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Some properties of solutions for a class of metaparabolic equations *

Changchun Liu, Yujing Guan and Zhao Wang

Department of Mathematics
Jilin University, Changchun 130012, P. R. China

liucc@jlu.edu.cn

Abstract. In this paper, we study the initial boundary value problem for a class of metaparabolic equations. We establish the existence of solutions by the energy techniques. Some results on the regularity, blow-up and existence of global attractor are obtained.
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1 Introduction

In this paper, we study a metaparabolic equation

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} + \gamma \Delta^2 u = \operatorname{div}(\varphi(\nabla u)), \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n (n \leq 3)$ with smooth boundary, where $\gamma > 0$ is the interfacial energy parameter, $k > 0$ is the viscosity coefficient, $\varphi(\nabla u)$ is an intrinsic chemical potential with typical example as

$$\varphi(\nabla u) = \gamma_1 |\nabla u|^2 \nabla u - \nabla u,$$

where γ_1 is a constant. The term $\Delta^2 u$ denotes the capillarity-driven surface diffusion, and $\operatorname{div}(\varphi(\nabla u))$ denotes the upward hopping of atoms.

The equation (1.1) is a typical higher order equation, which has an extensive physical background and a rich theoretical connotation. A. Novick-Cohen [10] derived the following equation to study the dynamics of viscous first order phase transitions in cooling binary solutions such as alloys, glasses and polymer mixtures

$$\frac{\partial u}{\partial t} = \Delta \left[\mu + k \frac{\partial \Delta u}{\partial t} \right],$$

where $u(x, t)$ is a concentration, μ is the intrinsic chemical potential. If we take $\mu = \varphi(u) - \gamma \Delta u$, we obtain the viscous Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} + \gamma \Delta^2 u = \Delta \varphi(u).$$

Many authors have paid much attention to the viscous Cahn-Hilliard equation, among which some numerical approaches and basic existence results have been

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developed [1, 3, 9, 12]. The well-known Cahn-Hilliard equation is obtained by setting $k = 0$, which has been well studied; see for example [2, 4, 15]. In fact, when the influence of many factors, such as the molecular and ion effects, are considered, one has the nonlinear relation $\operatorname{div}(\varphi(\nabla u))$ in stead of $\Delta\varphi(u)$ in right-hand side of above equation, so we obtain the equation (1.1).

On the basis of physical consideration, the equation (1.1) is supplemented by the zero mass flux boundary condition

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = \frac{\partial \Delta u}{\partial n}\Big|_{\partial\Omega} = 0, \tag{1.2}$$

and initial value condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.3}$$

During the past years, many authors have paid much attention to the equation (1.1) for a special case with $k = 0$ [11, 14]. B. B. King, O. Stein and M. Winkler [5] studied the equation (1.1) for a special case with $k = 0$, namely,

$$\frac{\partial u}{\partial t} + \Delta^2 u - \operatorname{div}(f(\nabla u)) = g(x,t),$$

where reasonable choice of $f(z)$ is $f(z) = |z|^{p-2}z - z$. They proved the existence, uniqueness and regularity of solutions in an appropriate function space for initial-boundary value problem. Liu considered the equation with nonlinear principal part

$$\frac{\partial u}{\partial t} + \operatorname{div}\left[m(u)k\nabla\Delta u - |\nabla u|^{p-2}\nabla u\right] = 0.$$

He proved the existence of solutions for one dimension [6] and two dimensions [7, 8].

The purpose of the present paper is devoted to the investigation of properties of solutions with γ_1 not restricted to be positive. We first discuss the regularity. We show that the solutions might not be classical globally. In other words, in some cases, the solutions exist globally, while in some other cases, such solutions blow up at a finite time. The main difficulties for treating the problem (1.1)–(1.3) are caused by the nonlinearity of $\operatorname{div}(\varphi(\nabla u))$ and the lack of maximum principle. To prove the existence of solutions, the method we use is based on the energy estimates and the Schauder type a priori estimates. In order to prove the blow-up result, we construct a new Lyapunov functional.

Throughout the paper we use Q_T to denote $\Omega\times(0,T)$. The norms of $L^\infty(\Omega)$, $L^2(\Omega)$ and $H^s(\Omega)$ are denoted by $\|\cdot\|_\infty$, $\|\cdot\|$ and $\|\cdot\|_s$.

2 Global existence

Let $\Omega = (0,1)$ and consider the following initial-boundary value problem

$$\frac{\partial u}{\partial t} + \gamma D^4 u - k\frac{\partial D^2 u}{\partial t} = D\varphi(Du), \qquad 0 < x < 1, 0 < t < T, \tag{2.1}$$

$$Du(0,t) = Du(1,t) = D^3u(0,t) = D^3u(1,t) = 0, \qquad t > 0, \tag{2.2}$$
$$u(x,0) = u_0(x), \qquad 0 < x < 1, \tag{2.3}$$

where $D = \frac{\partial}{\partial x}$, $\varphi(z) = -z + \gamma_1 z^3$ with γ, γ_1 and k being constants with $\gamma, k > 0$. From the classical approach, it is not difficult to conclude that the problem admits a unique classical solution local in time. So, it is sufficient to make a priori estimates.

Theorem 2.1 *If $\gamma_1 > 0$, then for any initial data $u_0 \in H^2(\Omega)$ with $Du_0(0) = Du_0(1) = 0$ and $T > 0$, the problem (2.1)–(2.3) admits one and only one solution u with u_t, D^4u, $D^2u_t \in L^2(Q_T)$, where $Q_T = \Omega \times (0,T)$.*

Proof. Multiplying both sides of the equation by u and then integrating resulting relation with respect to x over $(0,1)$, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2dx + \gamma\int_0^1 (D^2u)^2dx + k\int_0^1 Du_tDudx = -\int_0^1 \varphi(Du)Dudx.$$

That is

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\int_0^1 u^2dx + k\int_0^1 (Du)^2dx\right) + \gamma\int_0^1 (D^2u)^2dx \\ +\gamma_1\int_0^1 (Du)^4dx = \int_0^1 (Du)^2dx.\end{aligned} \tag{2.4}$$

The Gronwall inequality implies that

$$\sup_{0<t<T}\int_0^1 u^2dx \le C, \tag{2.5}$$
$$\sup_{0<t<T}\int_0^1 |Du|^2dx \le C, \tag{2.6}$$
$$\iint_{Q_T} (D^2u)^2dxdt \le C. \tag{2.7}$$

By the Sobolev imbedding theorem, it follows from (2.5), (2.6) that

$$\sup_{Q_T}|u(x,t)| \le C. \tag{2.8}$$

Let

$$F(t) = \int_0^1 \left(\frac{\gamma}{2}(D^2u)^2 + H(Du)\right)dx,$$

where $H(Du) = \frac{\gamma_1}{4}(Du)^4 - \frac{1}{2}(Du)^2$. Then, we have

$$\frac{dF(t)}{dt} = \int_0^1 (\gamma D^2uD^2u_t + \varphi(Du)Du_t)dx.$$

Integrations by parts and (2.1)-(2.2) yield

$$\begin{aligned}\frac{dF(t)}{dt} &= \int_0^1 [\gamma D^4 u - D(\varphi(Du))]u_t dx \\ &= -\int_0^1 (u_t - kD^2 u_t)u_t dx \\ &= -\int_0^1 [(u_t)^2 + k(Du_t)^2]dx \le 0\end{aligned}$$

and

$$F(t) \le F(0) = \int_0^1 \Big(\frac{\gamma}{2}(D^2 u_0)^2 + H(Du_0)\Big)\,dx.$$

By Young's inequality

$$(Du)^2 \le \varepsilon (Du)^4 + C,$$

we have

$$\sup_{0<t<T} \int_0^1 (D^2 u)^2 dx \le C. \tag{2.9}$$

By the Sobolev imbedding theorem, it follows from (2.5), (2.6), (2.9) that

$$\sup_{Q_T} |Du(x,t)| \le C. \tag{2.10}$$

Again multiplying both sides of the equation (2.1) by $D^4 u$ and integrating the resulting relation with respect to x over $(0,1)$, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (D^2 u)^2 dx + \gamma \int_0^1 (D^4 u)^2 dx + k\int_0^1 D^3 u_t D^3 u dx = \int_0^1 \varphi'(Du) D^2 u D^4 u dx.$$

Hence

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 (D^2 u)^2 dx + \gamma \int_0^1 (D^4 u)^2 dx + k\int_0^1 D^3 u_t D^3 u dx \\ \le\ & C\left(\sup_{Q_T} |Du|^2 + 1\right)\left(\varepsilon \int_0^1 (D^4 u)^2 dx + C_\varepsilon \int_0^1 (D^2 u)^2 dx\right) \\ \le\ & \frac{\gamma}{2}\int_0^1 (D^4 u)^2 dx + C\int_0^1 (D^2 u)^2 dx.\end{aligned} \tag{2.11}$$

By Gronwall's inequality

$$\iint_{Q_T} (D^4 u)^2 dx dt \le C. \tag{2.12}$$

The a priori estimates (2.8), (2.9) and (2.12) complete the proof of global existence.

Remark 2.1 *The corresponding problem for* $n = 2, 3$ *is (1.1)-(1.3). For* $u_0 \in H^2(\Omega)$ *with* $\frac{\partial u_0}{\partial n}\Big|_{\partial\Omega} = 0$*, there exists a unique global solution* u*. The proof is the same as that of Theorem 2.1 with minor changes. Without loss of generality, assume that*

$$\int_\Omega u_0(x)dx = 0 = \int_\Omega u(x,t)dx. \tag{2.13}$$

By the boundary conditions, (2.13) and the Poicaré-Friedrichs inequalities $\|\Delta u\|$*,* $\|\Delta^2 u\|$ *are equivalent to* $\|u\|_2$ *and* $\|u\|_4$*. Now as before in (2.5)-(2.7) and (2.9), we have*

$$\|u(t)\|_2 \leq C. \tag{2.14}$$

By Sobolev's imbedding theorem and (2.14), we have

$$\|\nabla u\|_{L^q} \leq C, \quad \text{for any } q < \infty, \ (n = 2), \tag{2.15}$$

$$\|\nabla u\|_{L^6} \leq C, \quad (n = 3). \tag{2.16}$$

By the Nirenberg inequality, we have

$$\|\nabla u\|_\infty \leq C\|\Delta^2 u\|^a \|\nabla u\|_{L^q}^{1-a}, \quad \text{where } a = \frac{1}{1+q}, \ (n = 2), \tag{2.17}$$

$$\|\nabla u\|_\infty \leq C\|\Delta^2 u\|^{1/4} \|\nabla u\|_{L^q}^{3/4}, \quad (n = 3). \tag{2.18}$$

Applying Young's inequality to the right-hand side of

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [(\Delta u)^2 + k|\nabla \Delta u|^2]dx + \gamma \int_\Omega (\Delta^2 u)^2 dx = \int_\Omega (3\gamma_1|\nabla u|^2 - 1)\Delta u \Delta^2 u dx.$$

We have

$$\begin{aligned}\int_\Omega (3\gamma_1|\nabla u|^2 - 1)\Delta u \Delta^2 u dx &\leq C(\|\nabla u|_\infty^2 + 1)\|\Delta u\|\|\Delta^2 u\| \\ &\leq C\|\Delta^2 u\|^{2a+1} \quad (n = 2),\end{aligned}$$

and

$$\begin{aligned}\int_\Omega (3\gamma_1|\nabla u|^2 - 1)\Delta u \Delta^2 u dx &\leq C(\|\nabla u|_\infty^2 + 1)\|\Delta u\|\|\Delta^2 u\| \\ &\leq C\|\Delta^2 u\|^{3/2} \quad (n = 3).\end{aligned}$$

Here, we have used (2.14), (2.17) and (2.18). Hence, we obtain

$$\|\Delta u\|^2 + k\|\nabla \Delta u\|^2 + \int_\Omega \|\Delta^2 u\|^2 dt \leq C.$$

This completes the proof of global existence.

3 Regularity

Now, we turn our discussion to the regularity of solutions.

Theorem 3.1 *Assume that* $\gamma_1 > 0$, $u_0 \in C^{4+\alpha}(\overline{\Omega})$ *and* $Du_0(0) = Du_0(1) = 0$. *Then the problem (2.1)–(2.3) admits a classical solution* $u \in C^{4+\alpha,1+\alpha/4}(\overline{Q_T})$.

Proof. Set $L = (I - kD^2)^{-1}$, and $Lg = w$, namely, w satisfies

$$\begin{cases} (I - kD^2)w = g, \quad x \in \Omega, \\ Dw = 0, \quad x \in \partial\Omega. \end{cases}$$

It is easily seen that $\int_\Omega |w|^2 dx \le \int_\Omega |g|^2 dx$, i. e.

$$\int_\Omega |Lg|^2 dx \le \int_\Omega |g|^2 dx.$$

Now, we rewrite the equation (2.1) into the form

$$(I - kD^2)\frac{\partial u}{\partial t} + \gamma D^4 u = D\varphi(Du).$$

Using the operator L for the above equation, we have

$$\frac{\partial u}{\partial t} + LD[\gamma D^3 u - \varphi(Du)] = 0. \tag{3.1}$$

Integrating the equation (3.1) with respect to x over $(y, y + (\Delta t)^{1/4}) \times (t_1, t_2)$, where $0 < t_1 < t_2 < T$, $\Delta t = t_2 - t_1$, we see that

$$\begin{aligned} & \int_y^{y+(\Delta t)^{1/4}} [u(z, t_2) - u(z, t_1)]dz \\ = & -\int_{t_1}^{t_2} L[(\gamma D^3 u(y', s) - \varphi(Du(y', s))) \\ & -(\gamma D^3 u(y, s) - \varphi(Du(y, s)))]ds. \end{aligned} \tag{3.2}$$

Set

$$N(s, y) = L[(\gamma D^3 u(y', s) - \varphi(Du(y', s))) - (\gamma D^3 u(y, s) - \varphi(Du(y, s)))]ds.$$

Then (3.2) is converted into

$$\begin{aligned} & (\Delta t)^{1/4} \int_0^1 u(y + \theta(\Delta t)^{1/4}, t_2) - u(y + \theta(\Delta t)^{1/4}, t_1))d\theta \\ = & -\int_{t_1}^{t_2} N(s, y)ds. \end{aligned}$$

Integrating the above equality with respect to y over $(x, x+(\Delta t)^{1/4})$, we get

$$(\Delta t)^{1/2}(u(x^*,t_2)-u(x^*,t_1)) = -\int_{t_1}^{t_2}\int_x^{x+(\Delta t)^{1/4}} N(s,y)dyds.$$

Here, we have used the mean value theorem, where $x^* = y^* + \theta^*(\Delta t)^{1/4}$, $y^* \in (x, x+(\Delta t)^{1/4})$, $\theta \in (0,1)$. Hence by Hölder inequality and (2.10), (2.12), we get

$$|u(x^*,t_2)-u(x^*,t_1)|^2\Delta t \le C(\Delta t)^{5/4},$$

that is

$$|u(x^*,t_2)-u(x^*,t_1)| \le C(\Delta t)^{1/8}. \tag{3.3}$$

Again integrating the equation (2.1) over $(0,x)\times(t_1,t_2)$ yields

$$Du(x,t_2)-Du(x,t_1) = \int_0^x [u(z,t_2)-u(z,t_1)]dz$$

$$+\int_{t_1}^{t_2}\gamma D^3u(x,s)ds - \int_{t_1}^{t_2}\varphi(Du(x,s))ds.$$

Integrating the above equation with respect to x over $(y, y+(\Delta t)^{3/4})$, using the mean value theorem, and (3.3) we have

$$(\Delta t)^{3/4}|Du(x^*,t_2)-Du(x^*,t_1)| \le C(\Delta t)^{7/8},$$

that is

$$|Du(x^*,t_2)-Du(x^*,t_1)| \le C(\Delta t)^{1/8}. \tag{3.4}$$

By (2.6) and (2.9), we get

$$|u(x_1,t)-u(x_2,t)| \le C|x_1-x_2|^{1/2} \tag{3.5}$$

and

$$|Du(x_1,t)-Du(x_2,t)| \le C|x_1-x_2|^{1/2}. \tag{3.6}$$

Again using (3.5), (3.6) we have

$$|u(x_1,t_1)-u(x_2,t_2)| \le (|t_1-t_2|^{1/8}+|x_1-x_2|^{1/2}) \tag{3.7}$$

and

$$|Du(x_1,t_1)-Du(x_2,t_2)| \le (|t_1-t_2|^{1/8}+|x_1-x_2|^{1/2}). \tag{3.8}$$

Define the linear spaces

$$X = \left\{ u \in C^{1+\alpha,1+\alpha/4}(\overline{Q_T}); Du\Big|_{x=0,1} = 0, u(x,0) = u_0(x) \right\}$$

and the associated operator $T : X \to X,\ u \to w$, where w is determined by the following linear problem

$$\frac{\partial w}{\partial t} - k\frac{\partial D^2 w}{\partial t} + \gamma D^4 w - 3\gamma_1(Du)^2D^2w + D^2w = 0,$$

$$Dw(0,t) = Dw(1,t) = D^3w(0,t) = D^3w(1,t) = 0,$$
$$w(x,0) = u_0(x).$$

By (3.7) and (3.8) we see that

$$a(x,t) = (Du(x,t))^2$$

is Hölder continuous, hence from the classical linear theory, the above problem admits a unique solution $w \in C^{4+\beta,1+\beta/4}(\overline{Q_T})$, $\dfrac{\partial D^2 w}{\partial t} \in C^{\beta}(\overline{Q_T})$. So, the operator T is well-defined and compact. Moreover, if $u = \sigma Tu$, for some $\sigma \in (0,1]$, then u satisfies (2.1), (2.2) and $u(x,0) = \sigma u_0(x)$. Thus from the discussion above, we see that the norm of u in $C^{4+\beta,1+\beta/4}(\overline{Q_T})$ and norm of $\dfrac{\partial D^2 u}{\partial t} \in C^{\beta}(\overline{Q_T})$ can be estimated by some constant C depending only on the known quantities. By Leray–Schauder principle of fixed point, the operator T has a fixed point u, which is the desired classical solution of the problem (2.1)–(2.2). The proof is complete.

4 The case of small initial data

In §2, we have proved the global existence of solution of the problem (1.1)-(1.3) for $\gamma_1 > 0$. We turn now to the proof of global existence for $\gamma_1 < 0$. Without loss of generality, we assume that (2.13) holds.

Theorem 4.1 *If $\gamma > \frac{3}{2\pi^2}$, and $\|u_0\|_2$ is sufficiently small, then there exists a unique global solution u with $u_t, D^4u, D^2u_t \in L^2(Q_T)$ to (2.1)-(2.3).*

Proof. As mentioned before, it needs only to obtain a priori estimates for smooth solution u. In what follows $C_j (j = 1, 2 \cdots)$ denote the constants independent of u and t. Set

$$f = D(\gamma_1 (Du)^3).$$

The equation (2.1) may be rewritten as

$$\frac{\partial u}{\partial t} + \gamma D^4 u - k\frac{\partial D^2 u}{\partial t} + D^2 u = f. \tag{4.1}$$

For any fixed $t > 0$ we define

$$N(t) = \sup_{0<\tau<t} \|u(\tau)\|_2^2 + \int_0^t \|u(\tau)\|_2^2 d\tau. \tag{4.2}$$

Our goal is to show that $N(t)$ is bounded above.

Firstly, multiplying (4.1) by u and integrating with respect to x, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 - \gamma\int_0^1 D^3uDudx + k\int_0^1 DuDu_tdx - \int_0^1 (Du)^2dx = \int_0^1 fudx.$$

Thus

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2 + k\|Du\|^2) + \gamma\|D^2u\|^2 - \|Du\|^2 = \int_0^1 fudx.$$

Since $\int_0^1 u(x,t)dx = 0$, by Poincaré inequality and Friedrichs inequality, we have

$$\|u\|^2 \le \|Du\|^2 \le \frac{1}{\pi^2}\|D^2u\|^2.$$

It follows that

$$\frac{d}{dt}(\|u\|^2 + k\|Du\|^2) + (2\gamma - \frac{3}{\pi^2})\|D^2u\|^2 \le C_3\|f\|^2. \tag{4.3}$$

Integrating the (4.3) over (0, t), we have

$$\|u\|^2 + k\|Du\|^2 + C_2\int_0^t \|D^2u\|^2 \le \|u_0\|^2 + k\|Du_0\|^2 + C_3\int_0^t \|f\|^2ds. \tag{4.4}$$

Next, multiplying (4.1) by u_t and integrating with respect to x over (0, 1), and integrating by parts, we have

$$\|u_t\|^2 + \frac{d}{dt}(\gamma\|D^2u\|^2 + \|Du\|^2) + k\|Du_t\|^2 \le \|f\|^2.$$

Hence

$$\int_0^t (\|u_t\|^2 + k\|Du_t\|^2)ds + C_1\|D^2u\|^2 \le \|D^2u_0\| + \|Du_0\| + \int_0^t \|f\|^2ds. \tag{4.5}$$

By (4.4), (4.5) we obtain

$$N(t) \le C_4\Big\{\|u_0\|_2^2 + \int_0^t \|f\|^2ds\Big\}. \tag{4.6}$$

Noticing

$$\|f\|^2 \le C_5\|Du\|_\infty^4\|D^2u\|^2, \tag{4.7}$$

and using the Sobolev inequality and the Poincaré inequality, we obtain

$$\int_0^t \|f\|^2ds \le C_8 \sup_{0<s<t}\|u\|_2^4\int_0^t \|u\|_2^2ds. \tag{4.8}$$

Taking (4.6) and (4.8) together yields

$$N(t) \le C_9\{\|u_0\|_2^2 + N(t)^3\}, \qquad t > 0. \tag{4.9}$$

We conclude that there is a constant C_{10} such that

$$N(t) \le C_{10}\|u_0\|_2^2, \qquad \forall t > 0, \tag{4.10}$$

provided that $\|u_0\|_2$ is sufficiently small. To show this, we set

$$M(t) = C_9 N(t)^2.$$

Assume that $\|u_0\|_2$ is small enough, such that

$$M(0) = C_9\|u_0\|_2^4 < \frac{1}{2}, \quad 8{C_9}^3\|u_0\|_2^4 < 1. \tag{4.11}$$

Then we have the assertion

$$M(t) < \frac{1}{2}, \qquad \forall t > 0. \tag{4.12}$$

In fact, if (4.12) were not sure, then there would exist a $t_0 > 0$, such that $M(t_0) = \frac{1}{2}$ and $M(t) < \frac{1}{2}$ for $t \in (0, t_0)$. By (4.9) we obtain

$$N(t_0) \le \frac{C_9\|u_0\|_2^2}{1 - C_9 N(t_0)^2} \le \frac{C_9\|u_0\|_2^2}{1 - M(t_0)} \le 2C_9\|u_0\|_2^2. \tag{4.13}$$

Using the second inequality in (4.11), we have

$$M(t_0) = C_9 N(t_0)^2 \le 4(C_9)^3\|u_0\|_2^4 < \frac{1}{2}. \tag{4.14}$$

The contradiction shows that (4.12) and hence (4.10) holds. Finally multiplying (4.1) by $-D^2u$ and D^4u yield the following inequalities

$$\|Du\|^2 + k\|D^2u\|^2 + \int_0^t \|D^3u\|^2 ds \le C_{11}\{\|u_0\|_2^2 + \int_0^t \|f\|^2 ds\},$$

and

$$\|D^2u\|^2 + k\|D^3u\|^2 + \int_0^t \|D^4u\|^2 ds \le C_{12}\{\|u_0\|_2^2 + \int_0^t \|f\|^2 dx\}.$$

As did in section 2, we may easily show that the global solution u satisfies $u_t, D^4u, D^2u_t \in L^2(Q_T)$. The proof is complete.

5 Blow-up

In the previous sections, we have seen that the solution of the problem (1.1)–(1.3) is globally classical, provided that $\gamma_1 > 0$ or $\gamma_1 < 0$ and $\|u_0\|_2$ sufficiently small. The following theorem shows that the solution of the problem (1.1)–(1.3) blows up at a finite time for $\gamma_1 < 0$ and $F(0) \le 0$.

Theorem 5.1 *Assume that $u_0 \not\equiv 0$, $\gamma_1 < 0$ and $-\int_\Omega\{H(\nabla u_0) + \frac{\gamma}{2}|\Delta u_0|^2\}dx \ge 0$. Then the solution of the problem (1.1)–(1.3) must blow up at a finite time, namely, for some $T > 0$*

$$\lim_{t\to T}\|u(t)\|_1 = +\infty,$$

where $H(\nabla u) = \frac{\gamma_1}{4}|\nabla u|^4 - \frac{1}{2}|\nabla u|^2$.

Proof. As in the proof of Theorem 2.1,

$$2\int_\Omega H(\nabla u)dx - 2F(0) \le -\gamma\int_\Omega |\Delta u|^2 dx, \tag{5.1}$$

where

$$F(0) = \int_\Omega \left(\frac{\gamma}{2}(\Delta u_0)^2 + H(\nabla u_0)\right) dx.$$

Now multiplying (1.1) by u and integrating with respect to x, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|u\|_1^2 = -2\gamma\int_\Omega (\Delta u)^2 dx - 2\int_\Omega \varphi(\nabla u)\cdot\nabla u dx \\
\ge\ & 4\int_\Omega H(\nabla u)dx - 4F(0) - 2\int_\Omega \varphi(\nabla u)\nabla u dx \\
=\ & 4\int_\Omega \left(\frac{\gamma_1}{4}|\nabla u|^4 - \frac{1}{2}|\nabla u|^2\right) dx - 2\int_\Omega \left(\gamma_1|\nabla u|^4 - |\nabla u|^2\right) dx - 4F(0) \\
=\ & -\gamma_1\int_\Omega |\nabla u|^4 dx - 4F(0) \\
\ge\ & -\frac{\gamma_1}{2|\Omega|}\left(\int_\Omega |\nabla u|^2 dx\right)^2 - 4F(0),
\end{aligned}$$

by (2.13) and Poincaré inequality

$$\|u\|^2 \le C(\Omega)\int_\Omega |\nabla u|^2 dx.$$

Therefore

$$\|u\|_1^2 \le C\int_\Omega |\nabla u|^2 dx.$$

Again $-F(0) \ge 0$, hence

$$\frac{d}{dt}\|u\|_1^2 \ge -\frac{\gamma_1}{2|\Omega|C^2}\|u\|_1^4,$$

that is

$$\|u\|_1^2 \ge \frac{\|u_0\|_1^2}{1 + \frac{\gamma_1}{2|\Omega|C^2}t\|u_0\|_1^2}.$$

By $u_0 \not\equiv 0$, it follows that u must blow up in a finite time T.

6 Global attractor

In this section, we are going to prove the existence of attractor as $\gamma_1 > 0$. By Theorem 6.1, we can define the operator semigroup $\{S(t)\}_{t\ge 0}$ in H^2 space as

$$S(t)u_0 = u(t), \quad t \ge 0,$$

where $u(t)$ is the solution of (1.1)-(1.3) corresponding to initial value u_0.

To study the existence of a global attractor, we have to find a closed metric space and prove that there exists a global attractor in the closed metric space. Since the total mass is conserved for all time, it is not possible for us to have a global attractor for the whole space without any constraints. Instead, we consider a series of subspaces with constraints as follow

$$\left| \int_{\Omega} u dx \right| \leq \kappa,$$

for any given positive constant κ.

We let

$$X_{\kappa} = \left\{ u | u \in H^2(\Omega), \left| \int_{\Omega} u dx \right| \leq \kappa \right\},$$

where $\kappa > 0$ is a constant. It is easy to see that the restriction of $\{S(t)\}$ on the affined space X_{κ} is a well defined semigroup.

Theorem 6.1 *For every κ chosen as above, the semiflow associated with the solution u of the problem (1.1)- (1.3) possesses in X_{κ} a global attractor $\mathcal{A}_{\kappa}$ which attracts all the bounded set in X_{κ}.*

In order to prove Theorem 6.1, here we establish some a priori estimates for the solution u of problem (1.1)-(1.3). In what follows, we always assume that $\{S(t)\}_{t \geq 0}$ is the semigroup generated by the weak solutions of problem (1.1)-(1.3) with initial data $u_0 \in H^2(\Omega)$.

Lemma 6.1 *There exists a bounded set $\mathcal{B}_{\kappa}$ whose size depends only on κ and Ω, in X_{κ} such that for all the orbits starting from any bounded set B in X_{κ}, $\exists\, t_1 = t_1(B) \geq 0$ such that $\forall\, t \geq t_1$ all the orbits will stay in $\mathcal{B}_{\kappa}$.*

Proof. By Young inequality, it follows from (2.4) that

$$\frac{1}{2} \frac{d}{dt} \left(\int_0^1 u^2 dx + k \int_0^1 (Du)^2 dx \right) + \gamma \int_0^1 (D^2 u)^2 dx \leq C_0.$$

By Poincaré inequality, we have

$$\|u\|^2 \leq \|Du\|^2 + \left(\int_0^1 u dx \right)^2 \leq \|Du\|^2 + \kappa^2.$$

Using Friedrichs inequality, we get $\|Du\|^2 \leq \|D^2 u\|^2$. So, if γ sufficiently large, we have

$$\frac{1}{2} \frac{d}{dt} \left(\int_0^1 u^2 dx + k \int_0^1 (Du)^2 dx \right) + C_2 \left(\int_0^1 u^2 dx + k \int_0^1 (Du)^2 dx \right) \leq C_1,$$

which immediately yields

$$\int_0^1 u^2 dx + k \int_0^1 (Du)^2 dx \leq e^{-C_2 t} \left(\int_0^1 u^2 dx + k \int_0^1 (Du)^2 dx \right) + \frac{C_1}{C_2}.$$

Thus for initial in any bounded set $B \subset X_\kappa$, there is a uniform time $t_0(B)$ depending on B such that for $t \geq t_0(B)$,

$$\int_0^1 u^2 dx + k \int_0^1 (Du)^2 dx \leq \frac{2C_1}{C_2}. \tag{6.1}$$

Similar to the above, multiplying both sides of the equation by D^2u, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (Du)^2 dx + \frac{\gamma}{2}\int_0^1 (D^3u)^2 dx + k\int_0^1 D^2u_t D^2 u dx \leq C. \tag{6.2}$$

Hence there is a uniform time $t_1(B)$ depending on B such that for $t \geq t_1(B)$,

$$\int_0^1 (Du)^2 dx + k\int_0^1 (D^2u)^2 dx \leq \frac{2C_3}{C_4}. \tag{6.3}$$

The lemma is proved.

Lemma 6.2 *For any initial data u_0 in any bounded set $B \subset X_\kappa$, there is a $t_2(B) > 0$ such that*

$$\|u(t)\|_{H^3} \leq C, \quad \forall\, t \geq t_2 > 0,$$

which turns out that $\bigcup_{t\geq t_2} u(t)$ is relatively compact in X_κ.

Proof. Firstly, integrating (6.2) over $(t, t+1)$, we have

$$\int_t^{t+1}\int_0^1 |D^3u|^2 dx d\tau \leq C. \tag{6.4}$$

Combining (2.11) with (6.4) and using the uniform Gronwall inequality, we have that there is a uniform time $t_2(B)$ depending on B such that for $t \geq t_2(B)$,

$$\int_0^1 (D^3u)^2 dx \leq \frac{2C_5}{C_6}. \tag{6.5}$$

The lemma is proved.

Then by Theorem I.1.1 in [13], we immediately conclude that $\mathcal{A}_\kappa = \omega(\mathcal{B}_\kappa)$, the ω-limit set of absorbing set $\mathcal{B}_\kappa$ is a global attractor in X_κ. By Lemma 6.2, this global attractor is a bounded set in H^3. Thus the proof of Theorem 6.1 is complete.

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