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OSCILLATION CRITERIA OF SECOND ORDER NEUTRAL DELAY DYNAMIC EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

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ABSTRACT. In this paper we establish some oscillation theorems for second order neutral dynamic equations with distributed deviating arguments. We use the Riccati transformation technique to obtain sufficient conditions for the oscillation of all solutions. Further, some examples are provided to illustrate the results.

1. INTRODUCTION

In this paper we are concerned with the oscillatory behavior of solutions of second order neutral type dynamic equations with distributed deviating arguments of the form

$$(r(t)(x(t)+p(t)x(t-\tau))^{\Delta})^{\Delta}+\int_a^b q(t,\xi)f(x(g(t,\xi)))\Delta\xi=0, \qquad t\in\mathbb{T} \tag{1.1}$$

subject to the conditions:

(A_1) $r(t), p(t)$ are positive real valued rd-continuous functions on time scales with $0\le p(t)<1$;

(A_2) $f\in C(\mathbb{R},\mathbb{R})$ such that $uf(u)>0$ for $u\neq 0$, and $f(-u)=-f(u)$;

(A_3) $g(t,\xi)\in C_{rd}(\mathbb{T}\times[a,b]_{\mathbb{T}},\mathbb{T}), g(t,\xi)\le t, \xi\in[a,b]_{\mathbb{T}}$,where $[a,b]_{\mathbb{T}}=\{t\in\mathbb{T}: a\le t\le b\}$, g is strictly increasing with respect to t and decreasing with respect to ξ, and the integral of equation (1.1) is in the sense of Riemann (see [7]).

By a solution of equation (1.1), we mean a nontrivial real valued function $x(t)$ which has the properties $x(t)+p(t)x(t-\tau)\in C^1_{rd}([t_y,\infty)_{\mathbb{T}}$ and $r(t)[x(t)+p(t)x(t-\tau)]^{\Delta}\in C^1_{rd}([t_y,\infty)_{\mathbb{T}}$ and satisfying equation (1.1) for all $t\in[t_0,\infty)_{\mathbb{T}}$.

We restrict our attention to nontrivial solutions of equation (1.1) that exist on some half-line $[t_y,\infty)_{\mathbb{T}}$, and satisfying $\sup\{|\ x(t)\ |: t\in[t_1,\infty)_{\mathbb{T}}\}>0$ for any $t_1\in[t_y,\infty)_{\mathbb{T}}$. A solution $x(t)$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. The equation itself is

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called oscillatory if all its solutions are oscillatory. Since we are interested in oscillatory behavior of solutions, we will suppose that the time scale $\mathbb{T}$ under considerations is not bounded above, that is, it is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$.

We note that if $\mathbb{T} = \mathbb{R}$, then we have $f^{\Delta}(t) = f'(t)$, and equation (1.1) becomes the second order neutral differential equation with distributed deviating arguments of the form

$$(r(t)(x(t) + p(t)x(t-\tau))')' + \int_a^b q(t,\xi)f(x(g(t,\xi)))d\xi = 0, \qquad t \geq t_0. \tag{1.2}$$

If $\mathbb{T} = \mathbb{N}$, we have $f^{\Delta}(n) = f(n+1) - f(n)$, and the equations (1.1) becomes the second order neutral difference equation with distributed deviating arguments of the form

$$\Delta(r_n \Delta(x_n + p_n x_{n-\tau})) + \sum_{s=a}^{b} q_s(\xi) f(x(g_s(\xi))) = 0, \qquad t \in \mathbb{N}. \tag{1.3}$$

Recently there has been an increasing interest in studying the oscillation of solutions of dynamic equations with continuous deviating arguments, see for example [1,6,14,16-18] and the references cited therein. To the best of our knowledge no paper has been published in dynamic equations with distributed deviating arguments. This motivated us to study the oscillatory behavior of equation (1.1).

The purpose of this paper is to derive some sufficient conditions for the solutions of the equation (1.1) to be oscillatory under the conditions

$$(C_1) \qquad \int_{t_0}^{\infty} \frac{1}{r(s)} \Delta s = \infty, \quad \text{and} \quad (C_2) \qquad \int_{t_0}^{\infty} \frac{1}{r(s)} \Delta s < \infty.$$

In Section 2, we present some basic lemmas , and in Section 3 we will use the Riccati transformation technique to prove our oscillation results of the equations (1.1) under the condition (C_1). Also we derive sufficient condition for the equation (1.1) to be oscillatory under the condition (C_2). In Section 4, we present some examples to illustrate our main results.

2. Some basic lemmas

In this section, we give some preliminary lemmas which are useful to prove the main results.

Lemma 2.1. [4]*Assume that $v \in \mathbb{T} \to \mathbb{R}$ is strictly increasing, and $\widetilde{\mathbb{T}} = v(\mathbb{T})$ is a time scale. Let $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $v^{\Delta}(t)$, and $w^{\widetilde{\mathbb{T}}}(v(t)$ exist for $t \in \mathbb{T}^k$, then*

$$(w \circ v)^{\Delta} = (w^{\widetilde{\Delta}} \circ v)v^{\Delta}.$$

Lemma 2.2. *If $a > 0, b \geq 0$, then*

$$-ax^2 + bx \leq -\frac{1}{2}x^2 + \frac{b^2}{2a}, \quad x \in \mathbb{R}.$$

Proof. The proof is obvious. □

Lemma 2.3. *Assume that condition (C_1) holds. Let $x(t)$ be an eventually positive solution of equation (1.1), and let $y(t) = x(t) + p(t)x(t-\tau)$. Then there exists a $t_1 \geq t_0$ such that $y(t) > 0, y^\Delta(t) > 0$ and $(r(t)y^\Delta(t))^\Delta \leq 0,\ t \in [t_1, \infty)_\mathbb{T}$.*

Proof. Suppose that $x(t)$ is an eventually positive solution of equation (1.1) with $x(t) > 0, x(t-\tau) > 0$ and $x(g(t,\xi)) > 0$ for all $t \in [t_1, \infty)_\mathbb{T}$ and $\xi \in [a, b]_\mathbb{T}$. Set $y(t) = x(t) + p(t)x(t-\tau)$. Then $y(t) > 0$ for all $t \in [t_1, \infty)_\mathbb{T}$. In view of equation (1.1) we have $(r(t)y^\Delta(t))^\Delta \leq 0$, and this implies that $r(t)y^\Delta(t)$ is an eventually decreasing function, since $q(t,\xi) > 0$. We claim that $r(t)y^\Delta(t)$ is eventually nonnegative on $[t_1, \infty)_\mathbb{T}$. Suppose not, there is a $t_2 \in [t_1, \infty)_\mathbb{T}$ such that $r(t_2)y^\Delta(t_2) =: \alpha < 0$. Then

$$r(t)y^\Delta(t) \leq r(t_2)y^\Delta(t_2) = \alpha, t \in [t_2, \infty)_\mathbb{T}.$$

Hence

$$y(t) \leq y(t_2) + \alpha \int_{t_2}^{t} \frac{1}{r(s)} \Delta s$$

which implies by condition (C_1) that $y(t) \to -\infty$ as $t \to \infty$. This contradicts the fact that $y(t) > 0$ for all $t \in [t_i, \infty)_\mathbb{T}$. Hence $r(t)y^\Delta(t) > 0$ eventually. □

3. Oscillation results

In this section, first we derive some sufficient conditions for the solutions of equation (1.1) to be oscillatory when the condition (C_1) holds. We begin with the following theorem.

Theorem 3.1. *Assume that condition (C_1) holds, and further assume that there exist $g^\Delta(t,b), H(t,s) \in C^1_{rd}(D;\mathbb{R}), h(t,s) \in C^1_{rd}(D;\mathbb{R})$, and $\alpha(t) \in C^1_{rd}([t_0,\infty), (0,\infty))$, where $D_0 = \{(t,s)/t > s > t_0\}, D = \{(t,s)/t \geq s \geq t_0\}$, such that*

(A_4) *$H(\sigma(t), t) = 0, H(t,s) > 0$;*

(A_5) *$H^{\Delta_s}(t,s) \leq 0$ and $-H^{\Delta_s}(t,s) - H(t,s)\frac{\alpha^\Delta(s)}{\alpha^\sigma(s)} = h(t,s)\sqrt{H(t,s)}$.*

If

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)\alpha(s) \int_a^b Mq(s,\xi)[1 - p(g(s,\xi))]\Delta\xi - \frac{r(g(s,\xi))(\alpha^\sigma(s))^2 h^2(t,s)}{4\alpha(s)g^\Delta(s,a)}\Big]\Delta s = \infty, \quad (3.1)$$

then equation (1.1) is oscillatory.

Proof. Suppose to the contrary that equation (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality we may assume that $x(t)$ is an eventually positive solution of equation (1.1). Then there is a $t_1 \geq t_0$ such that $x(t) > 0$, $x(t-\tau) > 0$, and $x(g(t,\xi)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$, $\xi \in [a,b]_{\mathbb{T}}$. Now we define $y(t) = x(t) + p(t)x(t-\tau)$ for $t \in [t_1, \infty)_{\mathbb{T}}$.

Then using Lemma 2.3, there exists a $t_2 \geq t_0$ such that

$$y(t) > 0, \;\; y^{\Delta}(t) > 0 \quad \text{and} \quad (r(t)y^{\Delta}(t))^{\Delta} \leq 0, \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{3.2}$$

From Lemma 2.3 and using $y(t) \geq x(t), t \in [t_1, \infty)_{\mathbb{T}}$, we get $y(g(t,\xi)) \geq y(g(t,\xi) - \tau) \geq x(g(t,\xi) - \tau)$, and we write

$$\begin{aligned} 0 &= (r(t)(x(t) + p(t)x(t-\tau))^{\Delta})^{\Delta} + \int_a^b q(t,\xi) f(x(g(t,\xi)))\Delta\xi \\ &\geq (r(t)(x(t) + p(t)x(t-\tau))^{\Delta})^{\Delta} + \int_a^b q(t,\xi) M x(g(t,\xi))\Delta\xi \\ &= (r(t)(x(t) + p(t)x(t-\tau))^{\Delta})^{\Delta} \\ &\quad + \int_a^b Mq(t,\xi)[y(g(t,\xi)) - p(g(t,\xi))x(g(t,\xi) - \tau)\Delta\xi \\ &\geq (r(t)(x(t) + p(t)x(t-\tau))^{\Delta})^{\Delta} + \int_a^b Mq(t,\xi)[1 - p(g(t,\xi))]y(g(t,\xi))\Delta\xi, \end{aligned}$$

or

$$(r(t)(x(t) + p(t)x(t-\tau))^{\Delta})^{\Delta} + \int_a^b Mq(t,\xi)[1 - p(g(t,\xi))]y(g(t,\xi))\Delta\xi \leq 0. \tag{3.3}$$

Using the fact that $g(t,\xi)$ is decreasing with respect to ξ, we have from (3.3) that

$$(r(t)(x(t) + p(t)x(t-\tau))^{\Delta})^{\Delta} + y(g(t,b)) \int_a^b Mq(t,\xi)[1 - p(g(t,\xi))]\Delta\xi \leq 0. \tag{3.4}$$

Define

$$w(t) = \alpha(t) \frac{r(t)(y^{\Delta}(t))^{\gamma}}{y(g(t,b))}. \tag{3.5}$$

Then clearly $w(t) \geq 0$ for all $t \in [t_2, \infty)_{\mathbb{T}}$, and

$$\begin{aligned} w^{\Delta}(t) &= (r(t)y^{\Delta}(t))^{\sigma} \Big[\frac{\alpha(t)}{y(g(t,b))}\Big]^{\Delta} + \frac{\alpha(t)}{y(g(t,b))}(r(t)y^{\Delta}(t))^{\Delta} \\ &= \frac{\alpha(t)}{y(g(t,b))}(r(t)y^{\Delta}(t))^{\Delta} + (ry^{\Delta})^{\sigma} \frac{\alpha^{\Delta}(t)y(g(t,b)) - \alpha(t)[y(g(t,b))]^{\Delta}}{y(g(t,b))y(g(\sigma(t),b))} \\ &\leq -\alpha(t) \int_a^b Mq(t,\xi)[1 - p(g(t,\xi))]\Delta\xi + \alpha^{\Delta}(t) \frac{(ry^{\Delta})^{\sigma}}{y(g(\sigma(t)b))} \\ &\qquad - \frac{\alpha(t)(ry^{\Delta})^{\sigma}}{y(g(t,b))y(g(\sigma(t),b))}\{y(g(t,b))\}^{\Delta}. \end{aligned} \tag{3.6}$$

Let $g_b(t) = g(t,b)$. Then using Lemma 2.1, we have

$$(y \circ g_b)^{\Delta}(t) = y^{\Delta}(g_b(t))g_b^{\Delta}(t). \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$\begin{aligned} w^{\Delta}(t) &\le -\alpha(t)\int_a^b Mq(t,\xi)[1-p(g(t,\xi))]\Delta\xi + \frac{\alpha^{\Delta}(t)}{\alpha^{\sigma}(t)}w^{\sigma}(t) \\ &\quad -\frac{\alpha(t)(ay^{\Delta})^{\sigma}}{y(g(t,b))y(g(\sigma(t),b))}y^{\Delta}(g(t,b))g^{\Delta}(t,a). \end{aligned} \tag{3.8}$$

Since $y^{\Delta}(t) \ge 0$, and $(r(t)y^{\Delta}(t))^{\Delta} \le 0, \quad t \ge t_1$, we have

$$y(g(t,b)) \le y(g(t,b))^{\sigma}, \quad \text{and} \quad r(t)y^{\Delta}(t) \le r(g(t,b)y^{\Delta}(g(t,b)). \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$\begin{aligned} \alpha(t)\int_a^b Mq(t,\xi)[1-p(g(t,\xi))]\Delta\xi &\le -w^{\Delta}(t) + \frac{\alpha^{\Delta}(t)}{\alpha^{\sigma}(t)}w^{\sigma}(t) \\ &\quad -\frac{\alpha(t)g^{\Delta}(t,b)}{a(g(t,b))(\alpha^{\sigma}(t))^2}\{w^{\sigma}(t)\}^2. \end{aligned} \tag{3.10}$$

Multiplying (3.10) by $H(t,s)$ and then integrating from T to t , for any $t \ge T \ge t_2$, we have

$$\begin{aligned} \int_T^t H(t,s) \quad \alpha(s) \quad \int_a^b Mq(s,\xi)[1-p(g(s,\xi))]\Delta\xi\Delta s &\le -\int_T^t H(t,s)w^{\Delta}(s)\Delta s \\ + \int_T^t H(t,s)\frac{\alpha^{\Delta}(s)}{\alpha^{\sigma}(s)}w^{\sigma}(s)\Delta s &- \int_T^t H(t,s)\frac{\alpha(s)g^{\Delta}(s,b)}{a(g(s,b))\alpha^2(\sigma(s))}\{w^{\sigma}(s)\}^2\Delta s. \end{aligned}$$

Using integrating by parts, we have

$$\begin{aligned} \int_T^t H(t,s)w^{\Delta}(s)\Delta s &= H(t,s)w(s)|_T^t - \int_T^t H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s \\ &= H(t,T)w(T) - \int_T^t H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s. \end{aligned} \tag{3.11}$$

In view of conditions $(A_4), (A_5)$, and (3.11), we have

$$\int_T^t H(t,s)\alpha(s)\int_a^b Mq(s,\xi)[1-p(g(s,\xi))]\Delta\xi\Delta s$$
$$\le H(t,T)w(T) + \int_T^t \Big(H^{\Delta_s}(t,s) + H(t,s)\frac{\alpha^\Delta(s)}{\alpha^\sigma(s)}\Big)w^\sigma(s)\Delta s$$
$$- \int_T^t H(t,s)\frac{\alpha(s)g^\Delta(s,b)}{a(g(s,b))\alpha^2(\sigma(s))}\{w^\sigma(s)\}^2\Delta s$$
$$= H(t,T)w\Delta(T) - \int_T^t h(t,s)\sqrt{H(t,s)}w^\sigma(s)\Delta s$$
$$- \int_T^t H(t,s)\frac{\alpha(s)g^\Delta(s,b)}{a(g(s,b))\alpha^2(\sigma(s))}\{w^\sigma(s)\}^2\Delta s$$
$$\le H(t,T)w(T) - \int_T^t \Big[\sqrt{\frac{H(t,s)\alpha(s)g^\Delta(s,b)}{r(g(s,b))(\alpha^\sigma(t))^2}}w^\sigma(s) + \frac{\sqrt{r(g(s,b))}\alpha^\sigma(s)h(t,s)}{2\sqrt{\alpha(s)g^\Delta(s,b)}}\Big]\Delta s$$
$$+ \int_T^t \frac{r(g(s,b))(\alpha^\sigma(s))^2h^2(t,s)}{4\alpha(s)g^\Delta(s,b)}\Delta s,$$

or

$$\int_T^t \Big[H(t,s)\alpha(s)\int_a^b Mq(s,\xi)[1-p(g(s,\xi))]\Delta\xi - \frac{r(g(s,\xi))(\alpha^\sigma(s))^2h^2(t,s)}{4\alpha(s)g^\Delta(s,b)}\Big]\Delta s$$
$$\le H(t,T)w(T) - \int_T^t \Big[\sqrt{\frac{H(t,s)\alpha(s)g^\Delta(s,b)}{r(g(s,b))(\alpha^\sigma(t))^2}}w^\sigma(s) + \frac{\sqrt{r(g(s,b))}\alpha^\sigma(s)h(t,s)}{2\sqrt{\alpha(s)g^\Delta(s,b)}}\Big]^2\Delta s$$

which implies that

$$\frac{1}{H(t,t_0)}\int_{t_0}^t \Big[H(t,s)\alpha(s)\int_a^b Mq(s,\xi)[1-p(g(s,\xi))]\Delta\xi - \frac{r(g(s,b))(\alpha^\sigma(s))^2h^2(t,s)}{4\alpha(s)g^\Delta(s,b)}\Big]\Delta s$$
$$\le w(t_0) + \int_{t_0}^T \{\alpha(s)\int_a^b q(t,\xi)[1-p(g(s,\xi))]\Delta\xi\}\Delta s = M < \infty, \tag{3.12}$$

where M is a constant, which contradicts (3.1).

Suppose that $x(t)$ is an eventually negative solution of equation (1.1). Then by taking $z(t) = -x(t)$, we have that $z(t)$ is eventually positive solution of the equation (1.1), since $f(-u) = -f(u)$. Similar to the proof as above we obtain a contradiction. This completes the proof. □

As a consequence of Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Assume that the hypotheses of Theorem 3.1 hold. If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\alpha(s) \int_a^b Mq(s,\xi)[1-p(g(s,\xi))]\Delta\xi\Delta s = \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) \frac{r(g(s,b))(\alpha^\sigma(s))^2 h^2(t,s)}{4\alpha(s)g^\Delta(s,b)}\Big]\Delta s < \infty,$$

then every solution of equation (1.1) is oscillatory.

Remark 3.3. By introducing different choices of $H(t,s)$ and $\alpha(t)$ in Theorem 3.1, we can obtain several different oscillation criteria for the equation (1.1). For instance, let $H(t,s) = 1$ and $H(t,s) = (t-s)^m, t \geq s \geq t_0$, in which $m > 1$ is an integer, we obtain the following criteria respectively.

Corollary 3.4. *Assume that condition* (C_1) *holds. Further assume that there exist* $g^\Delta(t,b)$, *and* $\alpha(t)$ *as in Theorem 3.1 such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \Big[\alpha(s) \int_a^b Mq(s,\xi)[1-p(g(s,\xi))]\Delta\xi - \frac{r(g(s,b))(\alpha^\Delta(s))_+^2}{4\alpha(s)g^\Delta(s,b)}\Big]\Delta s = \infty. \quad (3.13)$$

Then every solution of equation (1.1) is oscillatory.

If we choose $H(t,s) = (t-s)^m, m > 1$, then it is easy to verify that $H^{\Delta_s} \leq -m(t-s)^{m-1}, t > s \geq t_0$. FromTheorem 3.1, we have

Corollary 3.5. *Assume that condition* (C_1) *holds, and there exist* $g^\Delta(t,b)$ *and* $\alpha(t)$ *as in Theorem 3.1. If, for* $m > 1$,

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} \Big[(t-s)^m\alpha(s) \int_a^b Mq(s,\xi)[1-p(g(s,\xi))]\Delta\xi - \frac{r(g(s,\xi))(\alpha^\sigma(s))^2 h^2(t,s)}{4\alpha(s)g^\Delta(s,b)}\Big]\Delta s = \infty,$$

where $h(t,s) = (t-s)^{m/2-1}\Big(m - (t-s)\frac{(\alpha^\Delta(s))_+^2}{(\alpha^\sigma(s))^2}\Big)$, *then every solution of equation (1.1) is oscillatory.*

Next, if we consider $\alpha(t) = t$ and $\alpha(t) = 1$ for $t \geq t_0$ in Corollary 3.4, we can obtain few more oscillation criteria as corollaries of Theorem 3.1.

Corollary 3.6. *Assume that condition* (C_1) *holds, and there exists* $g^\Delta(t,b)$ *as in Theorem 3.1. If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \Big[s \int_a^b Mq(s,\xi)[1-p(g(s,\xi))]\Delta\xi - \frac{r(g(s,b))}{4sg^\Delta(s,b)}\Big]\Delta s = \infty, \quad (3.14)$$

then every solution of equation (1.1) is oscillatory.

Corollary 3.7. *Assume that condition* (C_1) *holds, and there exists* $g^{\Delta}(t,b)$ *as in Theorem 3.1. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\int_a^b Mq(s,\xi)[1-p(g(s,\xi))]\Delta\xi\Delta s=\infty, \tag{3.15}$$

then every solution of equation (1.1) is oscillatory.

Theorem 3.8. *Assume that the hypotheses of Theorem 3.1 hold. Further assume that*

$$0<\liminf_{s\geq t_0}\big[\liminf_{t\to\infty}\frac{H(\sigma(t),s)}{H(\sigma(t),t_0)}\big]\leq\infty, \tag{3.16}$$

and there exists a positive delta differentiable function $\alpha(t)$ *such that*

$$\lim_{t\to\infty}\frac{1}{H(\sigma(t),t_0)}\int_{T_0}^{t}\frac{r(g(s,b))(\alpha^{\sigma}(s))^2}{\alpha(s)g^{\Delta}(s,b)}\Delta s<\infty, \tag{3.17}$$

hold. If there exists a function $\varphi\in C_{rd}([t_0,\infty),\mathbb{R})$ *such that*

$$\int_{t_1}^{\infty}\frac{\alpha(s)g^{\Delta}(s,b)}{a(g(s,b))\alpha^2(\sigma(s))}\varphi_+^2(s)\Delta s=\infty, \tag{3.18}$$

and for every $T\geq t_0$,

$$\liminf_{t\to\infty}\frac{1}{h(t,u)}\int_u^t\Big[H(t,s)\alpha(s)\int_a^b Mq(s,\xi)[1-p(g(s,\xi))]\Delta\xi - \frac{r(g(s,\xi))(\alpha^{\sigma}(s))^2h^2(t,s)}{4\alpha(s)g^{\Delta}(s,b)}\Big]\Delta s\geq\varphi(T), \tag{3.19}$$

where $\varphi_+(t)=\max\{\varphi(t),0\}$, *then every solution of equation (1.1) is oscillatory.*

Proof. On the contrary, we assume that (1.1) has a nonoscillatory solution $x(t)$. We suppose without loss of generality that $x(t)>0$ for all $t\in[t_0,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 3.1, for $t>u\geq t_1\geq t_0$, we have

$$\frac{1}{H(t,u)}\int_u^t\Big[H(t,s)\alpha(s)\int_a^b Mq(s,\xi)[1-p(g(s,\xi))]\Delta\xi - \frac{r(g(s,\xi))(\alpha^{\sigma}(s))^2h^2(t,s)}{4\alpha(s)g^{\Delta}(s,b)}\Big]\Delta s\leq w(u)-\frac{1}{H(t,u)}\int_u^t\Big[\sqrt{\frac{H(t,s)\alpha(s)g^{\Delta}(s,b)}{r(g(s,b))(\alpha^{\sigma}(t))^2}}w^{\sigma}(s) + \frac{\sqrt{r(g(s,b))}\alpha^{\sigma}(s)h(t,s)}{2\sqrt{\alpha(s)g^{\Delta}(s,b)}}\Big]^2\Delta s.$$

Let $t \to \infty$ and taking the upper limit, we have

$$\limsup_{t\to\infty} \frac{1}{h(t,u)} \int_u^t \Big[H(t,s)\alpha(s)\int_a^b Mq(s,\xi)[1-p(g(s,\xi))]\Delta\xi$$

$$-\frac{r(g(s,\xi))(\alpha^\sigma(s))^2h^2(t,s)}{4\alpha(s)g^\Delta(s,b)}\Big]\Delta s \le w(u) - \liminf_{t\to\infty}\frac{1}{h(t,u)}\int_T^u \Big[\sqrt{\frac{H(t,s)\alpha(s)g^\Delta(s,b)}{r(g(s,b))(\alpha^\sigma(t))^2}}w^\sigma(s)$$

$$+\frac{\sqrt{r(g(s,b))}\alpha^\sigma(s)h(t,s)}{2\sqrt{\alpha(s)g^\Delta(s,b)}}\Big]^2\Delta s.$$

From (3.19), we have

$$w(u) \ge \varphi(u) \text{ for all } u \ge t_0, \tag{3.20}$$

and

$$\liminf_{t\to\infty}\frac{1}{h(t,u)}\int_u^t \Big[\sqrt{\frac{H(t,s)\alpha(s)g^\Delta(s,b)}{r(g(s,b))(\alpha^\sigma(t))^2}}w^\sigma(s) \quad + \quad \frac{\sqrt{r(g(s,b))}\alpha^\sigma(s)h(t,s)}{2\sqrt{\alpha(s)g^\Delta(s,b)}}\Big]^2\Delta s$$
$$\le w(u)-\varphi(u) \quad = \quad M<\infty, \tag{3.21}$$

where M is a constant. Let

$$F(t) \quad = \quad \frac{1}{H(t,t_1)}\int_{t_1}^t \frac{H(t,s)\alpha(s)g^\Delta(s,b)}{a(g(s,b))\alpha^2(\sigma(s))}(w^\sigma(s))^2\Delta s, \tag{3.22}$$

and

$$G(t) \quad = \quad \frac{1}{H(t,t_1)}\int_{t_1}^t h(t,s)\sqrt{H(t,s)}w^\sigma(s)\Big]\Delta s \tag{3.23}$$

for $t > t_0$. Then by (3.21), (3.22) and (3.23), we obtain

$$\liminf_{t\to\infty}[G(t)+F(t)] < \infty. \tag{3.24}$$

Now we claim that

$$\int_{t_1}^\infty \frac{\alpha(s)g^\Delta(s,b)}{a(g(s,b))\alpha^2(\sigma(s))}(w^\sigma(s))^2\Delta s < \infty. \tag{3.25}$$

Suppose to the contrary that

$$\int_{t_1}^\infty \frac{\alpha(s)g^\Delta(s,b)}{a(g(s,b))\alpha^2(\sigma(s))}(w^\sigma(s))^2\Delta s = \infty. \tag{3.26}$$

From (3.16), there is a positive $\eta > 0$ satisfying

$$\liminf_{s\ge t_0}\Big[\liminf_{t\to\infty}\frac{H(\sigma(t),s)}{H(\sigma(t),t_0)}\Big] > \eta > 0. \tag{3.27}$$

On the other hand by (3.26) for any positive number $\mu > 0$, there exists $T \ge t_1$ such that

$$\int_{t_1}^\infty \frac{\alpha(s)g^\Delta(s,b)}{a(g(s,b))\alpha^2(\sigma(s))}(w^\sigma(s))^2\Delta s \ge \frac{\mu}{\eta} \qquad \text{for all } t \ge T. \tag{3.28}$$

So, for all $t \geq T > t_1$, integration by parts yields that

$$
\begin{aligned}
F(t) &= \frac{1}{H(t,)} \int_{t_1}^{t} \frac{H(t,s)\alpha(s)g^{\Delta}(s,b)}{a(g(s,b))\alpha^2(\sigma(s))}(w^{\sigma}(s))^2 \Delta s \\
&= \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s)\Delta\Big[\int_{t_1}^{s} \frac{\alpha(u)g^{\Delta}(u,b)}{a(g(u,b))\alpha^2(\sigma(u))}(w^{\sigma}(u))^2\Delta u\Big] \\
&= \frac{1}{H(t,t_1)} \int_{T}^{t} \Big[\int_{t_1}^{s} \frac{\alpha(u)g^{\Delta}(u,b)}{a(g(u,b))\alpha^2(\sigma(u))}(w^{\sigma}(u))^2\Delta u\Big] H^{\Delta_s}(t,s)\Delta s \\
&\geq \frac{\mu}{\eta}\frac{1}{H(t,t_1)} \int_{T}^{t} H^{\Delta_s}(\sigma(t),s)\Delta s \\
&= \frac{\mu}{\eta}\frac{H(t,T)}{H(t,t_1)} \geq \frac{\mu}{\eta}\frac{H(t,T)}{H(t,t_0)}. \qquad (3.29)
\end{aligned}
$$

It follows from (3.27) that

$$\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \geq \eta > 0, \qquad s \geq t_0. \qquad (3.30)$$

Therefore, there exists a $t_2 \geq T$ such that

$$\frac{H(t,T)}{H(t,t_0)} \geq \eta \qquad (3.31)$$

for all $t \geq t_2$. From (3.29) and (3.31),we have

$$G(t) \geq \eta \qquad \text{for } \ t \geq t_2. \qquad (3.32)$$

Since μ is arbitrary, we conclude that

$$\lim_{t\to\infty} F(t) = \infty. \qquad (3.33)$$

From (3.24), there is a sequence $\{t_n\}_{n=1}^{\infty}$ in $[t_1,\infty)_{\mathbb{T}}$ with $\lim_{n\to\infty} t_n = \infty$, and such that

$$\lim_{n\to\infty}(F(t_n)+G(t_n)) = \lim_{t\to\infty}(F(t_n)+G(t_n)) < \infty.$$

Thus there exist constants N_1 and M such that

$$F(t_n)+G(t_n) \leq M \qquad \text{for } \ n > N_1. \qquad (3.34)$$

It follows from (3.33) that

$$\lim_{n\to\infty} F(t_n) = \infty. \qquad (3.35)$$

Further, from (3.34) and (3.35), we obtain

$$\lim_{n\to\infty} G(t_n) = -\infty. \qquad (3.36)$$

Then for any $\epsilon \in (0,1)$, there exists a positive integer N_2 such that

$$\frac{G(t_n)}{F(t_n)} + 1 < \epsilon, \qquad \text{for } \ n > N_2,$$

or

$$\frac{G(t_n)}{F(t_n)} < \epsilon - 1 < 0. \tag{3.37}$$

From (3.36) and (3.37),we have

$$\lim_{n\to\infty} \frac{G(t_n)}{F(t_n)} G(t_n) = \infty. \tag{3.38}$$

On the other hand, by Schwarz inequality,we have

$$\begin{aligned}
0 \ \leq \ & G^2(t_n) = \frac{1}{H^2(t_n,t_1)}\{\int_{t_1}^{t_n} \sqrt{H(t_n,s)}h(t_n,s)w^\sigma(s)\Delta s\}^2 \\
& \leq \{\frac{1}{H(t_n,t_1)}\int_{t_1}^{t_n} \frac{H(t_n,s)\alpha(s)g^\Delta(s,b)}{a(g(s,b))\alpha^2(\sigma(s))}(w^\sigma(s))^2\Delta s\}^2 \\
& \times\{\frac{1}{H(t_n,t_1)}\int_{t_1}^{t_n} \frac{a(g(s,b))\alpha^2(\sigma(s))}{\alpha(s)g^\Delta(s,b)}h(t_n,s)(\Delta s\} \\
= \ & F(t_n)\frac{1}{H(t_n,t_1)}\int_{t_1}^{t_n} \frac{a(g(s,b))\alpha^2(\sigma(s))}{\alpha(s)g^\Delta(s,b)}h(t_n,s)(\Delta s,
\end{aligned}$$

$$\frac{G^2(t_n)}{F(t_n)} \leq \frac{1}{H(t_n,t_1)}\int_{t_1}^{t_n} \frac{a(g(s,b))\alpha^2(\sigma(s))}{\alpha(s)g^\Delta(s,b)}h(t_n,s)\Delta s, \tag{3.39}$$

for all large n. In view of (3.31), we obtain

$$\frac{1}{H(t_n,t_1)} \leq \frac{1}{H(t_n,T)} = \frac{H(t_n,t_0}{H(t_n,T)}\frac{1}{H(t_n,t_0)} \leq \frac{1}{LH(t_n,t_0)}, \tag{3.40}$$

and therefore, from (3.39) and (3.40), we have

$$\frac{G^2(t_n)}{F(t_n)} \leq \frac{1}{LH(t_n,t_1)}\int_{t_1}^{t_n} \frac{a(g(s,b))\alpha^2(\sigma(s))}{\alpha(s)g^\Delta(s,b)}h(t_n,s)\Delta s. \tag{3.41}$$

From (3.41) and(3.38), we have

$$\lim_{n\to\infty} \frac{1}{H(\sigma(t_n),t_0)}\int_{t_0}^{t_n} \frac{a(g(s,b))\alpha^2(\sigma(s))}{\alpha(s)g^\Delta(s,b)}h(t_n,s)\Delta s = \infty. \tag{3.42}$$

This implies that

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),t_0)}\int_{t_0}^{t} \frac{a(g(s,b))\alpha^2(\sigma(s))}{\alpha(s)g^\Delta(s,b)}h(t,s)\Delta s = \infty, \tag{3.43}$$

which contradicts (3.17). Therefore, from (3.25) and $w^\sigma(s) \geq \varphi(s)$, we have

$$\int_{t_1}^{\infty} \frac{\alpha(s)g^\Delta(s,b)}{a(g(s,b))\alpha^2(\sigma(s))}\varphi_+^2(s)\Delta s \leq \int_{t_1}^{\infty} \frac{\alpha(s)g^\Delta(s,b)}{a(g(s,b))\alpha^2(\sigma(s))}w^2(\sigma(s))\Delta s < \infty, \tag{3.44}$$

which contradicts (3.18). This completes the proof. ☐

Next we obtain sufficient condition for the equation (1.1) to be oscillatory when condition (C_2) holds.

Lemma 3.9. *Assume condition (C_2) holds. If $x(t)$ is an eventually positive solution of equation (1.1) and $y(t) > 0$, then $y(t)$ satisfies the following inequality*

$$r(t)\pi(t)y^{\Delta}(t) + y(t) \geq 0 \tag{3.45}$$

where

$$\pi(t) = \int_t^{\infty} \frac{1}{r(s)} \Delta s$$

for all sufficiently large $t \geq t_0$.

Proof. From (3.3), it is clear that $r(t)y^{\Delta}(t)$ is nonincreasing on $[t_0, \infty)_{\mathbb{T}}$ for some $t_0 > 0$.. Then

$$r(u)y^{\Delta}(u) \leq r(s)y^{\Delta}(s) \quad \text{for all } u > s > t_0.$$

Dividing the last inequality by $r(u)$ and integrating over $[s, t]_{\mathbb{T}}$, we obtain

$$0 < y(t) \leq y(s) + r(s)y^{\Delta}(s) \int_s^t \frac{1}{r(u)} \Delta u, \quad t > s > t_0. \tag{3.46}$$

Then we consider the following two cases:
Case(I): $y^{\Delta}(t) \geq 0$. From (3.46), we find

$$0 < y(t) \leq y(s) + r(s)y^{\Delta}(s) \int_s^t \frac{1}{r(u)} \Delta u, \tag{3.47}$$

$$\leq y(s) + r(s)y^{\Delta}(s) \int_s^{\infty} \frac{1}{r(u)} \Delta u \tag{3.48}$$

which implies (3.45).
Case(II): $y^{\Delta}(t) < 0$. From (3.46), the condition (C_2) implies that $y(t)$ is bounded from above. Letting $t \to \infty$ in (3.46), we have

$$0 \leq r(t)\pi(t)(y^{\Delta}(t)) + y(t)$$

which gives (3.45). ☐

Theorem 3.10. *Assume that condition (C_2) holds. If*

$$\int^{\infty} \pi(\sigma(t)) \int_a^b kQ(t, \xi)\pi(g(t, \xi)) \Delta\xi \Delta s = \infty, \tag{3.49}$$

then every solution of equation (1.1) is oscillatory.

Proof. Suppose to the contrary that $x(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. Then there exists a point t_1 in $[t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0, x(t-\tau) > 0$ and $x(g(t, \xi)) > 0$ for all $\xi \in [a, b]_{\mathbb{T}}$, and for all $t \in [t_1, \infty)_{\mathbb{T}}$. From (3.3), we write $(r(t)y^{\Delta}(t)) \leq 0$. Therefore $r(t)y^{\Delta}(t) \geq 0$ or $r(t)y^{\Delta}(t) < 0$ eventually. Since $r(t) > 0$, we have $y^{\Delta}(t) \geq 0$ or $y^{\Delta}(t) < 0$. Now we consider the case $y^{\Delta}(t) \geq 0$. We see that $y(t) \geq y(t_2)$ for some $t_2 \in [t_1, \infty)_{\mathbb{T}}$. Since $\lim_{t\to\infty} \pi(t) = 0$, we find that $y(t_2) \geq \pi(t)$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. Hence $y(t) \geq \pi(t)$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. Suppose that $y^{\Delta}(t) < 0$. Then since $(r(t)y^{\Delta}(t))^{\Delta} \leq 0$ for all $t \in [t_2, \infty)_{\mathbb{T}}$,

$$r(t)y^{\Delta}(t) \leq r(t_2)y^{\Delta}(t_2) = -c_0 < 0, \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{3.50}$$

Substituting (3.45) in the above inequality we obtain

$$y(t) \geq c_0\pi(t) \ \text{ for } \ t \in [t_2, \infty)_{\mathbb{T}}, \tag{3.51}$$

or

$$y(g(t, \xi)) \geq c_0\pi(g(t, \xi)) \ \text{ for } \ t \in [t_2, \infty)_{\mathbb{T}}. \tag{3.52}$$

Then from equation (1.1), we have

$$(r(t)y^{\Delta}(t))^{\Delta} + \int_a^b Mc_0 q(t, \xi)[1 - p(g(t, \xi))]\pi(g(t, \xi))\Delta\xi \leq 0. \tag{3.53}$$

Now multiplying the last inequality by $\pi(\sigma(t))$ and then integrating, we obtain

$$\pi(t)r(t)y^{\Delta}(t) + y(t) - \pi(t_2)r(t_2)y^{\Delta}(t_2) - y(t_2) \\ + \int_{t_2}^t \pi(\sigma(t)) \int_a^b Mc_0 q(s, \xi)[1 - p(g(s, \xi))]\pi(g(s, \xi))\Delta\xi\Delta s \leq 0.$$

This implies that

$$\int_{t_2}^t \pi(\sigma(t)) \int_a^b Mc_0 q(s, \xi)[1 - p(g(s, \xi))]\pi(g(s, \xi))\Delta\xi\Delta s \leq \pi(t_2)r(t_2)y^{\Delta}(t_2) + y(t_2).$$

This contradicts (3.49) as $t \to \infty$. Hence every solution of equation (1.1) is oscillatory. The proof is now complete. □

4. Examples

In this section, we present some examples to illustrate our main results when the conditions (C_1) and (C_2) hold.

Example 4.1. Consider the dynamic equation

$$((x(t) + \frac{1}{2}x(t-1))^{\Delta\Delta} + \int_0^1 x(t-\xi)\Delta\xi = 0. \tag{4.1}$$

where $r(t) = 1, \quad p(t) = 1, \quad a = 0, \quad b = 1, \quad \tau = 1, \quad q(t, \zeta) = 1$. By Corollary 3.6, every solution of equation (4.1) is oscillatory.

Example 4.2. Consider the dynamic equation

$$((x(t)+\frac{1}{t+1}x(t-1)))^{\Delta\Delta}+\int_0^1 \gamma\frac{t-\zeta+1}{t(t-\xi)}x(t-\xi)\Delta\xi=0. \qquad (4.2)$$

Here $p(t)=\frac{1}{t+1}, r(t)=1, \tau=1, a=1/2, b=1, f(u)=u, M=1, q(t,\xi)=\frac{t-\xi+1}{t(t-\xi)}$ and $g(t,\xi)=t-\xi$. Then by Corollary 3.7, we have

$$\begin{aligned}\limsup_{t\to\infty}\int_{t_0}^t\int_a^b Mq(s,\xi)[1-p(g(s,\xi))] \quad \Delta\xi\Delta s \\ &= \limsup_{t\to\infty}\int_{t_0}^t\int_{1/2}^1\frac{t-\xi+1}{t(t-\xi)}[1-\frac{1}{t-\xi+1}]\Delta\xi\Delta s\\ &= \limsup_{t\to\infty}\int_{t_0}^t\frac{1}{t}\Delta\xi\\ &= \infty.\end{aligned}$$

Therefore every solution of equation (4.2) is oscillatory.

Example 4.3. Consider the dynamic equation

$$(t\sigma(t)(x(t)+p(t)x(t-\tau))^{\Delta})^{\Delta}+\int_0^1\frac{\sigma(t)}{t}(\sigma(t)-\xi)x(t-\xi)\Delta\xi=0, \quad t\in[1,\infty)_{\mathbb{T}}. \quad (4.3)$$

Here $p(t)=1/2, r(t)=t\sigma(t), q(t,\xi)=\frac{\sigma(t)}{t}(\sigma(t)-\xi)$, and $g(t,\xi)=t-\xi$. Since (C_2) holds, from Theorem 3.10, we have

$$\begin{aligned}\lim_{t\to\infty}\int_1^t\pi(s)\int_0^1 q(s,\xi)\pi(g(s,\xi))\Delta\xi\Delta s &= \lim_{t\to\infty}\int_1^t\frac{1}{\sigma(s)}\int_0^1\frac{\sigma(s)}{s}(\sigma(s)-\xi)(g(t,\xi))\Delta\xi\Delta s\\ &= \lim_{t\to\infty}\int_1^t\frac{1}{s}\Delta s=\infty.\end{aligned}$$

Hence every solution of equation (1.1) is oscillatory.

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