed boundary value problems arise naturally in various applications, and have received more and more attention in recent years. Contrast structures, namely solutions which have internal transition layers, are initially investigated by using the boundary function method by Butuzov and Vasil'eva in 1987 [1], and have recently been one of the hot topics in singular perturbation problems; see [2-8], for instance. The step-type contrast structure of the equation (1.2) with two-point boundary conditions $y(0,\mu) = y^0$ and $y(1,\mu) = y^1$ was considered by Butuzov and Vasil'eva [1]. They gave the conditions which ensure the existence of step-type contrast structure and applied the boundary function method to construct the corresponding asymptotic solution. Recently, Ni and Lin [7] proved rigorously the uniform validity of asymptotic solution by using Nagumo's Theorem.

In 2009, Ni and Wang [8] extended the equation (1.2) to higher dimension and studied the following semilinear singularly perturbed system

$$\begin{aligned} \mu^2 y_1'' &= f_1(y_1, y_2, \ldots, y_n, t), \\ \mu^2 y_2'' &= f_2(y_1, y_2, \ldots, y_n, t), \\ &\vdots \\ \mu^2 y_n'' &= f_n(y_1, y_2, \ldots, y_n, t), \end{aligned}$$

subject to the conditions

$$\begin{gathered} y_k(0,\mu) = y_k^0, \quad k = 1, 2, \ldots, n, \\ y_j'(0,\mu) = z_j^1, \quad k = 1, 2, \ldots, n-1, \\ y_n'(1,\mu) = z_{n'}^1. \end{gathered}$$

The authors gave the conditions under which there exists an internal transition layer, and constructed the uniformly valid asymptotic expansion of a solution with a step-type contrast structure.

To our knowledge, contrast structures of singularly perturbed problems with integral boundary conditions have not been investigated. Boundary value problems with integral boundary conditions have significant applications in thermal conduction [9], semiconductor problems [10], biomedical science [11], and so on. In [12], Cakir and Amiraliyev studied the singularly perturbed nonlocal boundary value problem

$$\varepsilon^2 y'' + \varepsilon a(t) y' - b(t) y = f(t), \quad 0 < t < l, \quad 0 < \varepsilon \ll 1,$$

$$y(0) = y^0, \quad y(1) = y^1 + \int_{l_0}^{l_1} g(s) y(s) ds, \quad 0 \le l_0 < l_1 \le l,$$

where y^0, y^1 are given constants, and $a(t) \ge \alpha > 0$, $b(t) \ge \beta > 0$, $g(t)$ and $f(t)$ are sufficiently smooth functions in $[0,1]$. The authors presented a finite difference method for numerical solutions of the problem which exhibited two boundary layers at $t = 0$ and $t = l$. In [13], Xie and Zhang extended the above problem to general weakly nonlinear singular perturbation problems with integral boundary conditions by using the boundary function method.

The present paper is devoted to investigate the existence of step-type contrast structure for the problem (1.1)-(1.2). Integral boundary conditions (1.2) can be viewed as the generalization of two-point and nonlocal boundary conditions. The boundary function method and the theory of differential inequalities will be applied to obtain the uniformly valid asymptotic solution of the problem (1.1) -(1.2). The main difficulty different from the corresponding two-point boundary value problem lies in that the integral boundary conditions of the two associated problems are coupled, which will be overcome with the aid of the property that boundary layer functions decay exponentially.

The remainder of this paper is organized as follows. In section 2 we give some assumptions and construct the formal asymptotic solution of the original problem. In section 3 the uniform validity of formal solution is proved by the theory of differential equalities.

2. Basic Assumptions and Construction of Asymptotic Solution

Let us begin with two basic assumptions.

$[H_1]$ $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$, $h_i : \mathbb{R} \to \mathbb{R}\,(i = 1, 2)$ are $C^{(2)}$-functions, and $h_i'(x) \geq 0$.

$[H_2]$ The reduced equation $f(t, y) = 0$ has three isolated solutions $\varphi_i(t)$ $(i = 1, 2, 3)$ on $[0, 1]$, satisfying

$$\varphi_1(t) < \varphi_2(t) < \varphi_3(t), \quad \frac{\partial f}{\partial y}(t, \varphi_i(t)) > 0 \ (i = 1, 3), \quad \frac{\partial f}{\partial y}(t, \varphi_2(t)) < 0.$$

Assumption $[H_2]$ is a so-called stability condition. It follows from the assumption $[H_2]$ that in the phase plane (y, y'), the equilibria $(\varphi_{1,3}, 0)$ is a saddle point and $(\varphi_2, 0)$ is a center. We are interested in the solution of step-type which has a transition from the vicinity of $\varphi_1(t)$ to that of $\varphi_3(t)$ at some point $t = t^*$. That is, for some $t^* \in (0, 1)$ the following limit holds:

$$\lim_{\mu \to 0} y(t, \mu) = \begin{cases} \varphi_1(t), & 0 < t < t^*, \\ \varphi_3(t), & t^* < t < 1. \end{cases}$$

$t = t^*$ is called the transition point.

We shall adopt the following strategy which is due to Butuzov and Vasil'eva [1]. We divide the original problem into the two associated pure boundary layer problems, that is, the associated left problem

$$\mu^2 \frac{d^2 y^{(-)}}{dt^2} = f(t, y^{(-)}), \quad 0 < t < t^*,$$
$$y^{(-)}(0,\mu) = \int_0^{t^*} h_1(y^{(-)}(s,\mu))ds + \int_{t^*}^1 h_1(y^{(+)}(s,\mu))ds,$$
$$y^{(-)}(t^*,\mu) = \varphi_2(t^*),$$

and the associated right problem

$$\mu^2 \frac{d^2 y^{(+)}}{dt^2} = f(t, y^{(+)}), \quad t^* < t < 1,$$
$$y^{(+)}(1,\mu) = \int_0^{t^*} h_2(y^{(-)}(s,\mu))ds + \int_{t^*}^1 h_2(y^{(+)}(s,\mu))ds,$$
$$y^{(+)}(t^*,\mu) = \varphi_2(t^*).$$

Considering that the solution $y(t,\mu)$ is smooth at $t = t^*$, it follows that

$$\frac{dy^{(-)}}{dt}(t^*,\mu) = \frac{dy^{(+)}}{dt}(t^*,\mu), \tag{2.1}$$

which is the condition determining the position of transition point.

Here the main difficulty different from the corresponding two-point boundary value problem lies in that the integral boundary conditions of the two associated problems are coupled. In order to overcome this difficulty, we need to handle these two associated problems simultaneously, and have the aid of properties of boundary layer functions to uncouple the boundary conditions.

Let us describe the formal scheme of seeking an asymptotic solution of the problem (1.1)-(1.2) of the form

$$\overline{y}(t,\mu) = \begin{cases} \overline{y}^{(-)}(t,\mu) + \Pi^{(-)}y(\tau_0,\mu) + Q^{(-)}y(\tau,\mu), & 0 \le t \le t^*, \\ \overline{y}^{(+)}(t,\mu) + R^{(+)}y(\tau_1,\mu) + Q^{(+)}y(\tau,\mu), & t^* \le t \le 1, \end{cases} \tag{2.2}$$

where

$$\overline{y}^{(\mp)}(t,\mu) = \overline{y}_0^{(\mp)}(t) + \mu \overline{y}_1^{(\mp)}(t) + \mu^2 \overline{y}_2^{(\mp)}(t) + \cdots$$

are the regular parts of the left problem and the right problem, respectively,

$$\Pi^{(-)}y(\tau_0,\mu) = \Pi_0^{(-)}y(\tau_0) + \mu\Pi_1^{(-)}y(\tau_0) + \mu^2\Pi_2^{(-)}y(\tau_0) + \cdots, \quad \tau_0 = \frac{t}{\mu}$$

is the left boundary layer part of the left problem,

$$Q^{(\mp)}y(\tau,\mu) = Q_0^{(\mp)}y(\tau) + \mu Q_1^{(\mp)}y(\tau) + \mu^2 Q_2^{(\mp)}y(\tau) + \cdots, \quad \tau = \frac{t-t^*}{\mu}$$

are the right boundary layer part of the left problem and the left boundary layer part of the right problem, respectively, and

$$R^{(+)}y(\tau_1,\mu) = R_0^{(+)}y(\tau_1) + \mu R_1^{(+)}y(\tau_1) + \mu^2 R_2^{(+)}y(\tau_1) + \cdots, \quad \tau_1 = \frac{t-1}{\mu}$$

is the right boundary layer part of the right problem. We also seek the asymptotic expansion of the transition point t^*:

$$t^* = t_0 + \mu t_1 + \mu^2 t_2 + \cdots.$$

Substituting (2.2) into (1.1) and equating the coefficients in like powers of μ, we get a recurrent sequence of algebraic equations for the functions $\overline{y}_i^{(\mp)}(t)$ $(i = 1, 2, \ldots)$.

$$f\left(t, \overline{y}_0^{(\mp)}(t)\right) = 0,$$

$$\frac{d^2\overline{y}_0^{(\mp)}}{dt^2} = \frac{\partial f}{\partial y}\left(t, \overline{y}_0^{(\mp)}(t)\right)\overline{y}_2^{(\mp)}(t),$$

$$\frac{d^2\overline{y}_2^{(\mp)}}{dt^2} = \frac{\partial f}{\partial y}\left(t, \overline{y}_0^{(\mp)}(t)\right)\overline{y}_4^{(\mp)}(t) + g_4^{(\pm)},$$

$$\cdots$$

$$\frac{d^2\overline{y}_{2k-2}^{(\mp)}}{dt^2} = \frac{\partial f}{\partial y}\left(t, \overline{y}_0^{(\mp)}(t)\right)\overline{y}_{2k}^{(\mp)}(t) + g_{2k}^{(\pm)},$$

where $g_{2k}^{(\pm)}$ are the determined functions of $\overline{y}_i^{(\mp)}$ $(0 \le i \le 2k-2)$. From the assumption $[H_2]$, the coefficients $\overline{y}_i^{(\mp)}$ can be obtained recurrently. In particular, we have

$$\overline{y}_0^{(-)} = \varphi_1(t), \quad \overline{y}_0^{(+)} = \varphi_3(t).$$

For simplicity, we only consider the approximation of first order for the boundary layer series. The left boundary layer functions $\Pi_0^{(-)}y(\tau_0,\mu)$ and $\Pi_1^{(-)}y(\tau_0,\mu)$ satisfy

$$\begin{cases} \dfrac{d^2\Pi_0^{(-)}y}{d\tau_0^2} = f\left(0, \varphi_1(0) + \Pi_0^{(-)}y\right), \\ \Pi_0^{(-)}y(0) = \displaystyle\int_0^{t_0} h_1(\varphi_1(s))ds + \int_{t_0}^1 h_1(\varphi_3(s))ds - \varphi_1(0), \\ \Pi_0^{(-)}y(+\infty) = 0, \end{cases} \tag{2.3}$$

and

$$\frac{d^2\Pi_1^{(-)}y}{d\tau_0^2} = \frac{\partial f}{\partial y}\left(0, \varphi_1(0) + \Pi_0^{(-)}y\right)\Pi_1^{(-)}y + \Delta_1^{(-)},$$

$$\Pi_1^{(-)}y(0) = \int_0^{+\infty} h_1'(\varphi_1(0))\Pi_0^{(-)}y(s)ds + \int_{-\infty}^0 h_1'(\varphi_1(t_0))Q_0^{(-)}y(s)ds$$
$$+ \int_0^{+\infty} h_1'(\varphi_3(t_0))Q_0^{(+)}y(s)ds + \int_{-\infty}^0 h_1'(\varphi_3(1))R_0^{(+)}y(s)ds,$$

$$\Pi_1^{(-)}y(+\infty) = 0,$$

respectively, where

$$\Delta_1^{(-)} = \frac{\partial f}{\partial y}\left(0, \varphi_1(0) + \Pi_0^{(-)}y\right)\left(\overline{y}_0^{(-)\prime}(0)\tau_0 + \overline{y}_1^{(-)}(0)\right) + \frac{\partial f}{\partial t}\left(0, \varphi_1(0) + \Pi_0^{(-)}y\right)\tau_0$$
$$-\frac{\partial f}{\partial y}\left(0, \varphi_1(0)\right)\left(\overline{y}_0^{(-)\prime}(0)\tau_0 + \overline{y}_1^{(-)}(0)\right) - \frac{\partial f}{\partial t}\left(0, \varphi_1(0)\right)\tau_0.$$

Analogously, the right boundary layer functions $R_0^{(+)}y(\tau_1)$ and $R_1^{(+)}y(\tau_1)$ satisfy the following boundary value problems

$$\begin{cases} \dfrac{d^2R_0^{(+)}y}{d\tau_1^2} = f\left(1, \varphi_3(1) + R_0^{(+)}y\right), \\ R_0^{(+)}y(1) = \displaystyle\int_0^{t_0} h_2(\varphi_1(s))ds + \int_{t_0}^1 h_2(\varphi_3(s))ds - \varphi_3(1), \\ R_0^{(+)}y(-\infty) = 0, \end{cases} \tag{2.4}$$

and

$$\frac{d^2R_1^{(+)}y}{d\tau_1^2} = \frac{\partial f}{\partial y}\left(1, \varphi_3(1) + R_0^{(+)}y\right)R_1^{(+)}y + \Delta_1^{(+)},$$
$$R_1y^{(+)}(1) = \int_0^{+\infty} h_2'(\varphi_1(0))\Pi_0^{(-)}y(s)ds + \int_{-\infty}^0 h_2'(\varphi_1(t_0))Q_0^{(-)}y(s)ds$$
$$+\int_0^{+\infty} h_2'(\varphi_3(t_0))Q_0^{(+)}y(s)ds + \int_{-\infty}^0 h_2'(\varphi_3(1))R_0^{(+)}y(s)ds,$$
$$R_1^{(+)}y(-\infty) = 0,$$

respectively, where

$$\Delta_1^{(+)} = \frac{\partial f}{\partial y}\left(1, \varphi_3(1) + R_0^{(+)}y\right)\left(\overline{y}_0^{(+)\prime}(1)\tau_1 + \overline{y}_1^{(+)}(1)\right) + \frac{\partial f}{\partial t}\left(1, \varphi_3(1) + R_0^{(+)}y\right)\tau_1$$
$$-\frac{\partial f}{\partial y}\left(1, \varphi_3(1)\right)\left(\overline{y}_0^{(+)\prime}(1)\tau_1 + \overline{y}_1^{(+)}(1)\right) - \frac{\partial f}{\partial t}\left(1, \varphi_3(1)\right)\tau_1.$$

In order to ensure the existence of solutions for the boundary value problems (2.3) and (2.4), we need the following assumptions. See [3] for their geometrical interpretation.

$[H_3]$ In the phase plane $\left(\Pi_0^{(-)}y, \dfrac{d\Pi_0^{(-)}y}{d\tau_0}\right)$, let the straight line $\dfrac{d\Pi_0^{(-)}y}{d\tau_0} = \displaystyle\int_0^{t_0} h_1(\varphi_1(s))ds + \int_{t_0}^1 h_1(\varphi_3(s))ds$ intersect the separatrix entering the saddle $(0, \varphi_1(0))$ as $\tau_0 \to +\infty$;

$[H_4]$ In the phase plane $\left(R_0^{(+)}y, \dfrac{dR_0^{(+)}y}{d\tau_1}\right)$, let the straight line $\dfrac{dR_0^{(+)}y}{d\tau_1} = \displaystyle\int_0^{t_0} h_2(\varphi_1(s))ds + \int_{t_0}^1 h_2(\varphi_3(s))ds$ intersect the separatrix entering the saddle $(0, \varphi_3(1))$ as $\tau_1 \to -\infty$.

Note that t_0 in the above assumptions is unknown, which is determined by (2.10).

From the assumptions $[H_1]$, $[H_2]$ and $[H_3]$, we have the following estimates of decaying exponentially for the boundary layer functions $\Pi_0^{(-)}y(\tau_0)$, $\Pi_1^{(-)}y(\tau_0)$, $R_0^{(+)}y(\tau_1)$ and $R_1^{(+)}y(\tau_1)$.

Lemma 2.1 *Under the assumptions* $[H_1]$, $[H_2]$ *and* $[H_3]$, *the following estimates*

$$\left|\Pi_0^{(-)}y\right| \le c_1\exp(-\kappa_1\tau_0), \quad \left|\frac{d\Pi_0^{(-)}y}{d\tau_0}\right| \le c_1\exp(-\kappa_1\tau_0), \quad \tau_0 \ge 0,$$

$$\left|R_0^{(+)}y\right| \le c_2\exp(\kappa_2\tau_1), \quad \left|\frac{dR_0^{(+)}y}{d\tau_1}\right| \le c_2\exp(\kappa_2\tau_1), \quad \tau_1 \le 0,$$

$$\left|\Pi_1^{(-)}y\right| \le c_3\exp(-\kappa_3\tau_0), \quad \tau_0 \ge 0; \quad \left|R_1^{(+)}y\right| \le c_4\exp(\kappa_4\tau_1), \quad \tau_1 \le 0$$

hold, where c_i *and* κ_i $(i = 1, 2, 3, 4)$ *are positive constants.*

Proof. The proof is essential similar to that of [3], and we omit it here.

Let us now consider the right boundary layer of the left problem and the left boundary layer of the right problem, that is, the interior layer of the original problem. We rewrite the equation (1.1) into the equivalent system

$$\mu\frac{dz}{dt} = f(t, y), \quad \mu\frac{dy}{dt} = z. \tag{2.5}$$

Substituting (2.2) into (2.5), and separate the equations according to the scales t and τ, we obtain

$$\mu\frac{d\overline{z}^{(\mp)}}{dt} = f\left(t, \overline{y}^{(\mp)}(t,\mu)\right), \quad \mu\frac{d\overline{y}^{(\mp)}}{dt} = \overline{z}^{(\mp)},$$

$$\frac{dQ^{(\mp)}z}{d\tau} = f\left(t^* + \mu\tau, \overline{y}^{(\mp)}(t^* + \mu\tau, \mu) + Q^{(\mp)}y\right) - f\left(t^* + \mu\tau, \overline{y}^{(\mp)}(t^* + \mu\tau, \mu)\right),$$

$$\frac{dQ^{(\mp)}y}{d\tau} = Q^{(\mp)}z.$$

Therefore, the coefficients $Q_0^{(\mp)}y$ and $Q_0^{(\mp)}z$ are determined by the following boundary value problems

$$\begin{cases} \dfrac{dQ_0^{(\mp)}z}{d\tau} = f\left(t_0, \varphi_{1,3}(t_0) + Q_0^{(\mp)}y\right), \\ \dfrac{dQ_0^{(\mp)}y}{d\tau} = Q_0^{(\mp)}z, \\ Q_0^{(\mp)}y(0) = \varphi_2(t_0) - \varphi_{1,3}(t_0), \\ Q_0^{(\mp)}y(\mp\infty) = Q_0^{(\mp)}z(\mp\infty) = 0. \end{cases} \tag{2.6}$$

By the transformations $\widetilde{y}^{(\mp)} = \overline{y}_0^{(\mp)}(t_0) + Q_0^{(\mp)}y$, $\widetilde{z}^{(\mp)} = \overline{z}_0^{(\mp)}(t_0) + Q_0^{(\mp)}z = Q_0^{(\mp)}z$, the problem (2.6) becomes

$$\begin{cases} \dfrac{d\widetilde{z}^{(\mp)}}{d\tau} = f\left(t_0, \widetilde{y}^{(\mp)}\right), \\ \dfrac{d\widetilde{y}^{(\mp)}}{d\tau} = \widetilde{z}^{(\mp)}, \\ \widetilde{y}^{(\mp)}(0) = \varphi_2(t_0), \\ \widetilde{y}^{(\mp)}(\mp\infty) = \varphi_{1,3}(t_0), \\ \widetilde{z}^{(\mp)}(\mp\infty) = 0. \end{cases} \tag{2.7}$$

It follows from the assumption $[H_2]$ that there exists a solution of the problem (2.7) for given t_0. In what follows we will give the condition determining t_0. Integrating the first two equations in (2.7) we have

$$\left[\widetilde{z}^{(-)}(\tau)\right]^2 = 2\int_{\varphi_1(t_0)}^{\widetilde{y}^{(-)}(\tau)} f(t_0, y)dy, \quad \left[\widetilde{z}^{(+)}(\tau)\right]^2 = 2\int_{\varphi_3(t_0)}^{\widetilde{y}^{(+)}(\tau)} f(t_0, y)dy. \tag{2.8}$$

Note that the zero order approximation of the smooth connection condition (2.1) becomes

$$\widetilde{z}^{(-)}(0) = \widetilde{z}^{(+)}(0). \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$I(t_0) \equiv \int_{\varphi_1(t_0)}^{\varphi_3(t_0)} f(t_0, y)dy = 0, \tag{2.10}$$

which is the equation determining the dominant term t_0 of t^*.

$[H_4]$ Assume that the equation (2.10) has a root $t = t_0$ with $I'(t_0) < 0$.

In a similar way, we can also get the expression t_1 which is closely related to the equations for $\overline{y}^{(\mp)}, Q_1^{(\mp)}y$ and $Q_1^{(\mp)}z$, and the details are omitted here.

Similar to Lemma 2.1, for the boundary layer functions $Q_i^{(\mp)}y$ and $Q_1^{(\mp)}z$ $(i = 0, 1)$ we have the following estimates of decaying exponentially.

Lemma 2.2 *Under the assumptions $[H_1]$, $[H_2]$ and $[H_3]$, the following estimates*

$$\left|Q_0^{(-)}y(\tau)\right| \le c_5 \exp(\kappa_5\tau), \quad \left|Q_0^{(-)}z(\tau)\right| \le c_5 \exp(\kappa_5\tau), \quad \tau \le 0,$$

$$\left|Q_1^{(-)}y(\tau)\right| \le c_6 \exp(\kappa_6\tau), \quad \left|Q_1^{(-)}z(\tau)\right| \le c_5 \exp(\kappa_5\tau), \quad \tau \le 0,$$

$$\left|Q_0^{(+)}y(\tau)\right| \le c_7 \exp(-\kappa_7\tau), \quad \left|Q_0^{(+)}z(\tau)\right| \le c_7 \exp(-\kappa_7\tau), \quad \tau \ge 0,$$

$$\left|Q_1^{(+)}y(\tau)\right| \le c_8 \exp(-\kappa_8\tau), \quad \left|Q_1^{(+)}z(\tau)\right| \le c_8 \exp(-\kappa_8\tau), \quad \tau \ge 0$$

hold, where c_i and κ_i $(i = 5, 6, 7, 8)$ are positive constants.

3. Existence of Step-type Solution

In this section we will prove the existence of step-type solution for the original problem and give the estimate for the remainder term.

Theorem 3.3 *Under the assumptions* $[H_1]-[H_4]$*, there exists a step-type contrast structure solution* $y(t,\mu)$ *of the problem (1.1)-(1.2) for sufficiently small* $\mu > 0$*. Moreover, the following asymptotic expansion holds*

$$y(t,\mu) = \begin{cases} \varphi_1(t) + \Pi_0^{(-)}y(\tau_0) + Q_0^{(-)}y(\tau) + O(\mu), & 0 \le t < t_0 + \mu t_1; \\ \varphi_3(t) + R_0^{(+)}y(\tau_1) + Q_0^{(+)}y(\tau) + O(\mu), & t_0 + \mu t_1 < t \le 1. \end{cases} \tag{3.1}$$

In order to prove Theorem 3.3 we need the following lemma which is a slight modification of Theorem 2.2 in [14].

Lemma 3.4 *Assume that the assumption* $[H_1]$ *holds and the continuous functions* $\alpha(t,\mu)$ *and* $\beta(t,\mu)$ *are of* $C^{(2)}$ *class on the intervals* $(0,t_\alpha)\cup(t_\alpha,1)$ *and* $(0,t_\beta)\cup(t_\beta,1)$*, respectively, having the following properties*

(1) $\alpha(t,\mu) \le \beta(t,\mu), \quad t \in [0,1];$

(2) $\mu^2 \dfrac{d^2\alpha}{dt^2} \ge f(t,\alpha), \quad t \in (0,t_\alpha)\cup(t_\alpha,1); \quad \mu^2 \dfrac{d^2\beta}{dt^2} \le f(t,\beta), \quad t \in (0,t_\beta)\cup(t_\beta,1);$

(3) $\alpha(0,\mu) \le \displaystyle\int_0^1 h_1(\alpha(s,\mu))ds, \quad \alpha(1,\mu) \le \int_0^1 h_2(\alpha(s,\mu))ds,$

$\beta(0,\mu) \ge \displaystyle\int_0^1 h_1(\beta(s,\mu))ds, \quad \beta(1,\mu) \ge \int_0^1 h_2(\beta(s,\mu))ds;$

(4) $\dfrac{d\alpha}{dt}(t_\alpha-) \le \dfrac{d\alpha}{dt}(t_\alpha+), \quad \dfrac{d\beta}{dt}(t_\beta-) \ge \dfrac{d\beta}{dt}(t_\beta+),$

where $t_\alpha, t_\beta \in (0,1)$*. Then, there exists a solution* $y(t,\mu)$ *of the problem (1.1)-(1.2) such that*

$$\alpha(t,\mu) \le y(t,\mu) \le \beta(t,\mu), \quad t \in [0,1].$$

Remark 3.5 The functions $\alpha(t,\mu)$ and $\beta(t,\mu)$ satisfying the above conditions are called lower and upper solutions of the problem (1.1)-(1.2), respectively.

Remark 3.6 It is requested in [14] that the functions $\alpha(t,\mu), \beta(t,\mu) \in C^{(2)}[0,1]$. Here we only need the functions $\alpha(t,\mu)$ and $\beta(t,\mu)$ to be piecewise $C^{(2)}$- smooth and an additional condition (4). It should be noted that the proof of Lemma 3.4 has no essential difference from that of Theorem 2.2 in [14], but some slight modifications.

Proof of Theorem 3.3. We select the auxiliary functions

$$\alpha(t,\mu)=\begin{cases}\varphi_1(t)+\Pi_0^{(-)}y(\tau_0)+\mu\Pi_1^{(-)}y(\tau_0)+Q_{0\alpha}^{(-)}y(\tau_\alpha)+\mu Q_{1\alpha}^{(-)}y(\tau_\alpha)-\gamma\mu, \ 0\le t\le t_\alpha,\\ \varphi_3(t)+R_0^{(+)}y(\tau_1)+\mu R_1^{(+)}y(\tau_1)+Q_{0\alpha}^{(+)}y(\tau_\alpha)+\mu Q_{1\alpha}^{(+)}y(\tau_\alpha)-\gamma\mu, \ t_\alpha\le t\le 1,\end{cases}$$

and

$$\beta(t,\mu)=\begin{cases}\varphi_1(t)+\Pi_0^{(-)}y(\tau_0)+\mu\Pi_1^{(-)}y(\tau_0)+Q_{0\beta}^{(-)}y(\tau_\beta)+\mu Q_{1\beta}^{(-)}y(\tau_\beta)+\gamma\mu, \ 0\le t\le t_\beta,\\ \varphi_3(t)+R_0^{(+)}y(\tau_1)+\mu R_1^{(+)}y(\tau_1)+Q_{0\beta}^{(+)}y(\tau_\beta)+\mu Q_{1\beta}^{(+)}y(\tau_\beta)+\gamma\mu, \ t_\beta\le t\le 1,\end{cases}$$

where

$$t_\alpha=t_0+\mu\delta,\quad t_\beta=t_0-\mu\delta,\quad \tau_\alpha=\frac{t-t_\alpha}{\mu},\quad \tau_\beta=\frac{t-t_\beta}{\mu},$$

while γ,δ are sufficiently large positive parameters. The functions $Q_{0\alpha}^{(\mp)}y$ and $Q_{1\alpha}^{(\mp)}y$ satisfy respectively the following boundary value problems

$$\frac{d^2Q_{0\alpha}^{(\mp)}y}{d\tau_\alpha^2}=f\left(\tau_\alpha,\varphi_{1,3}(t_\alpha)+Q_{0\alpha}^{(\mp)}y\right),$$
$$Q_{0\alpha}^{(\mp)}y(0)=\varphi_2(t_\alpha)-\varphi_{1,3}(t_\alpha),\quad Q_{0\alpha}^{(\mp)}y(\mp\infty)=0,$$

and

$$\frac{d^2Q_{1\alpha}^{(\mp)}y}{d\tau_\alpha^2}=\frac{\partial f}{\partial y}\left(\tau_\alpha,\varphi_{1,3}(t_\alpha)+Q_{0\alpha}^{(\mp)}y\right)Q_{1\alpha}^{(\mp)}y+\Delta_{1\alpha}^{(\mp)}-\omega\exp(\pm\kappa_0\tau_\alpha),$$
$$Q_{1\alpha}^{(\mp)}y(0)=\left(\varphi_2'(t_0)-\varphi_{1,3}'(t_0)\right)t_1,\quad Q_{1\alpha}^{(\mp)}y(\mp\infty)=0,$$

where ω,κ_0 are positive constants and

$$\Delta_{1\alpha}^{(\mp)}=\frac{\partial f}{\partial y}\left(\tau_\alpha,\varphi_{1,3}(t_\alpha)+Q_{0\alpha}^{(\mp)}y\right)\varphi_{1,3}'(t_\alpha)\tau_\alpha+\frac{\partial f}{\partial t}\left(\tau_\alpha,\varphi_{1,3}(t_\alpha)+Q_{0\alpha}^{(\mp)}y\right)\tau_\alpha.$$

The functions $Q_{0\beta}^{(\mp)}y$ and $Q_{1\beta}^{(\mp)}y$ are also determined by the corresponding boundary value problems.

To verify the conditions in Lemma 3.4 we divide the interval $[0,1]$ into five subintervals $[0,t_\beta/2]$, $[t_\beta/2,t_\beta]$, $[t_\beta,t_\alpha]$, $[t_\alpha,(t_\alpha+1)/2]$ and $[(t_\alpha+1)/2,1]$.

Let us first check the condition (1). On the intervals $[0,t_\beta/2]$ and $[(t_\alpha+1)/2,1]$, $\beta(t,\mu)-\alpha(t,\mu)=2\gamma\mu+EST>0$, where EST denotes exponentially small terms. On the interval $[t_\beta,t_\alpha]$, $\beta(t,\mu)-\alpha(t,\mu)=\varphi_3(t)-\varphi_1(t)+Q_{0\beta}^{(+)}y(\tau_\beta)-Q_{0\alpha}^{(-)}y(\tau_\alpha)+\mu\left(Q_{1\beta}^{(+)}y(\tau_\beta)-Q_{1\alpha}^{(-)}y(\tau_\alpha)\right)+2\gamma\mu+EST>0$. On the interval $[t_\beta/2,t_\beta]$,

$$\beta(t,\mu)-\alpha(t,\mu)=2\gamma\mu+Q_{0\beta}^{(-)}y(\tau_\beta)-Q_{0\alpha}^{(-)}y(\tau_\alpha)+\mu\left(Q_{1\beta}^{(-)}y(\tau_\beta)-Q_{1\alpha}^{(-)}y(\tau_\alpha)\right)>0,$$

where we have used the following formula

$$Q^{(-)}_{0\beta}y(\tau_\beta) - Q^{(-)}_{0\alpha}y(\tau_\alpha) = \frac{dQ^{(-)}_0 y}{d\tau}(\tau^*)2\delta > 0, \quad \tau_\alpha \le \tau^* \le \tau_\beta.$$

Similarly, on the interval $[t_\alpha, (t_\alpha+1)/2]$,

$$\beta(t,\mu) - \alpha(t,\mu) = 2\gamma\mu + Q^{(+)}_{0\beta}y(\tau_\beta) - Q^{(+)}_{0\alpha}y(\tau_\alpha) + \mu\left(Q^{(+)}_{1\beta}y(\tau_\beta) - Q^{(+)}_{1\alpha}y(\tau_\alpha)\right) > 0.$$

Next we check the condition (2). Here we only verify the condition (2) on the interval $[t_\beta/2, t_\beta]$, which are similar on other subintervals.

$$\begin{aligned}\mu^2\frac{d^2\beta}{dt^2} - f(t,\beta) = \mu^2\varphi_1''(t) &+ \frac{d^2Q^{(-)}_{0\beta}y}{d\tau^2} + \mu\frac{d^2Q^{(-)}_{1\beta}y}{d\tau^2} \\ &-f\left(t, \varphi_1(t) + \gamma\mu + Q^{(-)}_{0\beta}y + \mu Q^{(-)}_{1\beta}y\right) + EST. \end{aligned} \tag{3.2}$$

We rewrite f in $f = \widetilde{f}(\mu) + \overline{f}(\mu)$, where

$$\begin{aligned}\widetilde{f}(\mu) &= f\left(\tau\mu, \varphi_1(\tau\mu) + \gamma\mu + Q^{(-)}_{0\beta}y + \mu Q^{(-)}_{1\beta}y\right) - f\left(\tau\mu, \varphi_1(\tau\mu) + \gamma\mu\right) \\ &= \frac{d^2Q^{(-)}_{0\beta}y}{d\tau^2} + \mu\left(\frac{d^2Q^{(-)}_{1\beta}y}{d\tau^2} - \omega\exp(\kappa_0\tau)\right) + O\left(\mu^2\right),\end{aligned} \tag{3.3}$$

and

$$\overline{f}(\mu) = f\left(t, \varphi_1(t) + \gamma\mu\right) = \frac{\partial f}{\partial y}\left(t, \varphi_1(t)\right)\gamma\mu + O\left(\mu^2\right). \tag{3.4}$$

Inserting (3.3) and (3.4) into (3.2) we have

$$\mu^2\frac{d^2\beta}{dt^2} - f(t,\beta) = -\frac{\partial f}{\partial y}\left(t, \varphi_1(t)\right)\gamma\mu + \mu\omega\exp(\kappa_0\tau) + O\left(\mu^2\right) < 0.$$

In a similar way we can show that

$$\mu^2\frac{d^2\alpha}{dt^2} \ge f(t,\alpha), \quad t \in (0, t_\alpha) \cup (t_\alpha, 1).$$

We now show that

$$\beta(0,\mu) \ge \int_0^1 h_1(\beta(s,\mu))ds.$$

It follows from the construction of asymptotic solution that

$$\begin{aligned}&\beta(0,\mu) - \int_0^1 h_1(\beta(s,\mu))ds \\ &= \varphi_1(0) + \Pi^{(-)}_0 y(0) + \mu\Pi^{(-)}_1 y(0) + \gamma\mu - \int_0^{t_\beta} h_1(\beta(s,\mu))ds - \int_{t_\beta}^1 h_1(\beta(s,\mu))ds + EST\end{aligned}$$

$$
\begin{aligned}
&= \mu \Pi_1^{(-)} y(0) + \gamma\mu + \int_{t_0-\delta\mu}^{t_0} \left(h_1(\varphi_1(s)) - h_1(\varphi_3(s))\right) ds - \mu \int_0^{\frac{t_0-\delta\mu}{\mu}} h_1'(\varphi_1(0)) \Pi_0^{(-)} y(s) ds \\
&\quad -\mu \int_{\delta-\frac{t_0}{\mu}}^{0} h_1'(\varphi_1(t_0)) Q_0^{(-)} y(s) ds - \mu \int_0^{\frac{1-t_0}{\mu}-\delta} h_1'(\varphi_3(t_0)) Q_0^{(+)} y(s) ds \\
&\quad -\mu \int_{-\frac{1-t_0}{\mu}-\delta}^{0} h_1'(\varphi_3(1)) R_0^{(+)} y(s) ds + O\left(\mu^2\right) \\
&= \gamma\mu + O(\mu) > 0,
\end{aligned}
$$

provided that γ is large enough.

Other inequalities in the condition (3) can be proved analogously.

Finally, let us check the condition (4).

$$
\begin{aligned}
&\mu \frac{d\beta}{dt}(t_\beta -) - \mu \frac{d\beta}{dt}(t_\beta +) = \mu \left[\frac{d\beta}{dt}\right]_{(+)}^{(-)} \\
&= \left[\mu \overline{y}_0'(t_\beta) + \frac{dQ_{0\beta} y}{d\tau_\beta} + \mu \frac{dQ_{1\beta} y}{d\tau_\beta}\right]_{(+)}^{(-)} = \left[\frac{dQ_{0\beta} y}{d\tau_\beta}\right]_{(+)}^{(-)} + O(\mu). \qquad (3.5)
\end{aligned}
$$

From the process similar to (2.6)-(2.10) we can obtain

$$
\left[\frac{dQ_{0\beta} y}{d\tau_\beta}\right]_{(+)}^{(-)} = \int_{\varphi_1(t_\beta)}^{\varphi_3(t_\beta)} f(t_\beta, y) dy = -I'(t_0)\delta\mu + O\left(\mu^2\right) > 0. \qquad (3.6)
$$

It follows from (3.5) and (3.6) that

$$
\frac{d\beta}{dt}(t_\beta -) \geq \frac{d\beta}{dt}(t_\beta +).
$$

We can prove in a similar way that

$$
\frac{d\alpha}{dt}(t_\alpha -) \leq \frac{d\alpha}{dt}(t_\alpha +).
$$

Thus from Lemma 3.4 there exists a step-type contrast structure solution $y(t, \mu)$ of the problem (1.1)-(1.2) for sufficiently small $\mu > 0$, and the asymptotic formula (3.1) holds. The proof is completed.

Acknowledgment

F. Xie was supported by the National Natural Science Foundation of China (Nos 10701023, 10871040) and the Fundamental Research Funds for the Central Universities (No 2010B08-2-1); M. Ni was supported by the Natural Science Foundation of Shanghai (No 10ZR1409200), and in part by E-Institutes of Shanghai Municipal Education Commission (No N.E03004).

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(Received July 14, 2010)