Blow-up analysis for a semilinear parabolic equation with nonlinear memory and nonlocal nonlinear boundary condition *

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Abstract

In this paper, we consider a semilinear parabolic equation

$$u_t = \Delta u + u^q \int_0^t u^p(x,s)ds, \quad x \in \Omega, \quad t > 0$$

with nonlocal nonlinear boundary condition $u|_{\partial\Omega\times(0,+\infty)} = \int_\Omega \varphi(x,y)u^l(y,t)dy$ and nonnegative initial data, where p, $q \geq 0$ and $l > 0$. The blow-up criteria and the blow-up rate are obtained.

Keywords: Semilinear parabolic equation; Nonlinear memory; Nonlocal nonlinear boundary condition; Blow-up

Mathematics Subject Classification(2000) : 35B35, 35K50, 35K55

1 Introduction

In this paper, we deal with the following semilinear reaction-diffusion equation with nonlinear memory and nonlocal nonlinear boundary condition

$$\begin{cases} u_t = \Delta u + u^q \int_0^t u^p(x,s)ds, & x \in \Omega, t > 0, \\ u(x,t) = \int_\Omega \varphi(x,y)\, u^l(y,t)dy, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, p, $q \geq 0$, $l > 0$, the weight function $\varphi(x,y)$ in the boundary condition is nonnegative, continuous on $\partial\Omega \times \overline{\Omega}$,

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and $\int_\Omega \varphi(x,y)\,dy > 0$ on $\partial\Omega$, the initial data $u_0(x) \in C^{2+\nu}(\Omega) \cap C(\overline{\Omega})$ for some $0 < \nu < 1$, $u_0(x) \geq 0$, and satisfies the compatibility conditions $u_t|_{t=0} = \Delta u_0(x)$, $u_0(x) = \int_\Omega \varphi(x,y) u_0^l(y)\,dy$ for $x \in \partial\Omega$. It is known by the standard theory (see [12] and [21]) that there exist local nonnegative solutions to problem (1.1). Moreover, the uniqueness of the solution holds if p, q, $l \geq 1$.

From a physical point of view, (1.1) represents the slow-diffusion equations with memory. Problem (1.1) with $p = q = 1$ and $\varphi \equiv 0$ appears in the theory of nuclear reactor dynamics (see [13] and the references therein, where a more detailed physical background can be found). Parabolic equations with nonlinear memory and homogeneous Dirichlet boundary conditions have been studied by several authors (see [6], [20], [24], [25], [26] and the references therein). For instance, in [1], Bellout considered the following equation

$$u_t - \Delta u = \int_0^t (u+\lambda)^p\,ds + g(x), \quad x \in \Omega, t > 0 \tag{1.2}$$

with null Dirichlet boundary condition, where $g(x) \geq 0$ is a smooth function and $\lambda > 0$. In [27], Yamada investigated the stability properties of the global solutions of the following nonlocal Volterra diffusion equation

$$u_t - \Delta u = (a - bu)\,u - \int_0^t k(t-s)\,u(x,s)\,ds, \quad x \in \Omega, t > 0. \tag{1.3}$$

Recently, in [14], Li and Xie considered the following equation

$$\begin{cases} u_t = \Delta u + u^q \int_0^t u^p ds, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.4}$$

where $p, q \geq 0$. They established the conditions for global and non-global solutions. Moreover, under some appropriate hypotheses, they obtained the blow-up rate estimate for the special case $q = 0$.

On the other hand, parabolic equations with nonlocal boundary conditions are also encountered in other physical applications. For example, in the study of the heat conduction within linear thermoelastcity, in [3], [4], Day investigated a heat equation which is subjected to the following boundary conditions

$$u(-R,t) = \int_{-R}^{R} \varphi_1(x) u(x,t)\,dx, \quad u(R,t) = \int_{-R}^{R} \varphi_2(x) u(x,t)\,dx.$$

Friedman [7] generalized Day's result to the following general parabolic equation in n dimensions

$$u_t = \Delta u + g(x,u), \quad x \in \Omega, t > 0, \tag{1.5}$$

which is subjected to the following nonlocal boundary condition

$$u(x,t) = \int_\Omega \varphi(x,y)\,u(y,t)\,dy, \tag{1.6}$$

and studied the global existence of solution and its monotonic decay property under some hypotheses on $\varphi(x,y)$ and $g(x,u)$.

In addition, parabolic equations with both space-integral source terms and nonlocal boundary conditions have been studied as well (see [2], [5], [19], [23] and the references therein). For example, Lin and Liu [16] considered the problem of the form

$$u_t = \Delta u + \int_\Omega g(u)\,dx, \tag{1.7}$$

which is subjected to boundary condition (1.6). They established local existence, global existence and nonexistence of solutions, and discussed the blow-up properties of solutions. Furthermore, they derived the uniform blow-up estimate for some special $g(u)$.

However, to the authors' best knowledge, there is little literature on the study of the global existence and blow-up properties for the reaction-diffusion equations coupled with nonlocal nonlinear boundary condition. Recently, Gladkov and Kim [9] considered the following semilinear heat equation

$$\begin{cases} u_t = \Delta u + c(x,t)\,u^p, & x\in\Omega, t>0,\\ u(x,t) = \int_\Omega \varphi(x,y,t)\,u^l(y,t)\,dy, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), & x\in\overline{\Omega}, \end{cases} \tag{1.8}$$

where p, $l>0$. They obtained some criteria for the existence of global solution as well as for the solution to blow-up in finite time.

For other works on parabolic equations and systems with nonlocal nonlinear boundary, we refer readers to [10], [11], [15], [17], [18] and the references therein.

Motivated by those of works above, our main objectives of this paper are to investigate conditions for the occurrence of the blow-up in finite time or global existence and to estimate the blow-up rate of the blow-up solution. Due to the appearance of the nonlocal nonlinear boundary condition, the approaches used in [14] can not be extended to handle our problem (1.1). Meanwhile, our method is very different from those previously used in [16] because the space-integral source term $\int_\Omega g(u)\,dx$ is replaced by time-integral term $\int_0^t u^p ds$. By a modification of the methods used in [9], we show that the nonlinear memory term $\int_0^t u^p(x,s)\,ds$, the weight function $\varphi(x,y)$ and the nonlinear term $u^l(y,t)$ in the boundary condition play substantial roles in determining blow-up or not of the solution.

In order to state our results, we introduce some useful symbols. Throughout this paper, we let λ be the first eigenvalue of the eigenvalue problem

$$-\Delta\phi(x) = \lambda\phi, x\in\Omega; \quad \phi(x)=0, x\in\partial\Omega, \tag{1.9}$$

and $\phi(x)$ the corresponding eigenfunction with $\int_\Omega \phi(x)dx = 1$, $\phi(x)>0$ in Ω.

The main results of this paper are stated as follows.

Theorem 1.1. *Assume that $p+q\le 1$ and $l\le 1$, then the solution of problem (1.1) exists globally for any nonnegative φ and initial data u_0.*

Theorem 1.2. *Assume that $p+q>1$, $q\geq 1$ and $l>1$. If $\int_{\Omega}\varphi(x,y)\,dy\leq 1$ for all $x\in\partial\Omega$, then the solution of problem (1.1) is global for small initial data u_0.*

Theorem 1.3. *Assume that $l>1$, then for any positive φ, the solution of problem (1.1) blows up in finite time provided that the initial data u_0 satisfies $\int_{\Omega}u_0(x)\,\phi(x)\,dx\geq\varrho>1$ for some ϱ, where ϕ is given by (1.9).*

Theorem 1.4. *Assume that $p+q>1$. If $q\geq 1$, then the solution of problem (1.1) blows up in finite time for sufficiently large initial data u_0. If $q<1$, then the solution of problem (1.1) blows up in finite time for any nonnegative initial data u_0.*

Remark 1.5. *When $p=0$ in problem (1.1), our results are consistent with those in [9]. When $\varphi\equiv 0$ in problem (1.1), our results are consistent with those in [14].*

Consider problem (1.1) with $q=0$ and $l=1$. In order to obtain the blow-up rate, we need to add the following assumption on initial data u_0 (assume T^* is the blow-up time of the blow-up solution $u(x,t)$ to problem (1.1)):

(H1) There exists a constant $t_0\in(0,T^*)$ such that $u_t(x,t_0)\geq 0$ for all $x\in\overline{\Omega}$.

Theorem 1.6. *Assume that $p>1$, $\int_{\Omega}\varphi(x,y)dy\leq 1$ and (H1) hold, then there exist constants $0<C_2<C_1$ such that*

$$C_2\,(T^*-t)^{-\frac{2}{p-1}}\leq\max_{x\in\overline{\Omega}}u(x,t)\leq C_1\,(T^*-t)^{-\frac{2}{p-1}},\quad t\to T^*.$$

Remark 1.7. *From Theorem 1.4, we know that in the case $q=0$, $\int_{\Omega}\varphi(x,y)\,dy\leq 1\,(x\in\partial\Omega)$, the blow-up rate of equation (1.1) with nonlocal boundary condition is the same as that of problem (1.4) with $q=0$.*

Remark 1.8. *In [14], the authors proved the blow-up rate under the additional assumptions $\Omega=B_R$ and u_0 is radially symmetric decreasing. Motivated by the idea of Souplet in [22], we have no restriction on Ω and u_0 here.*

The rest of this paper is organized as follows. In Section 2, we shall establish the comparison principle for problem (1.1). In Section 3, we shall discuss the global existence of the solution and prove Theorems 1.1 and 1.2. In Section 4, we shall discuss the blow-up results of the solution and prove Theorems 1.3 and 1.4. Finally, we shall estimate the blow-up rate and give the proof of Theorem 1.6 in Section 5.

2 Preliminaries

In this section, we will give a suitable comparison principle for problem (1.1). Let $\Omega_T=\Omega\times(0,T)$, $S_T=\partial\Omega\times(0,T)$ and $\overline{\Omega}_T=\overline{\Omega}\times[0,T)$. We begin with the precise definitions of subsolutions and supersolutions of problem (1.1).

Definition 2.1. *A function* $\underline{u}(x,t)$ *is called a subsolution of the problem (1.1) if* $\underline{u}(x,t) \in C^{2,1}(\Omega_T) \cap C\left(\overline{\Omega}_T\right)$ *satisfies*

$$\begin{cases} \underline{u}_t \leq \Delta \underline{u} + \underline{u}^q \int_0^t \underline{u}^p(x,s)ds, & (x,t) \in \Omega_T, \\ \underline{u}(x,t) \leq \int_\Omega \varphi(x,y)\, \underline{u}^l(y,t)\, dy, & (x,t) \in S_T, \\ \underline{u}(x,0) \leq u_0(x), & x \in \overline{\Omega}. \end{cases} \tag{2.1}$$

A supersolution $\overline{u}(x,t)$ *is defined analogously by the above inequalities with "$\leq$" replaced by "$\geq$". We say that* $u(x,t)$ *is a solution of the problem (1.1) in* Ω_T *if it is both a subsolution and a supersolution of problem* (1.1) *in* Ω_T.

Now, let $g_i(x,t) \in C^{2,1}(\Omega_T) \cap C\left(\overline{\Omega}_T\right)$ $(i = 1,2)$, $\chi(x,y) \geq 0$ on $\partial\Omega \times \overline{\Omega}$. First of all, we give some hypotheses on $g_i(x,t)$ and $\chi(x,y)$ as follows, which will be used in the sequel.

(H2) For $x \in \partial\Omega$, $y \in \overline{\Omega}$, $t > 0$, $\chi(x,y)\, g_i^{l-1}(y,t) \geq 0$, $i = 1,2$. Furthermore,

$$\int_\Omega l\chi(x,y)\, g_i^{l-1}(y,t)\, dy < 1, \quad i = 1,2.$$

(H3) For $x \in \partial\Omega$, $y \in \overline{\Omega}$, $t > 0$, there exists $K > 0$ such that

$$0 \leq l\chi(x,y)\, g_i^{l-1}(y,t) \leq K, \quad i = 1,2.$$

Lemma 2.2. *Let (H2) hold,* a_{ij} $(i,j = 1,\cdots,n)$, b_i $(i = 1,\cdots,n)$, f_1, $f_2 \in C\left(\overline{\Omega}_T\right)$, *and* f_2, $f_3 \geq 0$ *in* Ω_T. *If* $\chi(x,y) \geq 0$ *on* $\partial\Omega \times \overline{\Omega}$, $g_i \in C^{2,1}(\Omega_T) \cap C\left(\overline{\Omega}_T\right)$ $(i = 1,2)$ *satisfy*

$$\begin{cases} g_{1t} - \mathcal{L}_1 g_1 \geq f_1 g_1 + f_2 \int_0^t f_3(s) g_1(s) ds, & (x,t) \in \Omega_T, \\ g_{2t} - \mathcal{L}_1 g_2 \leq f_1 g_2 + f_2 \int_0^t f_3(s) g_2(s) ds, & (x,t) \in \Omega_T, \\ g_1(x,t) \geq \int_\Omega \chi(x,y)\, g_1^l(y,t)\, dy, & (x,t) \in S_T, \\ g_2(x,t) \leq \int_\Omega \chi(x,y)\, g_2^l(y,t)\, dy, & (x,t) \in S_T, \\ g_1(x,0) \geq g_2(x,0), & x \in \overline{\Omega}. \end{cases} \tag{2.2}$$

Then $g_1(x,t) \geq g_2(x,t)$ *in* $\overline{\Omega}_T$, *where*

$$\mathcal{L}_1 = \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial}{\partial x_i}.$$

Proof. Let

$$M_1 = \max_{\overline{\Omega}_T} |f_1(x,t)|, \quad M_2 = \max_{\overline{\Omega}_T} f_2(x,t), \quad M_3 = \max_{\overline{\Omega}_T} f_3(x,t).$$

For any given $\varepsilon > 0$, define

$$\tilde{g}_1 = g_1 + \varepsilon e^{\gamma t} \text{ and } \tilde{g}_2 = g_2 - \varepsilon e^{\gamma t},$$

where $\gamma > M_1 + TM_2M_3$. Then, after a direct computation, we obtain

$$\begin{cases} \tilde{g}_{1t} - \mathcal{L}_1\tilde{g}_1 > f_1\tilde{g}_1 + f_2\int_0^t f_3(s)\tilde{g}_1(s)ds, & (x,t) \in \Omega_T, \\ \tilde{g}_{2t} - \mathcal{L}_1\tilde{g}_2 < f_1\tilde{g}_2 + f_2\int_0^t f_3(s)\tilde{g}_2(s)ds, & (x,t) \in \Omega_T. \end{cases} \tag{2.3}$$

On the other hand, for $(x,t) \in S_T$, we have

$$\begin{aligned} \tilde{g}_1(x,t) &\geq \varepsilon e^{\gamma t} + \int_\Omega \chi(x,y)\, g_1^l(y,t)\, dy \\ &= \int_\Omega \chi(x,y)\, \tilde{g}_1^l(y,t)\, dy + \varepsilon e^{\gamma t} - \int_\Omega \chi(x,y) \left(\tilde{g}_1^l(y,t) - g_1^l(y,t)\right) dy \\ &= \int_\Omega \chi(x,y)\, \tilde{g}_1^l(y,t)\, dy + \varepsilon e^{\gamma t} - \varepsilon e^{\gamma t} \int_\Omega l\chi(x,y)\, \theta_1^{l-1}(y,t)\, dy, \end{aligned}$$

here θ_1 is an intermediate value between g_1 and $\tilde{g}_1$. It follows from (H2) that

$$\tilde{g}_1(x,t) > \int_\Omega \chi(x,y)\, \tilde{g}_1^l(y,t)\, dy \quad \text{for } (x,t) \in S_T. \tag{2.4}$$

Likewise, for any $(x,t) \in S_T$, we have

$$\tilde{g}_2(x,t) < \int_\Omega \chi(x,y)\, \tilde{g}_2^l(y,t)\, dy. \tag{2.5}$$

In addition, it is obvious that $\tilde{g}_1(x,0) - \varepsilon \geq \tilde{g}_2(x,0) + \varepsilon$, which implies that

$$\tilde{g}_1(x,0) > \tilde{g}_2(x,0) \quad \text{for } x \in \overline{\Omega}. \tag{2.6}$$

Put

$$h(x,t) = \tilde{g}_1(x,t) - \tilde{g}_2(x,t).$$

Now, our goal is to show that

$$h(x,t) > 0 \quad \text{in } \overline{\Omega}_T. \tag{2.7}$$

Actually, if (2.7) is true, then we can immediately get

$$g_1(x,t) + \varepsilon e^{\gamma t} \geq g_2(x,t) - \varepsilon e^{\gamma t} \quad \text{for all } (x,t) \in \overline{\Omega}_T,$$

which means that $g_1(x,t) \geq g_2(x,t)$ in $\overline{\Omega}_T$ as desired.

In order to prove (2.7), we set

$$\tilde{h}(x,t) = e^{-\sigma t} h(x,t)$$

with $\sigma > \frac{M_1 + \sqrt{M_1^2 + 4M_2M_3}}{2}$. Then from (2.3)-(2.6), we have

$$\begin{cases} \tilde{h}_t - \mathcal{L}_1\tilde{h} > (f_1 - \sigma)\,\tilde{h}_1 + f_2\int_0^t f_3(s)\tilde{h}_1(s)ds, & (x,t) \in \Omega_T, \\ \tilde{h}(x,t) > \int_\Omega l\chi(x,y)\, \theta_2^{l-1}\tilde{h}(y,t)\, dy, & (x,t) \in S_T, \\ \tilde{h}(x,0) > 0, & x \in \overline{\Omega}, \end{cases} \tag{2.8}$$

where θ_2 is an intermediate value between $\tilde{g}_1$ and $\tilde{g}_2$.

Since $\tilde{h}(x,0) > 0$, there exists $\delta > 0$ such that $\tilde{h}(x,t) > 0$ for $(x,t) \in \overline{\Omega} \times (0,\delta)$. Suppose a contradiction that

$$\bar{t} = \sup\left\{t \in (0,T) : \tilde{h} > 0 \text{ on } \overline{\Omega} \times [0,t]\right\} < T.$$

Then $\tilde{h} \geq 0$ on $\overline{\Omega}_{\bar{t}}$, and there exists at least one point $(\bar{x},\bar{t})$ such that $\tilde{h}(\bar{x},\bar{t}) = 0$. If $(\bar{x},\bar{t}) \in \Omega_{\bar{t}}$, by virtue of the first inequality of (2.8) and the strong maximum principle,we conclude that $\tilde{h}(x,t) \equiv 0$ in $\Omega_{\bar{t}}$, a contradiction. If $(\bar{x},\bar{t}) \in S_{\bar{t}}$, by (H2), this also results in a contradiction, that

$$0 = \tilde{h}(\bar{x},\bar{t}) > \int_\Omega l\chi(\bar{x},y)\,\theta_2^{l-1}\tilde{h}(y,\bar{t})\,dy \geq 0.$$

This proves $\tilde{h} > 0$, and in turn $g_1(x,t) \geq g_2(x,t)$ in $\overline{\Omega}_T$. The proof of Lemma 2.2 is complete. $\square$

Lemma 2.3. *Let the hypotheses of Lemma 2.1, with (H2) replaced by (H3), be satisfied. Then*

$$g_1(x,t) \geq g_2(x,t) \ \text{ in } \overline{\Omega}_T.$$

Proof. Choose a positive function $\psi \in C^2\left(\overline{\Omega}\right)$ satisfying $\psi|_{x\in\partial\Omega} = 1$ and $\int_\Omega \psi(y)\,dy < \frac{1}{K}$. Set

$$g_i(x,t) = \psi(x)\,\rho_i(x,t), \quad i = 1,2.$$

Then from (2.2), we have

$$\begin{cases} \rho_{1t} - \mathcal{L}_2\rho_1 \geq \left(f_1 + \sum\limits_{i,j=1}^n a_{ij}\psi_{x_ix_j}\right)\rho_1 + f_2\int_0^t f_3(s)\rho_1(s)ds, & (x,t) \in \Omega_T, \\ \rho_{2t} - \mathcal{L}_2\rho_2 \leq \left(f_1 + \sum\limits_{i,j=1}^n a_{ij}\psi_{x_ix_j}\right)\rho_2 + f_2\int_0^t f_3(s)\rho_2(s)ds, & (x,t) \in \Omega_T, \\ \rho_1(x,t) \geq \int_\Omega \chi(x,y)\,\psi^l(y)\,\rho_1^l(y,t)\,dy, & (x,t) \in S_T, \\ \rho_2(x,t) \leq \int_\Omega \chi(x,y)\,\psi^l(y)\,\rho_2^l(y,t)\,dy, & (x,t) \in S_T, \\ g_1(x,0) \geq g_2(x,0), & x \in \overline{\Omega}, \end{cases} \tag{2.9}$$

where

$$\mathcal{L}_2 = \sum_{i,j=1}^n a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n\left(\sum_{j=1}^n 2a_{ij}\frac{\partial\psi}{\partial x_j} + b_i\psi\right)\frac{1}{\psi}\frac{\partial}{\partial x_i}$$

is a uniformly elliptic operator. By (H3), it is easy to see that

$$\int_\Omega l\chi(x,y)\,\rho_1^{l-1}(y,t)\,\psi^l(y)\,dy \leq K\int_\Omega \psi(y)\,dy \leq 1, \tag{2.10}$$

and

$$\int_\Omega l\chi\left(x,y\right)\rho_2^{l-1}\left(y,t\right)\psi^l\left(y\right)dy\le K\int_\Omega\psi\left(y\right)dy\le 1. \tag{2.11}$$

Combining now (2.9)-(2.11) and applying Lemma 2.2, we have

$$\rho_1\left(x,t\right)\ge\rho_2\left(x,t\right),$$

which implies that

$$g_1\left(x,t\right)\ge g_2\left(x,t\right).$$

The proof of Lemma 2.3 is complete. □

On the basis of the above lemmas, we obtain the following comparison principle for problem (1.1).

Proposition 2.4 (Comparison principle)**.** *Let* $\underline{u}\left(x,t\right)$ *and* $\overline{u}\left(x,t\right)$ *be a nonnegative subsolution and a nonnegative supersolution of problem* (1.1) *in* Ω_T*, respectively. Suppose that* $\underline{u}\left(x,t\right)$*,* $\overline{u}\left(x,t\right)>0$ *in* $\overline{\Omega}_T$ *if* $\min\left\{p,q,l\right\}<1$*. If* $\underline{u}\left(x,0\right)\le\overline{u}\left(x,0\right)$ *for* $x\in\overline{\Omega}$*, then* $\underline{u}\left(x,t\right)\le\overline{u}\left(x,t\right)$ *in* $\overline{\Omega}_T$*.*

Proof. It is easy to check that $\underline{u}$, $\overline{u}$ and φ satisfy hypotheses (H3). □

3 Global existence of the solution

In this section, we investigate the global existence of the solution to problem (1.1).

Proof of Theorem 1.1. Let T be any positive number. In order to prove our conclusion, according to Proposition 2.4, we only need to construct a suitable gloabl supersolution of problem (1.1) in Ω_T. Remember that λ and ϕ are the first eigenvalue and the corresponding normalized eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary condition. We choose ζ to satisfy that for some $0<\epsilon<1$,

$$\max_{\partial\Omega\times\overline{\Omega}}\varphi\left(x,y\right)\int_\Omega\frac{1}{\zeta\phi\left(y\right)+\epsilon}dy\le 1. \tag{3.1}$$

Now, let $v(x,t)$ be defined as

$$v\left(x,t\right)=\frac{\eta e^{\kappa t}}{\zeta\phi\left(x\right)+\epsilon}$$

with

$$\eta=\sup_{\overline{\Omega}}\left(u_0+1\right)\left(\zeta\phi+\epsilon\right),\quad \kappa=\max\left\{\sqrt{\frac{2}{p}},\quad 2\lambda+\sup_{\overline{\Omega}}\frac{4\zeta^2\left|\nabla\phi\right|^2}{\left(\zeta\phi+\epsilon\right)^2}\right\}. \tag{3.2}$$

A simple computation shows

$$
\begin{aligned}
Pv &\equiv v_t - \Delta v - v^q \int_0^t v^p ds \\
&= \kappa v - v\left(\frac{\lambda\zeta\phi}{\zeta\phi+\epsilon} + \frac{2\zeta^2 \left|\nabla\phi\right|^2}{\left(\zeta\phi+\epsilon\right)^2}\right) - \frac{\eta^q e^{q\kappa t}}{\left(\zeta\phi+\epsilon\right)^q} \int_0^t \frac{\eta^p e^{p\kappa s}}{\left(\zeta\phi+\epsilon\right)^p} ds \\
&= \kappa v - v\left(\frac{\lambda\zeta\phi}{\zeta\phi+\epsilon} + \frac{2\zeta^2 \left|\nabla\phi\right|^2}{\left(\zeta\phi+\epsilon\right)^2}\right) - \frac{\eta^{p+q} e^{(p+q)\kappa t}}{\kappa p \left(\zeta\phi+\epsilon\right)^{p+q}} + \frac{\eta^{p+q}}{\kappa p \left(\zeta\phi+\epsilon\right)^{p+q}} \\
&\geq \kappa v - v\left(\frac{\lambda\zeta\phi}{\zeta\phi+\epsilon} + \frac{2\zeta^2 \left|\nabla\phi\right|^2}{\left(\zeta\phi+\epsilon\right)^2}\right) - \frac{\eta^{p+q} e^{(p+q)\kappa t}}{\kappa p \left(\zeta\phi+\epsilon\right)^{p+q}}.
\end{aligned}
\tag{3.3}
$$

Noticing that $v\left(x,t\right) \geq 1$ and $p+q \leq 1$, then from (3.2) and (3.3), it follows that

$$
Pv \geq v\left(\frac{\kappa}{2} - \frac{1}{\kappa p}\right) + v\left[\frac{\kappa}{2} - \left(\frac{\lambda\zeta\phi}{\zeta\phi+\epsilon} + \frac{2\zeta^2 \left|\nabla\phi\right|^2}{\left(\zeta\phi+\epsilon\right)^2}\right)\right] \geq 0, \tag{3.4}
$$

and

$$
v\left(x,0\right) = \frac{\eta}{\zeta\phi(x)+\epsilon} \geq \frac{\sup\limits_{\overline{\Omega}} \left(u_0\left(x\right)+1\right)\left(\zeta\phi\left(x\right)+\epsilon\right)}{\zeta\phi(x)+\epsilon} > u_0\left(x\right). \tag{3.5}
$$

On the other hand, for any $(x,t) \in \partial\Omega \times (0,T)$, by virtue of (3.1), we have

$$
\begin{aligned}
v\left(x,t\right) &= \frac{\eta e^{\kappa t}}{\epsilon} > \eta e^{\kappa t} \geq \int_\Omega \varphi\left(x,y\right) \frac{\eta e^{\kappa t}}{\zeta\phi\left(y\right)+\epsilon} dy = \int_\Omega \varphi\left(x,y\right) v\left(y,t\right) dy \\
&\geq \int_\Omega \varphi\left(x,y\right) v^l\left(y,t\right) dy,
\end{aligned}
\tag{3.6}
$$

where the conditions $v(x,t) > 1$ and $l \leq 1$ are used.

Combining now from (3.3) to (3.6), we know that $v\left(x,t\right)$ is a supersolution of (1.1) in Ω_T and the solution $u\left(x,t\right) \leq v\left(x,t\right)$ by comparison principle, therefore the problem (1.1) has global solutions. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Let Ω_1 be a bounded domain in $\mathbb{R}^N$ such that $\Omega \subset\subset \Omega_1$, let λ_1 be the first eigenvalue of the following eigenvalue problem

$$
\begin{cases}
-\Delta\phi_1\left(x\right) = \lambda_1\phi_1, & x \in \Omega_1, \\
\phi_1\left(x\right) = 0, & x \in \partial\Omega_1,
\end{cases}
\tag{3.7}
$$

and ϕ_1 the corresponding eigenfunction. Denote $\sup\limits_{\Omega_1} \phi_1 = M$. It is obvious that there is a constant $\mu > 1$ such that

$$
\frac{\sup\limits_{\Omega_1} \phi_1}{\inf\limits_{\overline{\Omega}} \phi_1} < \mu. \tag{3.8}
$$

Let
$$\phi_2(x) = \frac{\mu\xi}{M}\phi_1(x), \quad x \in \overline{\Omega}_1,$$
where $0 < \xi \le \mu^{-\frac{l}{l-1}}$ is a constant. Then we can know that $\sup\limits_{\Omega_1}\phi_2 = \mu\xi$ and
$$\frac{\sup\limits_{\Omega_1}\phi_2}{\inf\limits_{\overline{\Omega}}\phi_2} = \frac{\frac{\mu\xi}{M}\sup\limits_{\Omega_1}\phi_1}{\frac{\mu\xi}{M}\inf\limits_{\overline{\Omega}}\phi_1} < \mu. \tag{3.9}$$
Furthermore, it is easy to verify that ϕ_2 satisfies (3.7). Then, from (3.9), it follows immediately that
$$\inf_{\partial\Omega}\phi_2 > \xi. \tag{3.10}$$

For $q > 1$, let
$$v(x,t) = \frac{\phi_2(x)}{(A+t)^{\alpha}},$$
where $\alpha > 0$ and $A > 1$ are constants to be determined later. Then after a simple computation, we have
$$\begin{aligned} Pv &= -\frac{\alpha\phi_2}{(A+t)^{\alpha+1}} + \frac{\lambda_1\phi_2}{(A+t)^{\alpha}} - \frac{\phi_2^{p+q}}{(A+t)^{\alpha q}}\int_0^t \frac{1}{(A+s)^{\alpha p}}ds \\ &= \frac{\phi_2}{(A+t)^{\alpha}}\left(\lambda_1 - \frac{\alpha}{A+t} - \frac{\phi_2^{p+q-1}}{(1-\alpha p)(A+t)^{\alpha(p+q-1)-1}} + \frac{A^{1-\alpha p}\phi_2^{p+q-1}}{(1-\alpha p)(A+t)^{\alpha(q-1)}}\right). \end{aligned}$$
Since that $q > 1$, we can choose α to satisfy
$$\frac{1}{p+q-1} < \alpha < \frac{1}{p}.$$
Then we have that $Pv \ge 0$ with A large enough.

On the other hand, since $\int_\Omega \varphi(x,y)\,dy < 1$ and $l > 1$, we have on the boundary that
$$v(x,t) > \frac{\xi}{(A+t)^{\alpha}} \ge \left(\frac{\mu\xi}{(A+t)^{\alpha}}\right)^l \ge \int_\Omega \varphi(x,y)\,v^l(x,t)\,dy. \tag{3.11}$$
Thus, by comparison principle, we know that the solution of problem (1.1) exists globally provided that
$$u_0(x) \le \frac{\phi_2(x)}{A^{\alpha}}.$$

For $q = 1$, let
$$v(x,t) = \frac{\beta\phi_2(x)}{e^{\tau t}},$$
where $\beta < 1$ and $\tau > 0$ are two constants to be determined later. Computing directly, we obtain
$$Pv = \frac{\beta\phi_2}{e^{\tau t}}\left(\lambda_1 - \tau + \frac{\beta^p\phi_2^p}{\tau p e^{\tau p t}} - \frac{\beta^p\phi_2^p}{\tau p}\right).$$

If $\tau < \lambda_1$ and β is sufficiently small, then we can conclude that $Pv \geq 0$. On the other hand, since $0 < \xi \leq \mu^{-\frac{l}{l-1}}$, we have on the boundary that

$$v\left(x,t\right) > \frac{\beta\xi}{e^{\tau t}} \geq \left(\frac{\mu\beta\xi}{e^{\tau t}}\right)^l \geq \int_\Omega \varphi\left(x,y\right) v^l\left(x,t\right) dy, \tag{3.12}$$

where the conditions $\int_\Omega \varphi\left(x,y\right) dy < 1$ are used. Therefore, $v\left(x,t\right)$ is a supersolution of problem (1.1) if $u_0\left(x\right) \leq \beta\phi_2\left(x\right)$. The proof of Theorem 1.2 is complete. $\square$

4 Blow-up of the solution

In this section, we will discuss the blow-up property of the solution to problem (1.1), and give the proofs of Theorems 1.3 and 1.4.

Proof of Theorem 1.3. We employ a variant of Kaplan's method to prove our blow-up result of the case $l > 1$. Let $u\left(x,t\right)$ be the unique solution to (1.1) and

$$J\left(t\right) = \int_\Omega \phi\left(x\right) u\left(x,t\right) dx, \quad 0 \leq t < T.$$

Taking the derivative of $J\left(t\right)$ with respect to t, and using Green's formula we could obtain

$$\begin{aligned}
J'\left(t\right) &= \int_\Omega \phi\left(\Delta u + u^q \int_0^t u^p(x,s)ds\right)dx \\
&= \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\phi dS - \int_\Omega \nabla\phi \cdot \nabla u dx + \int_\Omega u^q \int_0^t \phi(x)u^p(x,s)dsdx \\
&= \int_\Omega u\Delta\phi dx - \int_{\partial\Omega} \frac{\partial \phi}{\partial \nu} u dS + \int_\Omega u^q \int_0^t \phi(x)u^p(x,s)dsdx \\
&= -\lambda J(t) - \int_{\partial\Omega} \frac{\partial \phi}{\partial \nu}\left(\int_\Omega \varphi\left(x,y\right) u^l\left(y,t\right) dy\right) dS \\
&\quad + \int_\Omega u^q \int_0^t \phi(x)u^p(x,s)dsdx.
\end{aligned} \tag{4.1}$$

Applying $\int_{\partial\Omega} \frac{\partial\phi}{\partial\nu} dS = -\lambda \int_\Omega \phi dx = -\lambda$ to (4.1), we then have

$$J'\left(t\right) \geq -\lambda J(t) + \frac{\lambda \min\limits_{\partial\Omega\times\overline{\Omega}} \varphi}{\max\limits_{\overline{\Omega}} \phi} \int_\Omega \phi u^l dx. \tag{4.2}$$

From (4.2) and Jensen's inequality, it follows that

$$J'\left(t\right) \geq -\lambda J(t) + \frac{\lambda \min\limits_{\partial\Omega\times\overline{\Omega}} \varphi}{\max\limits_{\overline{\Omega}} \phi} J^l(t). \tag{4.3}$$

Next, we look for the solution $J(t)$ to (4.3) with $J(0) > 1$ on its interval of existence. Since the function $f(J) = J^l$ is convex, then there exists $\varrho > 1$ such that

$$\frac{\lambda \min\limits_{\partial\Omega\times\overline{\Omega}} \varphi}{\max\limits_{\overline{\Omega}} \phi} J^l \geq 2\lambda J \quad \text{for all } J \geq \varrho.$$

It follows easily that if $J(0) > \varrho$, then $J(t)$ is increasing on its interval of existence and

$$J'(t) \geq \frac{1}{2} J^l. \tag{4.4}$$

From the above inequality it follows that

$$\lim_{t\to T_0^-} J(t) = +\infty, \tag{4.5}$$

where

$$T_0 = \frac{2}{(l-1)\, J^{l-1}(0)}.$$

Then by assumptions in Theorem 1.3, the solution $u(x,t)$ becomes infinite in a finite time. The proof of Theorem 1.3 is complete. $\square$

Proof of Theorem 1.4. Consider the following equation

$$\begin{cases} v_t = \Delta v + v^q \int_0^t v^p(x,s)ds, & x \in \Omega, t > 0, \\ v(x,t) = 0, & x \in \partial\Omega, t > 0, \\ v(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{4.6}$$

and let $v(x,t)$ be the solution to problem (4.6). It is obvious that $v(x,t)$ is a subsolution of problem (1.1). For the case $q \geq 1$, from Theorem 3.1 in [14], we know that $v(x,t)$ blows up in finite time for sufficiently large $u_0(x)$. For the case $q < 1$, it is well-known that $v(x,t)$ blows up in a finite time for any nonnegative $u_0(x)$ (see [14], Theorem 3.3). By Proposition (2.4), we obtain our blow-up result immediately. The proof of Theorem 1.4 is complete. $\square$

5 Blow-up rate estimate

In this section, we consider problem (1.1) with $q = 0$ and $l = 1$, i.e.,

$$\begin{cases} u_t = \Delta u + \int_0^t u^p(x,s)ds, & x \in \Omega, t > 0, \\ u(x,t) = \int_\Omega \varphi(x,y)\, u(y,t)dy, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{5.1}$$

where $p > 1$. By Theorem 1.4, for any nonnegative nontrivial initial data u_0, u blows up in a finite time $T^* < \infty$. We first give the upper bounder of the blow-up rate near the blow-up time.

Lemma 5.1. *Suppose that* $\int_\Omega \varphi(x,y)dy \le 1$ *and assumptions (H1) hold, then for any* $t_1 \in (t_0, T^*)$, *the blow-up solution* $u(x,t)$ *to problem (5.1) satisfies*

$$u(x,t) \le C_1 (T^* - t)^{-\frac{2}{p-1}}, \quad t_1 < t < T^*, \tag{5.2}$$

where C_1 *is a positive constant.*

Proof. Let

$$J(x,t) = u_t - \delta \int_0^t u^p ds \quad \text{for} \quad (x,t) \in \Omega \times (t_1, T^*),$$

where δ is a sufficiently small positive constant. After straightforward computation, we then obtain

$$\begin{aligned} J_t - \Delta J &= u_{tt} - \delta u^p - \Delta u_t + \delta p \int_0^t u^{p-1} \Delta u ds + \delta p(p-1) \int_0^t u^{p-2} |\nabla u|^2 ds \\ &= (u_t - \Delta u)_t - \delta u^p + \delta p \int_0^t u^{p-1} \Delta u ds + \delta p(p-1) \int_0^t u^{p-2} |\nabla u|^2 ds \\ &\ge (1-\delta) u^p + \delta p \int_0^t u^{p-1} \left(u_t - \int_0^s u^p d\tau \right) ds \\ &= (1-\delta) u_0^p + p \int_0^t u^{p-1} \left(u_t - \delta \int_0^s u^p d\tau \right) ds \\ &\ge p \int_0^t u^{p-1} J ds. \end{aligned} \tag{5.3}$$

Fix $(x,t) \in \partial\Omega \times (t_1, T)$, we have

$$\begin{aligned} J(x,t) &= u_t - \delta \int_0^t u^p ds \\ &= \int_\Omega \varphi(x,y) u(y,t) u_t(y,t) dy - \delta \int_0^t \left(\int_\Omega \varphi(x,y) u(y,t) dy \right)^p ds. \end{aligned}$$

Differentiating the equation in (5.1) with respect to t, we obtain

$$u_{tt} = \Delta u_t + u^p \ge \Delta u_t. \tag{5.4}$$

Combining (H1) and (5.4), we know that $u_t > 0$ in $\Omega \times (t_1, T^*)$ for any $t_1 \in (t_0, T^*)$. Thus, according to $u_t(y,t) = J(y,t) + \delta \int_0^t u^p ds$, we have

$$\begin{aligned} &\int_\Omega \varphi(x,y) u_t(y,t) dy - \delta \int_0^t \left(\int_\Omega \varphi(x,y) u(y,t) dy \right)^p ds \\ &\quad = \int_\Omega \varphi(x,y) \left(J(y,t) + \delta \int_0^t u^p ds \right) dy - \delta \int_0^t \left(\int_\Omega \varphi(x,y) u(y,t) dy \right)^p ds \\ &\quad = \int_\Omega \varphi(x,y) J(y,t) dy + \delta \int_0^t \left[\int_\Omega \varphi(x,y) u^p(y,s) dy - \left(\int_\Omega \varphi(x,y) u(y,t) dy \right)^p \right] ds. \end{aligned}$$

Noticing that $0 < \Phi(x) = \int_\Omega \varphi(x,y)dy \le 1$, $x \in \partial\Omega$, we can apply Jensen's inequality to the last integral in the above inequality,

$$\begin{aligned}
&\int_\Omega \varphi(x,y)u^p(y,s)dy - \left(\int_\Omega \varphi(x,y)u(y,t)dy\right)^p \\
&\ge \Phi(x)\left(\int_\Omega \varphi(x,y)u(y,t)\frac{dy}{\Phi(x)}\right)^p - \left(\int_\Omega \varphi(x,y)u(y,t)dy\right)^p \\
&\ge 0.
\end{aligned}$$

Here, we used $p > 1$ and $0 < \Phi(x) \le 1$ in the last inequality. Hence

$$J(x,t) \ge \int_\Omega \varphi(x,y)J(y,t)dy \quad \text{for } (x,t) \in \partial\Omega \times (t_1, T). \tag{5.5}$$

On the other hand, (H1) implies that

$$J(x,t_1) = u_t\,(x,t_1) - \delta \int_0^{t_1} u^p\,(x,s)\,ds \ge 0 \quad \text{in } \overline{\Omega}. \tag{5.6}$$

Since φ and u are nonnegative bounded continuous for $(x,t) \in \Omega \times (t_1, T^*)$, it follows from (5.3), (5.5) and (5.6) that $J\,(x,t) \ge 0$ for $(x,t) \in \Omega \times (t_1, T^*)$, which implies

$$u_t \ge \delta \int_0^t u^p(x,s)ds. \tag{5.7}$$

Multiplying both sides of the inequality (5.7) by u^p and integrating over (t_1, t), we have

$$u^p\,(x,t) \ge (\delta\,(1+p))^{\frac{p}{1+p}} \left(\int_{t_1}^t u^p\,(x,s)\,ds\right)^{\frac{2p}{1+p}}, \quad t_1 < t < T^*. \tag{5.8}$$

Integrating above inequality from t to T^*, we deduce that

$$\int_{t_1}^t u^p\,(x,s)\,ds \le (\delta(1+p))^{-\frac{p}{p-1}}\,(T^*-t)^{-\frac{p+1}{p-1}}, \quad t_1 < t < T^*. \tag{5.9}$$

Taking a special $t' = \frac{T^*+t}{2}$ and applying $u_t \ge 0$, we discover that

$$\begin{aligned}
\frac{T^*-t}{2}u^p\,(x,t) &\le \int_t^{t'} u^p(x,s)ds \le \int_{t_1}^{t'} u^p(x,s)ds \\
&\le (\delta(1+p))^{-\frac{p}{p-1}}\,(T^*-t')^{-\frac{p+1}{p-1}} \\
&\le (\delta(1+p))^{-\frac{p}{p-1}}\left(\frac{T^*-t}{2}\right)^{-\frac{p+1}{p-1}},
\end{aligned}$$

which yields

$$u\,(x,t) \le C_1\,(T^*-t)^{-\frac{2}{p-1}}, \quad t_1 < t < T^*. \tag{5.10}$$

where $C_1 = \left(\frac{4}{\delta(1+p)}\right)^{\frac{1}{p-1}}$. $\square$

Proof of Theorem 1.6. Let $x_0 \in \overline{\Omega}$ such that $I(t) = u(x_0, t) = \max_{\overline{\Omega}} u(x, t)$. From the equation in (5.1), we have the following estimate (see [8], Theorem 4.5)

$$I'(t) \leq \int_0^t u^p ds \leq \int_0^t I^p ds, \quad \text{for} \quad 0 < t < T^*.$$

Similar to (5.9), we can easily get

$$\int_0^t I^p(x, s)\, ds \geq C_2'(T^* - t)^{-\frac{p+1}{p-1}}, \quad 0 < t < T^*. \tag{5.11}$$

For $t_1 \leq \varsigma < t < T^*$, by exploiting (5.10), (5.11) and I being nondecreasing on $[t_1, T^*)$, we obtain

$$C_2'(T^* - t)^{-\frac{p+1}{p-1}} \leq \int_0^\varsigma I^p ds + \int_\varsigma^t I^p ds \leq C_1(T^* - \varsigma)^{-\frac{p+1}{p-1}} + (t - \varsigma)\, I^p(t).$$

For t close enough to T^*, taking $\varsigma = at + (1-a)T^*$ with $a = \left(\frac{2C_1}{C_2'}\right)^{\frac{p-1}{p+1}} > 1$, we deduce that

$$I(t) \geq C_2(T^* - t)^{-\frac{2}{p-1}}, \tag{5.12}$$

which proves the lower estimate. Combining (5.12) with Lemma 5.1, we obtain the blow-up estimate. The proof of the Theorem 1.6 is complete. □

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